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SYSTEM RELIABILITIES IN BIVARIATE PARETO MODEL

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Abstract

In this paper, we consider two-components system which the lifetimes follow bivariate pareto model. We obtain the maximum likelihood estimates and natural estimates for the reliabilities with series system and stress-strength, respectively. Further, we construct approximated confidence intervals for the reliabilities based on asymptotic normal distributions. Also we present a numerical example by giving a data set which is generated by a computer.

1. Introduction

In many the aforementioned studies of two-components system data, the component lifetimes were assumed to be statistically independent for the sake of simplicity of mathematical treatment. However, the assumption of independence is unrealistic in many two-component

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systems which the component life lengths have a well-defined dependence structure.

Lindley and Singpurwalla [5] considered the model of life lengths measured in a laboratory environment as independent exponential distributions proved that, when they work in a different environment which may be harsher, the same or gentler than the original, the resulting density of life lengths has a bivariate pareto (BVP) model. Bandyapadhyay and Basu [1], and Veenus and Nair [6] introduced the BVP models corresponding to some well-known bivariate exponential models. Jeevanand [3] obtained Bayes estimation of the reliability of stress-strength in BVP model. Hanagal [2] introduced a new multivariate pareto model including interesting properties.

In this paper, we obtain the maximum likelihood estimates (MLE) and natural estimates for the reliability with series system and stress-strength, respectively. Further, we construct approximated confidence intervals for the reliabilities based on asymptotic normal distributions. Also we present a numerical example by giving a data set which is generated by a computer.

2. Preliminaries

Let (X, Y) be lifetimes of two-components that follow a BVP model with parameters $(\theta_1, \theta_2, \theta_3, \beta)$. Then the joint survival function of (X, Y) is given by

$$\overline{F}(x, y) = P(X > x, Y > y) = \left(\frac{x}{\beta}\right)^{\theta_1} \cdot \left(\frac{y}{\beta}\right)^{\theta_2} \cdot \max\left(\frac{x}{\beta}, \frac{y}{\beta}\right)^{\theta_3}, \tag{1}$$

where $\beta \leq \min(x, y) < \infty$. We assume $\beta = 1$ in BVP model, the joint survival function of (X, Y) is given by

$$\overline{F}(x, y) = x^{-\theta_1} \cdot y^{-\theta_2} \cdot (\max(x, y))^{-\theta_3}. \tag{2}$$

We call the survival function (2) as BVP type 2 and the survival function (1) as BVP type 1.

We note that the above BVP model is not absolutely continuous with respect to Lebesgue measure on R^2. That is, there is provision for simultaneous failure of the both components $P[X = Y] = \theta_3/\theta$, $\theta = \theta_1 + \theta_2 + \theta_3$. And the random variables X and Y are independent if and only if $\theta_3 = 0$.

Now, the reliability of series system for mission time $x_0 > \beta$ is given by

$$R_1 = P[\min(X, Y) > x_0] = \overline{F}(x_0, x_0) = [x_0/\beta]^{-\theta}. \tag{3}$$

Let k be the number of observations with $\min(x_i, y_i) > x_0$ in the sample. Then the distribution of k is binomial (n, R_1).

Now, the reliability for stress-strength is given by

$$R_2 = P[X < Y] = \theta_1/\theta. \tag{4}$$

Let n_1, n_2 and n_3 be the number of observations with $x_i < y_i$, $y_i < x_i$ and $x_i = y_i$ in the sample, respectively. Then (n_1, n_2, n_3) is multinomial distributed with parameters $(n, \theta_1/\theta, \theta_2/\theta, \theta_3/\theta)$.

The likelihood function of the sample of size n is given by

$$L = \theta_1^{n_1} \cdot \theta_2^{n_2} \cdot \theta_3^{n_3} \cdot (\theta_1 + \theta_3)^{n_2} \cdot (\theta_2 + \theta_3)^{n_1} \cdot \beta^{n\theta} \cdot \left[\prod_{i=1}^{n} x_i\right]^{-(\theta_1+1)}$$

$$\left[\prod_{i=1}^{n} y_i\right]^{-(\theta_2+1)} \cdot \left[\prod_{i=1}^{n} \max(x_i, y_i)\right]^{-\theta_3} \cdot \left[\prod_{\{i \mid x_i = y_i\}} x_i\right]^{-1}. \tag{5}$$

In this paper, we focus only on BVP type 2 model.

To obtain the MLE's of $(\theta_1, \theta_2, \theta_3)$, the likelihood equations are given by

$$\frac{n_1}{\theta_1} + \frac{n_2}{\theta_1 + \theta_3} - \sum_{i=1}^{n} \log(x_i) = 0, \tag{6}$$

$$\frac{n_2}{\theta_2} + \frac{n_1}{\theta_2 + \theta_3} - \sum_{i=1}^{n} \log(y_i) = 0, \tag{7}$$

$$\frac{n_3}{\theta_3} + \frac{n_2}{\theta_1 + \theta_3} + \frac{n_1}{\theta_2 + \theta_3} - \sum_{i=1}^{n} \log(\max(x_i,\ y_i)) = 0. \tag{8}$$

The likelihood equations (6)-(8) are not easy to solve. But we can obtain MLE's $(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ by either Newton-Raphson procedure or Fisher's method of scoring.

