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THE MAURER-CARTAN CONNECTION AND A COVARIANT VERSION OF THE LEMMA OF POINCARÉ

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Abstract

Let a vectorial bundle on a Lie group of matrices, so that for any two intersecting trivializing charts, an element g of the group exists such that the components of the transition functions are (i) right multiplication by g, and (ii) the linear transformation defined by g. Then a connection on the bundle exists such that on each trivializing chart, the forms of the connection are the Maurer-Cartan forms.

In this bundle, a vector-valued form on the group whose covariant derivative on a trivializing chart is zero, is locally "covariantly exact".

1. Introduction

Some applications of the Maurer-Cartan connection are:

1. The definition of the Chern-Simons invariant can be extended to a family of bundles and connections over a family of odd-dimensional manifolds with boundary [2]. In the case of a single bundle over an odd-dimensional manifold Y with boundary X, the Chern-Simons

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invariant of the connection must be defined relative to some "boundary conditions". For this, one takes a fixed manifold Y_0 with $\partial Y_0 = X$ together with a principal bundle $Y_0 \times G \to Y_0$ with the Maurer-Cartan connection extending the background connection on X.

2. A family of flat connections over a genus g surface X can be defined: Let Y be the corresponding handle body such that $\partial Y = X$. Choose the Maurer-Cartan connection in the trivial bundle over Y as boundary condition in the sense of [2]. Then these data define a line bundle $\mathcal{L} \to Z$ with a canonical flat connection and an everywhere non-zero Chern-Simons section which is parallel.

3. For a group G and a subgroup H, all non-trivial generators of the de Rham cohomology of G/H were constructed in [1], and may be conveniently expressed in terms of the components of the Maurer-Cartan connection.

In this paper we consider:

• a Lie group G of matrices of n by n, and

• a vectorial bundle B of rank n over G, so that for any two trivializing charts (U_1, φ_1) and (U_2, φ_2) such that $U_1 \cap U_2 \neq \emptyset$, exists $g \in G$ so that

$$\forall (u, r) \in U_1 \cap U_2 \times \mathbf{R}^n : \varphi_2^{-1} \circ \varphi_1(u, r) = (ug, gr). \tag{1}$$

Theorem 1. *A connection on the bundle B exists so that on each trivializing chart, the forms of the connection are the Maurer-Cartan forms.*

We call the connection of Theorem 1 the *Maurer-Cartan connection*.

Consider a vector-valued form on the group G, so that the covariant derivative of the form is zero on a trivializing chart. Then the form is locally "covariantly exact".

We denote by ∇ the covariant differentiation.

Theorem 2. *Consider a B-valued form f, so that a trivializing chart U* *of B exists so that on U,*

$$\nabla f = 0. \tag{2}$$

Then there exist

- *an open subset* $S \subset U$, *and*
- *a B-valued form* f' *on S,*

such that on S,

$$f = \nabla f'.$$

2. Proofs

We denote by inverse the inversion of the elements of G,

$$\forall g \in G : \text{inverse}(g) = g^{-1}.$$

2.1. Proof of Theorem 1

This is a consequence of (1), Lemma 1 and Proposition 35.4 in [4].

Suppose that for two intersecting charts of G, an element g in G exists so that in order to change coordinates, we multiply on the right by g. Then the Maurer-Cartan forms transform by conjugation by g.

Lemma 1. *Suppose that for two charts*

$$(U_1, \{h_{1\,j}^{\,i}\}_{i, j \in \{1, \dots, n\}}) \text{ and } (U_2, \{h_{2\,j}^{\,i}\}_{i, j \in \{1, \dots, n\}})$$

of G such that $U_1 \cap U_2 \neq \emptyset$, *exists* $g \in G$ *so that*

$$h_{1\,j}^{\,i} = h_{2\,k}^{\,i} g_j^k.$$

Then

$$(h_{1\,k}^{\,i} \circ \text{inverse}) dh_{1\,j}^{\,k} = g^{-1}{}_k^i((h_{2\,l}^{\,k} \circ \text{inverse}) dh_{2\,m}^{\,l}) g_j^m.$$

2.2. Proof of Theorem 2

There exist

- n forms $f^1, \dots, f^n$ on the chart U, and

- n sections $s_i : U \to B, i \in \{1, \ldots, n\}$,

such that on the chart U, the vector-valued form f is the sum of tensorial products of the sections s_i and the forms f^i,

$$f = s_i \otimes f^i. \tag{3}$$

We denote by h^i_j, $i, j \in \{1, \ldots, n\}$, the coordinates on U. Then the covariant derivative of the form f is the sum of tensorial products

$$\nabla f = s_i \otimes (df^i + (h^i_j \circ \text{inverse}) dh^j_k \wedge f^k). \tag{4}$$

By (2),

$$\forall i \in \{1, \ldots, n\} : df^i + (h^i_j \circ \text{inverse}) dh^j_k \wedge f^k = 0.$$

By (6),

$$\forall i \in \{1, \ldots, n\} : d(h^i_j f^j) = 0.$$

By the lemma of Poincaré (Theorem V.4.1 in [3]), there exist

- an open subset $S \subset U$, and
- n forms $f'^1, \ldots, f'^n$ on S,

such that

$$h^i_j f^j = d(h^i_j f'^j) = h^i_j (df'^j + (h^j_k \circ \text{inverse}) dh^k_l \wedge f'^l),$$

by (6). Then the forms f^j are equal to

$$f^j = df'^j + (h^j_k \circ \text{inverse}) dh^k_l \wedge f'^l. \tag{5}$$

Substituting (5) into (3), we obtain that the vector-valued form f is the covariant derivative of the sum of tensorial products of the sections s_i and the forms f'^i,

$$f = s_i \otimes (df'^i + (h^i_j \circ \text{inverse}) dh^j_k \wedge f'^k) = \nabla(s_i \otimes f'^i),$$

by (4).

Lemma 2. *Consider*

- *a chart* $(U, \{h_j^i\}_{i, j \in \{1, \ldots, n\}})$ *of* G, *and*

- n *forms* $f^1, \ldots, f^n$ *on* U.

Then

$$\forall i \in \{1, \ldots, n\} : d(h_j^i f^j) = h_j^i (df^j + (h_k^j \circ \text{inverse}) dh_l^k \wedge f^l). \tag{6}$$

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