Far East J. Appl. Math. **24**(1) (2006), 67-77
This paper is available online at http://www.pphmj.com

$\mathcal{A}$-QUASICONVEX INTEGRAL FUNCTIONALS AND EPICONVERGENCE

MESSAOUDI KHELIFA and ZAHI OUNASSA

Département de Mathématiques, Faculté des Sciences
Université de Batna, 05000, Algérie
e-mail: meskhelifa@hotmail.com

Abstract

We recall some properties of relaxation integral functional in the sense of the $\mathcal{A}$-quasiconvexity and prove by epiconvergence techniques that the sequence of the relaxation functionals epiconverges to an integral functional which depends on the operator $\mathcal{A}$.

1. Introduction

In our opinion the study of homogenization and the $\mathcal{A}$-quasiconvexity notion of the lower semicontinuity of the integral functional defined for every $(u, v) \in L^P(\Omega; \mathbb{R}^m) \times (L^q(\Omega; \mathbb{R}^d) \cap \ker \mathcal{A})$ by

$$F((u, v); \Omega) = \int_\Omega f(x, u(x), v(x))dx, \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$; $1 \leq p < +\infty$; $1 < q < +\infty$, $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0, +\infty[$ is a Carathéodory integrand and $\mathcal{A}$ is the

2000 Mathematics Subject Classification: 49J45, 65K10, 70G75, 74Qxx.

Keywords and phrases: $\mathcal{A}$-quasiconvexity, relaxation, lower semicontinuity, epiconvergence.

Received March 24, 2006

first order linear partial differential operator defined from $L^q(\Omega; \mathbb{R}^d)$ into $W^{-1,q}(\Omega; \mathbb{R}^l)$ by $\mathcal{A}v = \sum_{i=1}^{N} A^i \frac{\partial v}{\partial x_i}$ with linear transformations $A^i : \mathbb{R}^d \to \mathbb{R}^l$, $i = 1, \dots, N$, is motivated by the two following assertions: the first one is that, the lower semicontinuity over a compact space is the sufficient condition for the existence of the minimum, the second one is that, the *epi* lim when it exists, it is lsc (lower semicontinuous) and if $\bar{x}$ is the cluster point of the minimizing sequence of the minimizing problem associated to the sequence F_n, which epiconverges to F, then $\bar{x}$ is the solution of the minimizing problem associated to F (see Proposition 1). Recall that, the study of the lsc of the integral functional (1.1), in the sense of the 1-quasiconvexity; the k-quasiconvexity, $k \in \mathbb{N}^*$ and in the sense of the $\mathcal{A}$-quasiconvexity has been done by Acerbi and Fusco [1]; Braides et al. [4] and Fonseca and Mûller [7]. The goal of this paper is to study by epiconvergence techniques, the asymptotic behavior when $\varepsilon \to 0$ of the sequence of the integral functionals $\mathcal{F}_\varepsilon$ defined for every $D \in O$ (O denotes the family of the open subset of Ω) and for every $v \in (L^q(D; \mathbb{R}^d) \cap \ker \mathcal{A})$ by

$$\mathcal{F}_\varepsilon(v; D) = \int_D f\left(\frac{x}{\varepsilon}, v(x)\right) dx, \tag{1.2}$$

where $1 < q < +\infty$; $f : \mathbb{R}^N \times \mathbb{R}^d \to [0, +\infty[$ is the Carathéodory integrand which is Q-periodic, Q denotes the unit cube in $\mathbb{R}^N$ centered at the origin, and there exist $C > 0$ and $L > 0$, such that for every $(x, y, z) \in \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}^d$,

(H_1) $\frac{1}{C} \| y \|^q_{\mathbb{R}^d} - C \le f(x, y) \le C(1 + \| y \|^q_{\mathbb{R}^d})$;

(H_2) $| f(x, y) - f(x, z) | \le L(1 + \| y \|^{q-1} + \| z \|^{q-1}) \| y - z \|_{\mathbb{R}^d}$,

(H_3) $\mathcal{A}$ has the constant-rank.

