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SOME REMARKS ON THE n-PERMUTATIONAL PROPERTY

M. HASHEMI

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Abstract

The classification methods of n-permutational property are defined for the group G on the lower central series and elements. In this paper we study the relationship between them and the direct product in these properties.

1. Introduction

The classification methods of n-permutational property are defined for the group G on the lower central series as follows: For a given group G and an integer $n \geq 2$ we say that G satisfies the C_n -property if for every $(x_1, ..., x_n) \in G^n$ there exists a permutation $1 \neq \sigma \in S_n$ such that

$$[x_1, x_2, ..., x_n] = [x_{1\sigma}, x_{2\sigma}, ..., x_{n\sigma}],$$

and this will be written as $G \in C_n$.

MacDonald [6] investigated the class C_2 and proved that for every G in C_2, $[\gamma_3(G), \gamma_2(G)] = 1$ and the exponent of G is at most 4. Also Longobardi studied the class C_3 and showed that a finite group of odd order in C_3 is nilpotent of class at most 3, actually, of class ≤ 2 if 3 not

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divide $|G|$. But it is not true for any group G in C_3. For example, $S_3 \in C_3$ and is not nilpotent. Moreover, any finite group in C_3 is p-nilpotent for every prime $p \neq 2, 3$ (see [3]).

Let G be a group and $n \geq 2$ be an integer. Then we say that G satisfies the P_n-property if for every $(x_1, x_2, ..., x_n) \in G^n$ there exists $1 \neq \sigma \in S_n$ such that

$$x_1 x_2 \cdots x_n = x_{1\sigma} x_{2\sigma} \cdots x_{n\sigma},$$

this will be written as $G \in P_n$. Let $P = \bigcup\{P_n \mid n \geq 2\}$.

Clearly, P_2 is the class of abelian groups and P_n is closed with respect to forming the subgroups and images (i.e., quotient closed). Also if $|G| = n$, then $G \in P_n$. Selecting (a, b, c, d), where $a = (3, 4)$, $b = (2, 3)$, $c = (1, 2)$ and $d = (2, 3, 4)$, one may show that $S_4 \notin P_4$ and then $S_n \notin P_4$; $n \geq 4$.

Now, we consider a class of finitely presented groups as follows:

$$G_{mn} = \langle a, b \mid a^m = b^n = 1, [a, b]^a = [a, b], [a, b]^b = [a, b] \rangle, \quad m, n \geq 1.$$

Also, we recall the following lemma in [1].

Theorem 1.1. *Let $G = G_{mn}$ and $G_n = G_{nn}$. Then G is metabelian, $|G'| = d$ and $|G_n| = n^3$, where $d = $ g.c.d.(m, n).*

In Section 2 we study the relation between C_n-property and P_n-property. Section 3 is devoted to study of direct product on n-permutational property. First we need the following results. We start with:

Theorem 1.2 [2]. *Let G be a group. Then $G \in P_3$ if and only if $|G'| \leq 2$.*

The class P_4 was studied in [4, 5] and proved that

Theorem 1.3 [4]. *If a group G belongs to P_4, then G is metabelian.*

Theorem 1.4 [5]. *Let G be a finite group of odd order and $G \in P_4$. Then every commutator of G has order* 1, 3 *or* 5.

2. The Relation between C_n-property and P_n-property

In this section we prove that if G is metabelian, then $G \in C_4$.

Lemma 2.1. *Let G be metabelian group and $x, y, t, z \in G$. Then*

$$[x, y, z, t] = [x, y, t, z].$$

Proof. By the Jacobi's identity, we get

$$[[x, y], z, t][z, t, [x, y]][t, [x, y], z] = 1.$$

Since G is metabelian,

$$[z, t, [x, y]] = [[z, t], [x, y]] = 1.$$

So, we get

$$[x, y, z, t] = [t, [x, y], z]^{(-1)} = [[[x, y], t]^{(-1)}, z]^{(-1)}$$

$$= [z, [[x, y], t]^{(-1)}] = [[[x, y], t], z]^{[t, [x, y]]}$$

$$= [[x, y], t, z][[x, y], t, z, [t, a]] = [x, y, t, z].$$

This also shows that $G \in C_4$.

The following theorem asserts a relationship between the P_n-property and C_n-property, for every $n \in \{2, 3, 4\}$.

