Adv. in Fuzzy Sets & Systems **1**(1) (2006), 1-15
This paper is available online at http://www.pphmj.com

AN EXAMPLE IN FUZZY OPTIONS PRICING

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Abstract

To make a prediction on the interest rate and prices we only have imprecise data and the interaction between the economic agents and expectations that are not only random but also subjective and vague. It seems adequate and more realistic to consider fuzzy numbers. In this way, one can incorporate the subjectivity inherent in the financial process.

Triangular fuzzy numbers are very convenient kind of fuzzy numbers to deal with that imprecision. As an illustration, we present an example in options pricing. We use the fuzzy arithmetic to compute the levels of confidence for an option. This is the discrete version of the classical Black-Scholes formula. In the future we shall consider the fuzzy Black-Scholes formula.

2000 Mathematics Subject Classification: 91B24, 26E50.

Keywords and phrases: fuzzy logic, option pricing, Black-Scholes formula.

Research partially supported by Ministerio de Educación y Ciencia and FEDER, project MTM2004-06652-C03-01, and by Xunta de Galicia and FEDER, project PGIDIT05PXIC20702PN.

Communicated by Wieslaw A. Dudek

Received May 18, 2006

I. Introduction

Finance involves several levels of uncertainty, subjectivity and imprecision. The best and most precise description of any financial process uses linguistic terms that are also imprecise and vague.

To deal with imprecision and uncertainty, we have at our disposal fuzzy logic. Fuzzy logic introduces partial truth values, between *true* and *false*. Some recent approaches consider both concepts as complementary processes in the same continuum [9, 10]. According to Aristotelian logic, for a given proposition or state we only have two logical values: True-False, Black-White, 1-0. In real life and in finance, things are not either black or white, but most of the times are grey. Thus, in many practical situations, it is convenient to consider intermediate logical values.

Uncertainty is now considered essential to science and fuzzy logic is a way to model and deal with it using natural language. We can say that fuzzy logic is a qualitative computational approach. Fuzzy logic could be a method to render precise what is imprecise in the financial world.

Fuzzy logic and systems have been used in finance by several authors [1, 2, 3, 7, 13, 14, 15].

The paper is organized as follows. In Section II we recall some general concepts on fuzzy sets and fuzzy real numbers, and then we present in Section III the basic operations with fuzzy numbers.

In Section IV, we recall a simple model in option pricing, and then we present the fuzzy version to price the option. We obtain a fuzzy price and we have to make a decision based on the level of imprecision/risk we are decided to support. In the extremal case that we do not allow any imprecision nor fuzziness we, of course, obtain the price of the classical setting.

We finally present a numerical example. It is illustrated using *Mathematica*. We have also implemented it in *Excel*.

We note that this example brings all the flavor of the potentiality of the fuzzy technology applied to financial engineering (FE). We are now developing fuzzy versions and programs of most of the known methods in

FE and in particular the fuzzy Black-Scholes formula and equation to price different types of options. The results will appear elsewhere.

II. Fuzzy Numbers

Fuzzy sets originated with Zadeh's 1965 paper [16]. For the general theory of fuzzy sets and numbers see, for example, [5].

For a given set X, a subset $A \subset X$ can be identified with its membership or characteristic function $\chi_A : X \to \{0, 1\}$, $\chi_A(x) = 0$ if $x \notin A$, $\chi_A(x) = 1$ if $x \in A$.

A fuzzy subset is just a mapping

$$\mu : X \to I = [0, 1],$$

and the value $\mu(x)$ gives the grade of membership of the element x of X to the fuzzy subset μ.

A classical or crisp subset $A \subset X$ can be considered as a fuzzy subset: $A \equiv \chi_A$.

A fuzzy real number is a fuzzy subset of the real numbers. For a fuzzy number u, the support is

$$\text{Support}(u) = \{x \in \mathbf{R} : u(x) > 0\}.$$

We may consider the elements with a grade of membership greater than or equal to a given threshold α. For a fuzzy number u and $\alpha > 0$, we define the α-level set as

$$[u]^{\alpha} = \{x \in \mathbf{R} : u(x) \geq \alpha\},$$

and for $\alpha = 0$,

$$[u]^{0} = \overline{\{x \in \mathbf{R} : u(x) > 0\}},$$

i.e., the closure of the support of u.