$\mathcal{A}$ has the constant-rank, means that, there exists $r \in \mathbb{N}^*$, such that, for every $w = (w_1, \ldots, w_N) \in S^{N-1}$ (the unit sphere in $\mathbb{R}^N$),

$$\operatorname{rank}\left(\sum_{i=1}^{N} A^i w_i\right) = r.$$

We will prove that $\mathcal{F}_\varepsilon$ epiconverges towards F^{hom} for the weak topology of $L^q(\Omega; \mathbb{R}^d)$, where F^{hom} is defined, for every $D \in O$ and for every $v \in (L^q(D; \mathbb{R}^d) \cap \ker \mathcal{A})$ by

$$F^{\text{hom}}(v, D) = \int_D f^{\text{hom}}(v)dx. \tag{1.3}$$

For $z \in \mathbb{R}^d$,

$$f^{\text{hom}}(z) = \inf_{k \in \mathbb{N}^*} \frac{1}{k^N} \inf\left\{\int_{kQ} f(x, z + w(x))dx,\right.$$

$$\left. w \in L^q_{k-per}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker \mathcal{A}, \text{ and } \int_{kQ} w(x)dx = 0\right\},$$

$w \in L^q_{k-per}(\mathbb{R}^N; \mathbb{R}^d)$ means that, $w \in L^p_{loc}(\mathbb{R}^N; \mathbb{R}^d)$ and $w(k.)$, $k \in \mathbb{N}^*$ is Q-periodic. The paper is organized as follows: In Section 2, we give the definition and the variational properties of epiconvergence. In Section 3, we recall the definition and some properties of the relaxation function of (1.1) in the sense of the $\mathcal{A}$-quasiconvexity. In Section 4, we prove our main result.

2. Variational Properties of Epiconvergence

Let (X, τ) be a Banach space, and let $\{F_n, F, n \in \mathbb{N}\}$ be a family of functionals mapping X into $\mathbb{R} \cup \{+\infty\}$. Let us recall the following notion of convergence, which is called *epiconvergence* or in its general setting Γ-*convergence*. For overview about epiconvergence, we refer the reader to [2] and [5].

Definition 1. We say that the sequence $(F_n)_{n\in\mathbb{N}}$ τ-epiconvergences to F at x in X if and only if the following hold:

(i) There exists a sequence $(x_n)_{n\in\mathbb{N}}$ of X, τ-converging to x such that

$$F(x) \geq \limsup_{n\to+\infty} F_n(x_n).$$

(ii) For every sequence $(x_n)_{n\in\mathbb{N}}$, τ-converging to x in X,

$$F(x) \leq \liminf_{n\to+\infty} F_n(x_n).$$

When this property holds for every x in X, F_n is said to be τ-*convergent* to F in X, and we write $F = \tau - epi \lim F_n$.

We state now the variational properties of epiconvergence, see, for instance [2].

Proposition 1. *Assume that* $(F_n)\tau$*-epiconverges to* F, *and let* H *be a* τ*-continuous functional from* X *into* $\mathbb{R}$. *Then*

(i) F *is lsc and* τ-*epi* $\lim(F_n + H) = F + H$.

(ii) *If now,* $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* (X, τ) *such that* $F_n(x_n) \leq F_n(x) + \varepsilon_n$, *where* $\varepsilon_n > 0$, *and if furthermore* $(x_n)_{n\in\mathbb{N}}$ *is* τ*-relatively compact, then any cluster point* $\bar{x}$ *of* $(x_n)_{n\in\mathbb{N}}$ *is a minimizer of* F *and* $\lim_{n\to+\infty} \inf\{F_n(x);\ x \in X\} = \min\{F(x);\ x \in X\} = F(\bar{x})$.

3. The lsc of the $\mathcal{A}$-quasiconvex Integral Functionals

In order to prove that the $\mathcal{A}$-quasiconvexity is the necessary and sufficient condition of the lsc of the integral functional (1.1). We will assume in addition that there exists $K > 0$, such that for every $(y, z) \in \mathbb{R}^m \times \mathbb{R}^d$,

$$0 \leq f(x, y, z) \leq K(1 + \| y \|_{\mathbb{R}^m}^p + \| z \|_{\mathbb{R}^d}^q); \quad m_N \cdot p \cdot px \in \Omega, \tag{3.1}$$

$$f \text{ is } \mathcal{A}\text{-quasiconvex}, \tag{3.2}$$

$$\mathcal{A} \text{ is the constant-rank}, \tag{3.3}$$

f is $\mathcal{A}$-quasiconvex, i.e., for every $w \in C_{1-per}^{\infty}(\mathbb{R}^N; \mathbb{R}^d) \bigcap \ker \mathcal{A}$ with $\int_Q w(y)dy = 0$, we have