The Fisher information matrix is given by

$$I(\theta_1, \theta_2, \theta_3) = E\left[\frac{\partial^2 \log L}{\partial\theta_i \partial\theta_j}\right] = n((I_{ij})); \quad i,\ j = 1, 2, 3, \tag{9}$$

where

$$I_{11} = \frac{1}{\theta}\left(\frac{1}{\theta_1} + \frac{\theta_2}{(\theta_1 + \theta_3)^2}\right),\ I_{13} = \frac{1}{\theta}\frac{\theta_2}{(\theta_1 + \theta_3)^2},\ I_{22} = \frac{1}{\theta}\left(\frac{1}{\theta_2} + \frac{\theta_1}{(\theta_2 + \theta_3)^2}\right),$$

$$I_{23} = \frac{1}{\theta}\frac{\theta_1}{(\theta_2 + \theta_3)^2},\ I_{33} = \frac{1}{\theta}\left(\frac{\theta_1}{(\theta_2 + \theta_3)^2} + \frac{\theta_2}{(\theta_1 + \theta_3)^2} + \frac{1}{\theta_3}\right),\ I_{12} = 0.$$

Thus $\sqrt{n}(\hat{\underline{\theta}} - \underline{\theta})$ has asymptotic trivariate normal distribution with mean vector zero and covariance matrix $I^{-1}(\underline{\theta}) = \frac{1}{n}((I^{ij}));\ i,\ j = 1, 2, 3.$ Here, $\hat{\underline{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ and $\underline{\theta} = (\theta_1, \theta_2, \theta_3)$.

3. Estimations of System Reliability

In this section, we obtain MLE's and approximate confidence intervals for the reliabilities with series system and stress-strength, respectively. The exact distribution of $(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ is difficult to obtain but their asymptotic normal distribution can be obtained by using the results in Section 2.

3.1. Estimations of reliability for series system

We first estimate the reliability with series system and construct approximated confidence interval for the reliability. For mission time x_0, MLE for reliability with series system based on MLE's of $(\theta_1, \theta_2, \theta_3)$ is given by

$$\hat{R}_{1M} = [x_0/\hat{\beta}]^{-\hat{\theta}}, \quad \hat{\theta} = \hat{\theta}_1 + \hat{\theta}_2 + \hat{\theta}_3. \tag{10}$$

By consistency of MLE and delta method, we can see that the distribution of $\hat{R}_{1M}$ is asymptotic normal distribution with mean R_1 and variance $\Lambda \cdot [I^{-1}(\theta_1, \theta_2, \theta_3)/n] \cdot \Lambda'$ (see Lehmann [4, Chapter 5]), where $\Lambda = (-(x_0/\beta)^{-\theta} \cdot \log(x_0/\beta), -(x_0/\beta)^{-\theta} \cdot \log(x_0/\beta), -(x_0/\beta)^{-\theta} \cdot \log(x_0/\beta))$.

Therefore, $100(1-\alpha)\%$ approximated confidence interval for R_1 based on MLE is as follows:

$$\left(\hat{R}_{1M} - z_{\alpha/2} \cdot \sqrt{\hat{\Lambda} \cdot I(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) \cdot \hat{\Lambda}'/n}, \ \hat{R}_{1M} + z_{\alpha/2} \cdot \sqrt{\hat{\Lambda} \cdot I(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) \cdot \hat{\Lambda}'/n}\right), \tag{11}$$

where $\hat{R}_{1M}$ is given by (10).

We next estimate the reliability and construct approximate confidence interval based on k which follows binomial distributed. The natural estimate of R_1 based on k is given by $\hat{R}_{1N} = k/n$, which is asymptotic normal distribution with mean R_1 and variance $R_1(1-R_1)/n$. Therefore, $100(1-\alpha)\%$ approximated confidence interval for R_1 based on k is as follows:

$$\left(\hat{R}_{1N} - z_{\alpha/2} \cdot \sqrt{\hat{R}_{1N} \cdot (1-\hat{R}_{1N})/n}, \ \hat{R}_{1N} + z_{\alpha/2} \cdot \sqrt{\hat{R}_{1N} \cdot (1-\hat{R}_{1N})/n}\right), \tag{12}$$

where $\hat{R}_{1N} = k/n$.

3.2. Estimations of reliability for stress-strength

We first estimate the reliability and construct approximated confidence interval for the stress-strength. The MLE for R_2 based on MLE's of $(\theta_1, \theta_2, \theta_3)$ is given by $\hat{R}_{2M} = \frac{\hat{\theta}_1}{\hat{\theta}}$. By consistency of MLE and delta method, we can see that the asymptotic distribution of $\hat{R}_{2M}$ is normal distribution with mean R_2 and variance $\Delta \cdot [I^{-1}(\theta_1, \theta_2, \theta_3)/n] \cdot \Delta'$, where $\Delta = \left(\frac{\theta_2 + \theta_3}{\theta^2}, -\frac{\theta_1}{\theta^2}, -\frac{\theta_1}{\theta^2}\right)$.