A very adequate class of fuzzy numbers are triangular fuzzy numbers. They have an intuitive interpretation and very convenient from the computational point of view.

For example, suppose that we are informed that *the price of the stock will be around* 200 *with a deviation no greater than the* 1%. This is very simple qualitatively, but no so easy from the quantitative point of view. However, it can be quantified naturally with a triangular fuzzy number: (200; 2, 2).

A triangular fuzzy number is represented as $u = (u_0; l_u, r_u)$, where u_0, l_u, r_u are real numbers, u_0 is the center, and $l_u \geq 0$ and $r_u \geq 0$ are the left and the right spreads, respectively. If $l_u, r_u = 0$, then u is the real number u_0. The membership function of u is given by

$$u(x) = \frac{x - u_0 + l_u}{l_u}, \quad u_0 - l_u < x \leq u_0,$$

$$u(x) = \frac{u_0 + r_u - x}{r_u}, \quad u_0 \leq x < u_0 + r_u,$$

and $u(x) = 0$ otherwise.

The level sets are

$$[u]^\alpha = [u_0 - l_u + \alpha l_u, u_0 + r_u - \alpha r_u].$$

Note that $u(u_0) = 1$ and that the support of u is the interval $(u_0 - l_u, u_0 + r_u)$.

III. Basic Fuzzy Arithmetic

The arithmetic operations for two fuzzy numbers u, v are defined using the level sets [5]. For example, the sum:

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha.$$

This is equivalent to the sum obtained using the Zadeh extension principle:

$$(u + v)(x) = \sup_{x = x_1 + x_2} \{\min\{u(x_1), v(x_2)\}\}.$$

Thus, for two triangular fuzzy numbers $u = (u_0; l_u, r_u)$ and $v = (v_0; l_v, r_v)$, we have

$$u + v = (u_0 + v_0; l_u + l_v, r_u + r_v).$$

For a real number λ, we define $[\lambda u]^\alpha = \lambda[u]^\alpha$. For $u = (u_0; l_u, r_u)$ and $\lambda > 0$,

$$\lambda u = (\lambda u_0; \lambda l_u, \lambda r_u),$$

and for $\lambda < 0$,

$$\lambda u = (\lambda u_0; -\lambda r_u, -\lambda l_u).$$

We note that in finance we usually deal with positive fuzzy numbers and positive parameters. A fuzzy number u is said to be *positive* if $u(x) = 0$ for every $x \leq 0$.

The multiplication $u \cdot v$ is also defined levelwise: $[u \cdot v]^\alpha = [u]^\alpha \cdot [v]^\alpha$.

For u, v fuzzy numbers, one could define the difference $u - v$ as $u + (-1)v$, but this does not constitute a natural operation as shown by the following example. Let $u = (1; 1, 1)$. Then, $u - u = u + (-1)u = (1; 1, 1) + (-1; 1, 1) = (0; 2, 2) \neq 0$.

Instead we define the following difference. We say that $u - v = w$ if $u = v + w$. This is known as the Hukuhara difference and it need not exist. For example, for $u = \chi_{\{0\}}$ and $v = \chi_{\{[0,1]\}}$, there is no w with $\chi_{\{0\}} = \chi_{\{[0,1]\}} + w$.

On the other hand, if $u = (200; 10, 10)$ and $v = (150; 5, 5)$, then $u - v = (50; 5, 5)$ since $(150; 5, 5) + (50; 5, 5) = (200; 10, 10)$.

To define fuzzy division, assume that $0 \notin [v]^\alpha$, then it is possible to define $\frac{u}{v}$ by

$$\left[\frac{u}{v}\right]^\alpha = \frac{[u]^\alpha}{[v]^\alpha}.$$

It is not a natural operation. Indeed, for $u = \chi_{[10,12]}$ and $v = \chi_{[0.1,\,0.2]}$ we have

$$\frac{u}{v} = [50,\ 120] = w,$$

but we get

$$w \cdot v = [50,\ 120] \cdot [0.1,\ 0.2] = [5,\ 24] \neq u.$$

For division u/v, we proceed as follows. If there exists w such that $u = v \cdot w$, then we say that

$$\frac{u}{v} = w.$$

We point out that for fuzzy numbers, subtraction is not the inverse operation of addition nor division is the inverse function of multiplication. In consequence, a formula such as

$$P_1 = P_0(1 + r)$$

given the principal P_0 after 1 unit of time with a simple interest at rate r, is equivalent to

$$P_0 = \frac{P_1}{1 + r}$$

representing the present value of a capital P_1 after 1 unit of time at an interest rate r, in the crisp case, but not if we are dealing with fuzzy quantities.