$$\int_D f(x, u(x), v(x))dx \leq \int_D \int_Q f(x, u(x), v(x) + w(y))dydx.$$

Following Braides et al. [4], the relaxation formula of the integral functional (1.1) in the sense of the $\mathcal{A}$-quasiconvexity is given for every $D \in O$ and for every $(u, v) \in L^P(D; \mathbb{R}^m) \times (L^q(D; \mathbb{R}^d) \bigcap \ker \mathcal{A})$;

$$\begin{aligned}\mathcal{F}((u, v); D) = \inf\{&\liminf F((u_n, v_n); D);\\ &(u_n, v_n) \in L^P(D; \mathbb{R}^m) \times L^q(D; \mathbb{R}^d);\\ &u_n \to u \text{ in } L^P(D; \mathbb{R}^m), v_n \rightharpoonup v \text{ in } L^q(D; \mathbb{R}^d) \text{ and}\\ &\mathcal{A} v_n \to 0 \text{ in } W^{-1,q}(D; \mathbb{R}^l)\}.\end{aligned}$$

Or, in its equivalent form:

$$\begin{aligned}&\mathcal{F}((u, v); D)\\ &= \inf\left\{\liminf_{n\to+\infty} \int_D g(x, v_n(x))dx; (v_n) \subset L^q(D; \mathbb{R}^d) \bigcap \ker \mathcal{A},\right.\\ &\quad\left.(v_n) \text{ is } q\text{-equiintegrable}, v_n \rightharpoonup v \text{ in } L^q(D; \mathbb{R}^d) \text{ and } \int_D v_n dx = \int_D v dx\right\},\end{aligned}$$

where g is the Carathéodory function defined by $g(x, v) = f(x, u(x), v)$. Also, $\mathcal{F}((u, v); \cdot)$ is the trace of a Radon measure absolutely continuous with respect to m_N over Ω and its Radon-Nikodym derivative is given by

$$\frac{d\mathcal{F}((u, v); \cdot)}{dm_N}(x_0) = \mathcal{Q}_{\mathcal{A}} f(x_0, u(x_0), v(x_0)); \;\; m_N \text{ a.e. } x_0 \in \Omega,$$

where for each fixed $(x, y) \in \Omega \times R^m$, the function $\mathcal{Q}_{\mathcal{A}} f(x, y, \cdot)$ is the $\mathcal{A}$-quasiconvexification of $f(x, y, \cdot)$, defined, for every $z \in \mathbb{R}^d$ by

$$\mathcal{Q}_{\mathcal{A}} f(x, y, z) = \inf\left\{\int_Q f(x, y, z + w(t))dt; \right.$$

$$\left. w \in C^{\infty}_{1-per}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker \mathcal{A}, \int_Q w(t)dt = 0\right\}.$$

Hence $\mathcal{F}((u, v); D)$ admits the following integral representation: for every $D \in O$ and for every $(u, v) \in L^P(D; \mathbb{R}^m) \times (L^q(D; \mathbb{R}^d) \cap \ker \mathcal{A})$;

$$\mathcal{F}((u, v); D) = \int_D \mathcal{Q}_{\mathcal{A}} f(x, u(x), v(x))dx.$$

Theorem 1. *Under the assumptions* (3.1)-(3.3); *the functional* (1.1) *is* $L^P(D; \mathbb{R}^m) \times ((L^q(D; \mathbb{R}^d) \cap \ker \mathcal{A})$*-weak) lsc.*

Proof. Let (u_n, v_n) be a sequence in $L^P(D; \mathbb{R}^m) \times (L^q(D; \mathbb{R}^d) \cap \ker \mathcal{A})$ such that $u_n \to u$ in $L^P(D; \mathbb{R}^m)$, $v_n \rightharpoonup v$ in $L^q(D; \mathbb{R}^d)$, and $\mathcal{A} v_n \to 0$ in $W^{-1, q}(D; \mathbb{R}^l)$. Using the definition of $\mathcal{F}((u, v); \cdot)$, we have

$$\liminf_{n \to +\infty} F((u_n, v_n); D) \geq \mathcal{F}((u, v); D).$$

Since *f* is $\mathcal{A}$-quasiconvex,

$$\mathcal{F}((u, v); D) \geq F((u, v); D).$$

Therefore

$$\liminf_{n \to +\infty} F((u_n, v_n); D) \geq F((u, v); D).$$

4. The Main Result

Theorem 2. *Under the hypotheses* (H_1)-(H_3), $\mathcal{F}_\varepsilon$ *epiconverges towards* F^{hom} *for the weak topology of* $L^q(\Omega; \mathbb{R}^d)$.

Proof. It remains to verify by steps the assertions (i) and (ii) in Definition 1 of the epiconvergence.