Therefore, $100(1-\alpha)\%$ approximated confidence interval for R_2 based on MLE is as follows:

$$\left(\hat{R}_{2M} - z_{\alpha/2} \cdot \sqrt{\hat{\Delta} \cdot I(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) \cdot \hat{\Delta}'/n},\ \hat{R}_{2M} + z_{\alpha/2} \cdot \sqrt{\hat{\Delta} \cdot I(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) \cdot \hat{\Delta}'/n}\right). \tag{13}$$

We next estimate the reliability and construct approximate confidence interval based on n_1 which follows binomial $(n, \theta_1/\theta)$ distribution. The natural estimate of R_2 based on n_1 is given by $\hat{R}_{2N} = n_1/n$, which is asymptotic normal distribution with mean R_2 and variance $R_2(1-R_2)/n$. Therefore, $100(1-\alpha)\%$ approximated confidence interval for R_2 based on n_1 is as follows:

$$\left(\hat{R}_{2N} - z_{\alpha/2} \cdot \sqrt{\hat{R}_{2N} \cdot (1-\hat{R}_{2N})/n},\ \hat{R}_{2N} + z_{\alpha/2} \cdot \sqrt{\hat{R}_{2N} \cdot (1-\hat{R}_{2N})/n}\right), \tag{14}$$

where $\hat{R}_{2N} = n_1/n$.

4. Numerical Example

In this section, we present a numerical example by giving a data set which is generated by a computer. We generate a random sample of size 30 from BVP with parameter $(\theta_1 = 1.4, \theta_2 = 0.6, \theta_3 = 0.2, \beta = 1)$. And we set mission time $x_0 = 1.2$. Then the true reliabilities of series system and stress-strength are $R_1 = 0.6695$ and $R_2 = 0.6363$, respectively. The data are given as follows:

(3.2946, 1.9663), (2.0935, 1.0679), (1.0866, 18.2951), (1.2529, 10.1412), (7.3244, 5.6512), (1.1159, 1.7049), (1.6651, 5.4383), (3.0187, 23.5519), (1.2134, 80.6110), (1.7894, 1.3099), (1.1019, 2.0201), (1.1952, 5.8022), (1.0508, 5.2783), (1.4367, 6.7937), (1.0938, 1.3425), (2.2276, 1.0205), (1.3196, 2.5240), (1.0523, 5.6466), (1.2287, 2.0773), (1.3461, 1.2146), (1.5342, 1.5342), (72.0404, 17.4496), (3.7302, 1.2885), (4.3051, 11.6524), (1.0745, 1.0745), (1.6401, 8.0620), (1.8943, 11.4142), (2.2326, 2.2252), (1.8509, 1.7941), (2.4463, 4.7970).

From above data set, MLE's of the parameters in BVP model are given by $\hat{\theta}_1 = 1.3456$, $\hat{\theta}_2 = 0.6001$ and $\hat{\theta}_3 = 0.1328$, respectively. And n_1, n_2, n_3 and k are given by 18, 10, 2 and 20, respectively.

Then estimates of R_1 are given by $\hat{R}_{1M} = 0.6845$ and $\hat{R}_{1N} = 0.6667$ from subsection 3.1, respectively. Also approximated confidence intervals for R_1 are given by (0.6144, 0.7547) and (0.4979, 0.8353) by (11) and (12), respectively.

On the other hand, the estimates of R_2 are given by $\hat{R}_{2M} = 0.6473$ and $\hat{R}_{2N} = 0.6001$ from subsection 3.2, respectively. Also the approximate confidence intervals for R_2 are given by (0.5244, 0.7703) and (0.4246, 0.7753) by (13) and (14), respectively.

Therefore, we can see that the estimates and the approximated confidence intervals of R_1 and R_2 based on MLE's perform better than those of natural estimates based on k and n_1.

In our discussions, we have concentrated on the bivariate pareto model case. For censored samples or a more general model than above model, we can apply our results.

References

[1] D. Bandyapadhyay and A. P. Basu, On generalization of a model by Lindley and Singpurwalla, Advanced Applied Probability 22 (1990), 498-500.

[2] D. D. Hanagal, A multivariate pareto distribution, Communications in Statistics-Theory and Methods 25(7) (1996), 1471-1488.

[3] E. S. Jeevanand, Bayes estimation of $P(X_2 < X_1)$ for a bivariate pareto distribution, The Statistician 46(1) (1997), 93-99.

[4] E. L. Lehmann, Theory of Point Estimation, John Wiley and Sons, New York, 1983.

[5] D. V. Lindley and N. D. Singpurwalla, Multivariate distributions for the life lengths of components of a system sharing a common environment, J. Applied Probability 23 (1986), 418-431.

[6] P. Veenus and K. R. M. Nair, Characterization of a bivariate pareto distribution, J. Indian Statistical Association 32 (1994), 15-20.

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