Also, for $\lambda, \mu > 0$ if S_1^- and S_1^+ are possible prices of an stock, then

$$\lambda \cdot S_1^+ + \mu \cdot S_0^- = \lambda \cdot S_1^-,$$

is the same as

$$\lambda \cdot S_1^+ = \lambda \cdot S_1^- - \mu \cdot S_0^-$$

in the deterministic case, but they have different interpretations and solutions in the fuzzy case.

IV. Fuzzy Options Pricing

Consider a stock with present price S_0 per share and the following discrete model for pricing an option to purchase a stock at a future time at a fixed price K.

Suppose that after one time period, its price will be either S_1^+, or S_1^-.

Suppose further that at the present time 0 we can buy options to buy a number of shares of the stock at time 1 at a cost of C per option. Obviously we need $S_1^- < K < S_1^+$.

If the nominal interest rate is r, then the value of C, in order that there is no arbitrage, is given by [12]

$$C = \frac{S_1^+ - K}{S_1^+ - S_1^-}\left[S_0 - \frac{S_1^-}{1 + r}\right]. \tag{1}$$

Now, consider a stock with present price S_0 per share. This is known and hence it is a crisp real number. We assume that the interest rate r is a fuzzy number since we know its present value but its value in the future is imprecise. We also view the future prices S_1^-, S_1^+, and the price K to buy the option at time 1 as fuzzy quantities. To obtain the fuzzy price C as a fuzzy number, we need to perform the four basic operations: addition, subtraction, multiplication, and division. Addition and multiplication are easily computable using the level sets. However, subtraction and division are subtle. To proceed, we shall recall how to obtain the expression (1).

Suppose that we buy λ shares of the stock at time 0 at a cost of λS_0. Of course, for $\lambda > 0$ we buy and for $\lambda < 0$ we sell.

Each share will be worth either S_1^+ or S_1^- at time 1. Also, at the initial time we buy μ units of options at a cost (to be determined) C. Note that the cost of this transaction is

$$\lambda S_0 + \mu C. \tag{2}$$

The value of our holdings at time 1 will be:

- λS_1^- if the price at time 1 is S_1^-, or

- $\lambda S_1^+ + \mu(S_1^+ - K)$ if the price at time 1 is S_1^+ and hence we exercise our option.

If we choose λ, μ so that the value is the same independently of the value of the stock at time 1, then we impose that

$$\lambda S_1^- = \lambda S_1^+ + \mu(S_1^+ - K)$$

or

$$\mu K + \lambda S_1^- = \lambda S_1^+ + \mu S_1^+. \tag{3}$$

In the crisp case, this gives

$$\mu = -\lambda \frac{S_1^+ - S_1^-}{S_1^+ - K}.$$

Hence, with this proportion we have that our gain at time 1 is λS_1^-, and we will have gained the following amount:

$$\lambda S_1^- - (\lambda S_0 + \mu C)(1 + r).$$

This results by noting that we have to pay off our loan if the cost in (2) is positive or withdrawing our money from a bank if the cost (2) is negative. The only possibility that does not result in an arbitrage is, of course,

$$\lambda S_1^- - (\lambda S_0 + \mu C)(1 + r) = 0.$$

This last equality is equivalent to

$$\lambda S_1^- = (\lambda S_0 + \mu C)(1 + r). \tag{4}$$

In the fuzzy case, we shall use equations (3) and (4). We shall obtain the level sets $[C]^\alpha$ of fuzzy price C of the options. Each investor has to choose his/her level of subjective confidence. If we do not allow imprecision, or we have a total risk aversion, then we have to choose $\alpha = 1$ and, as expected, $[C]^1$ will coincide with the price in the crisp case.