Verification of (i)

Step 1. Let $D \in O$ and $u \in L^q(D; \mathbb{R}^d) \cap \ker A$ be an affine functional. Setting

$$u_\varepsilon(x) = u(x) + w_z\left(\frac{x}{\varepsilon}\right),$$

where w_z is the solution of the local minimizing problem (1.3), i.e.,

$$f^{\text{hom}}(z) = \frac{1}{k^N}\int_{kQ} f(x, z + w_z(x))dx;$$

with

$$w_z \in L^q_{k-per}(\mathbb{R}^N; \mathbb{R}^d) \cap \ker \mathcal{A} \text{ and } \int_{kQ} w_z(x)dx = 0.$$

Then

$$u_\varepsilon \rightharpoonup u \text{ in } L^q(D; \mathbb{R}^d), \text{ and } \mathcal{A}u_\varepsilon \to 0 \text{ in } W^{-1,q}(D; \mathbb{R}^d).$$

So

$$\mathcal{F}_\varepsilon(u_\varepsilon; D) \underset{\varepsilon\to 0}{\to} F^{\text{hom}}(u; D).$$

Step 2. Let u be a piecewise affine and continuous function. Then we define

$$u_\varepsilon^\delta = \begin{cases}(1-\varphi_\delta)u_\varepsilon^1 + \varphi_\delta u \text{ in } D_1,\\ (1-\varphi_\delta)u_\varepsilon^2 + \varphi_\delta u \text{ in } D_2.\end{cases}$$

Here u_ε^j; $j = 1, 2$ has the same form as u_ε in Step 1 and $\varphi_\delta \in \mathcal{D}(D; [0, 1])$ is such that

$$\begin{cases} \varphi_\delta = 1, \text{ in } \Sigma_\delta, \\ \varphi_\delta = 0, \text{ in } D \setminus \Sigma_{2\delta}, \\ 0 < \varphi_\delta < 1 \text{ if not.} \end{cases}$$

Hence, there exists a sequence $\hat{u}_\varepsilon := u_\varepsilon^{\delta(\varepsilon)}$ satisfying

$$\hat{u}_\varepsilon \rightharpoonup u \text{ in } L^q(D; \mathbb{R}^d) \text{ and } \mathcal{A}\hat{u}_\varepsilon \to 0 \text{ in } W^{-1,q}(D; \mathbb{R}^l).$$

Therefore, there exists $(u_\varepsilon) \in L^q(D; \mathbb{R}^d) \cap \ker \mathcal{A}$ which is q-equiintegrable

$$u_\varepsilon \rightharpoonup u \text{ in } L^q(D; \mathbb{R}^d),$$

and

$$\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon; D) \leq F^{\text{hom}}(u; D).$$

Step 3. For any $u \in L^q(D; \mathbb{R}^d) \cap \ker \mathcal{A}$, there exists a sequence of piecewise affine and continuous functions $u_\delta \in L^q(D; \mathbb{R}^d) \cap \ker \mathcal{A}$, such that

$$u_\delta \rightharpoonup u \text{ in } L^q(D; \mathbb{R}^d).$$

Setting

$$u_{\varepsilon,\delta}(x) = u_\delta(x) + w_z\left(\frac{x}{\varepsilon}\right).$$

Using the classical diagonalization argument, there exists $\hat{u}_\varepsilon = u_{\varepsilon,\delta(\varepsilon)}$ such that

$$\hat{u}_\varepsilon \rightharpoonup u \text{ in } L^q(D; \mathbb{R}^d) \text{ and } \mathcal{A}\hat{u}_\varepsilon \to 0 \text{ in } W^{-1,q}(D; \mathbb{R}^l),$$

hence, there exists $(u_\varepsilon) \in L^q(D; \mathbb{R}^d) \cap \ker \mathcal{A}$ which is q-equiintegrable

$$u_\varepsilon \rightharpoonup u \text{ in } L^q(D; \mathbb{R}^d),$$

and

$$\limsup_{\varepsilon\to 0}\int_D f\left(\frac{x}{\varepsilon},\, u_\varepsilon(x)\right)dx \le \liminf_{\varepsilon\to 0}\int_D f\left(\frac{x}{\varepsilon},\, \hat{u}_\varepsilon(x)\right)dx.$$

Since by Step 2,

$$\limsup_{\varepsilon\to 0}\int_D f\left(\frac{x}{\varepsilon},\, \hat{u}_\varepsilon(x)\right)dx \le F^{\text{hom}}(u;\, D),$$

$$\limsup_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u_\varepsilon;\, D) \le F^{\text{hom}}(u;\, D).$$

Verification of (ii)

The idea is to minor $\mathcal{F}_\varepsilon(u_\varepsilon;\, D)$ by $\mathcal{F}_\varepsilon(\hat{u}_\varepsilon;\, D)$ whenever $\hat{u}_\varepsilon = \hat{v}_\varepsilon + u_\varepsilon - u$, here $\hat{v}_\varepsilon$ is a sequence of piecewise affine and continuous functions.