V. A Numerical Example in Fuzzy Options Pricing

Suppose that

$$S_0 = 100$$

$$S_1^- = 50$$

$$S_1^+ = 200$$

$$K = 150$$

$$r = 3\%,$$

then

$$C = 17.1521.$$

If we consider fuzzy quantities,

$$S_1^- = (200;\ 20,\ 20)$$

$$S_1^- = (50;\ 5,\ 5)$$

$$K = (150;\ 15,\ 15)$$

$$r = (0.03;\ 0.003,\ 0.003).$$

Then we have

$$C = (17.1521;\ 1.5664,\ 1.5756).$$

For example, if we admit a total uncertainty, then we can choose the price C between 15.5857 and 18.7277 since

$$[C]^0 = [15.5857,\ 18.7277].$$

If we do not admit any uncertainty, i.e., $\alpha = 1$, then we have

$$[C]^1 = \{17.1521\}$$

and $C = 17.1521$ which coincides with the nonfuzzy situation.

For an intermediate situation, for example, $\alpha = 0.5$, we have the following range for the price:

$$[C]^{0.5} = [16.3677, 17.9387].$$

One should choose a subjective level of imprecision α ranging from 0 (total uncertainty: randomness and fuzziness) to 1 (randomness but not uncertainty).

In the file *PriceC.nb* we obtain, using *Mathematica*, the fuzzy price using the corresponding fuzzy calculus. There, we note that $S_0 = S0 = 100$, $S_1^- = S1 = 50$, $S_1^+ = S2 = 200$ and we have the graphs of the fuzzy quantities $S2 - K$, $S2 - S1$, $(S2 - K)/(S2 - S1)$, $S0 - (S1/(1 + r))$, and finally the fuzzy price C.

To obtain a level set of the price C, we have to choose a threshold between 0 and 1. In the file *LevelSetsC.nb*, we obtained the level sets for $\alpha = 0$, $\alpha = 0.5$, and $\alpha = 1$.

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PriceC.nb

Clear[S0, S1, S2, K, r]

S0 = 100

S2 = 200

S1 = 50

K = 150

r = 0.03

$$\text{Precio} = \frac{\text{S2 - K}}{\text{S2 - S1}} * \left(\text{S0} - \frac{\text{S1}}{1 + \text{r}}\right)$$

Tpc = 0.10

s0 = S0 * Tpc

s1 = S1 * Tpc

s2 = S2 * Tpc

k = K * Tpc

rf = r * Tpc

Plot[{S2 - s2 + x * s2 - (K - k + x * k), S2 + s2 - x * s2 - (K + k - x * k)}, {x, 0, 1},

PlotLabel → " S_1^+ - K"]

Plot[{S2 - s2 + x * s2 - (S1 - s1 + x * s1), S2 + s2 - x * s2 - (S1 + s1 - x * s1)}, {x, 0, 1}, PlotLabel $\to$ " S_1^+ - S_1^- "]

Plot[{ $\frac{S2 - s2 + x * s2 - (K - k + x * k)}{S2 - s2 + x * s2 - (S1 - s1 + x * s1)}$, $\frac{S2 + s2 - x * s2 - (K + k - x * k)}{S2 + s2 - x * s2 - (S1 + s1 - x * s1)}$ },

{x, 0, 1}, PlotRange –> {0, 1},

PlotLabel $\to$ " $\frac{S_1^+ - K}{S_1^+ - S_1^-}$ "]

Plot[{ $\frac{S1 - s1 + x * s1}{1 + r - rf + x * rf}$, $\frac{S1 + s1 - x * s1}{1 + r + rf - x * rf}$ }, {x, 0, 1},

PlotLabel $\to$ " $\frac{S_1^-}{1 + r}$ "]

Plot[{S0 - s0 + x * s0 - $\frac{S1 - s1 + x * s1}{1 + r - rf + x * rf}$, S0 + s0 - x * s0 - $\frac{S1 + s1 - x * s1}{1 + r + rf - x * rf}$ }, {x, 0, 1},

PlotLabel $\to$ " S0 - $\frac{S_1^-}{1 + r}$ "]

Plot[{ $\frac{S2 - s2 + x * s2 - (K - k + x * k)}{S2 - s2 + x * s2 - (S1 - s1 + x * s1)} * \left(S0 - s0 + x * s0 - \frac{S1 - s1 + x * s1}{1 + r - rf + x * rf}\right)$,

$\frac{S2 + s2 - x * s2 - (K + k - x * k)}{S2 + s2 - x * s2 - (S1 + s1 - x * s1)} * \left(S0 + s0 - x * s0 - \frac{S1 + s1 - x * s1}{1 + r + rf - x * rf}\right)$ }, {x, 0, 1},

PlotLabel $\to$ " C"]

100

200

50

150

0.03

17.1521

0.1

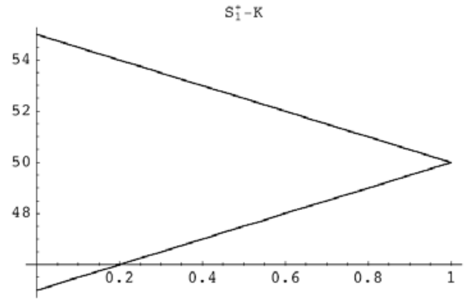
10.

5.

20.

15.

0.003



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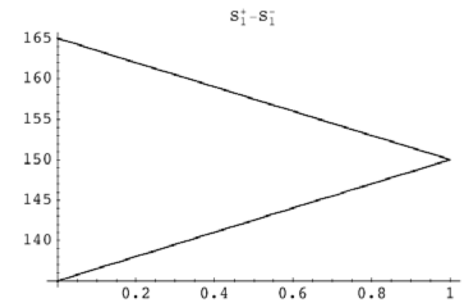

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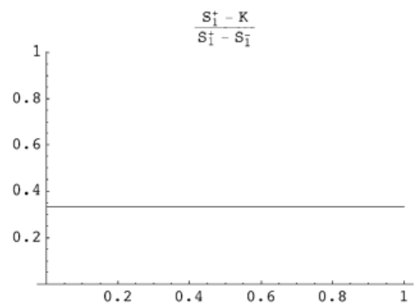

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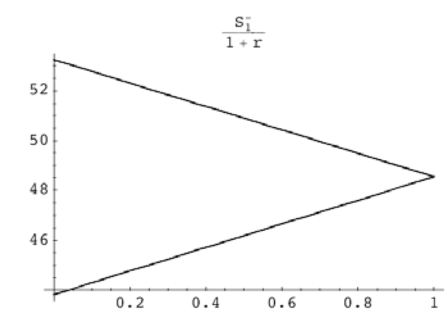

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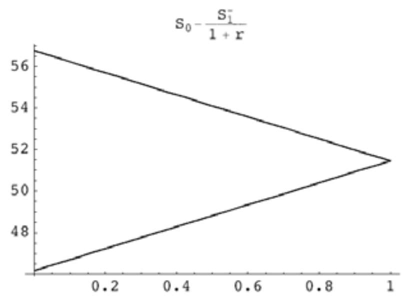

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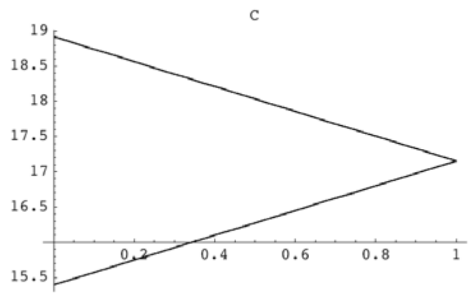

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LevelSetsC.nb

Clear[S0, S1, S2, K, r]

S0 = 100

S2 = 200

S1 = 50

K = 150

r = 0.03

$$\text{Precio} = \frac{\text{S2 - K}}{\text{S2 - S1}} * \left(\text{S0} - \frac{\text{S1}}{1+\text{r}}\right)$$

Tpc = 0.10

s0 = S0 * 0

s1 = S1 * Tpc

s2 = S2 * Tpc

k = K * Tpc

rf = r * Tpc

x = 1 / 2

$$\frac{S2 - s2 + x * s2 - (K - k + x * k)}{S2 - s2 + x * s2 - (S1 - s1 + x * s1)} * \left(S0 - s0 + x * s0 - \frac{S1 - s1 + x * s1}{1 + r - rf + x * rf} \right)$$

100

200

50

150

0.03

17.1521

0.1

0

5.

20.

15.

0.003

$\frac{1}{2}$

17.9387

$$\frac{S2 + s2 - x * s2 - (K + k - x * k)}{S2 + s2 - x * s2 - (S1 + s1 - x * s1)} * \left(S0 + s0 - x * s0 - \frac{S1 + s1 - x * s1}{1 + r + rf - x * rf} \right)$$

16.3677

17.1521 - 16.3677

0.7844

17.1521 - 17.9387

-0.7866

■