Step 1. Let u be a piecewise affine and continuous function. Consider

$$v_{\varepsilon,\delta} = \begin{cases}(1-\varphi_\delta)v_\varepsilon^1 + \varphi_\delta v \text{ in } D_1;\\ (1-\varphi_\delta)v_\varepsilon^2 + \varphi_\delta v \text{ in } D_2.\end{cases}$$

Then we have

$$F^{\text{hom}}(v;\, D) \le \liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(v_{\varepsilon,\delta};\, D),$$

and

$$F^{\text{hom}}(v;\, D) \le \liminf_{\delta\to 0}\,\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(v_{\varepsilon,\delta};\, D).$$

Using again the diagonalization argument, there exists an application $\varepsilon \mapsto \delta(\varepsilon)$ such that $\lim\limits_{\varepsilon\to 0}\delta(\varepsilon) = 0$ and

$$\liminf_{\delta\to 0}\,\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(v_{\varepsilon,\delta};\, D) \le \liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(\hat{v}_\varepsilon;\, D),$$

whenever

$$\hat{v}_\varepsilon = v_{\varepsilon,\delta(\varepsilon)},$$

then

$$F^{\hom}(v;\ D) \le \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\hat{v}_\varepsilon;\ D).$$

Step 2. For $u \in L^q(D;\ \mathbb{R}^d) \cap \ker \mathcal{A}$ and $u_\varepsilon \in L^q(D;\ \mathbb{R}^d) \cap \ker \mathcal{A}$,

$$u_\varepsilon \rightharpoonup u \text{ in } L^q(D;\ \mathbb{R}^d).$$

There exists $v_\delta \in L^q(D;\ \mathbb{R}^d)$ a sequence of piecewise affine and continuous functions such that

$$v_\delta \rightharpoonup u \text{ in } L^q(D;\ \mathbb{R}^d).$$

Letting

$$u_\varepsilon^\delta = v_\delta + u_\varepsilon - u,$$

using the lsc of the function $F^{\hom}$ and the diagonalization argument we get a sequence $\hat{u}_\varepsilon = u_\varepsilon^{\delta(\varepsilon)}$ satisfying

$$F^{\hom}(u;\ D) \le \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\hat{u}_\varepsilon;\ D).$$

By (H_2), we have

$$\mathcal{F}_\varepsilon(\hat{u}_\varepsilon;\ D) \le \mathcal{F}_\varepsilon(u_\varepsilon;\ D) + L(1 + \| \hat{u}_\varepsilon \|^{q-1} + \| u_\varepsilon \|^{q-1}) \| \hat{u}_\varepsilon - u_\varepsilon \|,$$

then

$$F^{\hom}(u;\ D) \le \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon).$$

References

[1] E. Acerbi and N. Fusco, Lower semicontinuity problems in the calculus of variations, Arch. Rat. Mech. Anal. 86 (1984), 125-145.

[2] H. Attouch, Variational Convergence for Functions and Operators, Appl. Math. Series, Pitman, London, 1984.

[3] J. M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, Arch. Rat. Mech. Anal. 63 (1997), 337-403.

[4] A. Braides, I. Fonseca and G. Leoni, $\mathcal{A}$-quasiconvexity relaxation and homoginization, ESAIM; COCV 58 (2000).

[5] G. Dal Maso, An Introduction to Γ-convergence, Birkhäuser, Boston, 1993.

[6] I. Fonseca and S. Mûller, Relaxation of quasiconvex functionals in $BV(\Omega; \mathbb{R}^p)$ for integrands $f(x, u, \nabla u)$, Arch. Rat. Mech. Anal. 123 (1993), 1-49.

[7] I. Fonseca and S. Mûller, Quasiconvex integrands and lower semicontinuity in L^1, SIAM J. Math. Anal. 23 (1992), 1081-1098.

[8] I. Fonseca and S. Mûller, $\mathcal{A}$-quasiconvexity, lower semicontinuity and Young measures, SIAM J. Math. Anal. 30 (1999), 1355-1390.

■