Proposition 2.2. *For $n \in \{2, 3, 4\}$, we have $P_n \subseteq C_n$.*

Proof. The case $n = 2$ is obvious. Now let $n = 3$ and $G \in P_3$. Then $|G'| \leq 2$ or for every $[x, y] \in G'$,

$$[x, y] = [x, y]^{-1} = [y, x].$$

And for every $x, y, z \in G$, we get

$$[x, y, z] = [[x, y], z] = [[y, x], z] = [y, x, z].$$

This shows that $G \in C_3$.

Let $n = 4$ and $G \in P_4$. Then G is metabelian and by Lemma 2.1 we get $G \in C_4$.

The following example shows that the inclusion in Proposition 2.2 is proper.

Example 2.3. Consider the group $G = S_3$. Then $|[G', G]| < 3$ and $G \in C_3$ (see [3]) however, $|G'| = 3$ and by Theorem 1.2 we get $G \notin P_3$. Also, by Theorem 1.1 and Lemma 2.1 we get that $G_{mn} \in C_4$ but $G_{11} \notin P_4$ (for, by Theorem 1.1 we have $|G_{11}| = 11^3$, $|G'_{11}| = 11$ and the result follows from Theorem 1.4).

3. The Direct Product on *n*-permutational Property

The following theorem shows that the class C_2 is closed under the direct product.

Proposition 3.1. *Let A and B be two groups in* C_2. *Then* $A \times B \in C_2$.

Proof. Suppose that $g_1, g_2 \in G$. Then we show that $[g_1, g_2] = [g_2, g_1]$. Also assume that $g_1 = a_1 b_1$ and $g_2 = a_2 b_2$, where $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Then

$$[g_1, g_2] = [a_1 b_1, a_2 b_2] = [[a_1, b_2][a_1, a_2]^{b_2}]^{b_1}[b_1, b_2][b_1, a_2]^{b_2},$$

and

$$[b_1, a_2] = 1, \quad [a_1, b_2] = 1, \quad [a_1, a_2]^{b_2} = [a_1, a_2].$$

Since $A, B \in C_2$,

$$[g_1, g_2] = [a_1, a_2][b_1, b_2] = [a_2, a_1][b_2, b_1]. \tag{1}$$

In a similar way one may show that

$$[g_2, g_1] = [a_2, a_1][b_2, b_1]. \tag{2}$$

And by the relations (1) and (2) we get $G \in C_2$.

Which we are interested in to consider is the class $P = \cup\{P_n \mid n \geq 2\}$. We recall the fundamental theorem of [6]:

Theorem 3.2. *Let G be a group. Then $G \in P$ if and only if G is finite-by-abelian-by-finite, i.e., G has a normal subgroup N such that N' and $|G : N|$ are finite.*

And as a result of this theorem, we get

Proposition 3.3. *The class P is closed under the direct product.*

Proof. Let $H, K \in P$ and $G = H \times K$. Then H and K are finite-by-abelian-by-finite, that is, there are $I \trianglelefteq K$ and $J \trianglelefteq H$ such that K/I, I', H/J and J' are finite. First we prove that $H \times K/J \times I \simeq H/J \times K/I$. Consider the function

$$\theta : H \times K \to H/J \times K/I,$$

where $\theta(h, k) = (Jh, IK)$. Then

$$\ker \theta = \{(h, k) | (Jh, Ik) = (J, I)\} \simeq I \times J$$

and

$$H \times K/J \times I \simeq H/J \times K/I.$$

So, $|H \times K/J \times I|$ is finite. Also

$$(J \times I)' = [J \times I, J \times I] = [J, J] \times [I, I] = J' \times I'.$$

That is, $(J \times I)'$ is finite. Then $G \times H$ is finite-by-abelian-by-finite, so $G \in P$.

The previous corollary dose not hold for P_n but it is true for C_2 (see 3.1).

Proposition 3.4. *The class P_3 is not closed under the direct product.*

Proof. By Theorem 1.3, $G \in P_3$ if and only if $|G'| \leq 2$. Suppose that $A \times B \in P_3$, where $A, B \in P_3$. Then $|A'| \leq 2$, $|B'| \leq 2$ and

$$|G'| = |(A \times B)'| = |A' \times B'| = |A'||B'| \leq 2$$

holds if and only if $|A'| = 1$ or $|B'| = 1$. This shows that $G = A \times B$ is in the class P_3 if and only if A or B is an abelian.

Finally, in the next example we obtain a group G that does not belong to the class P.

Example 3.5. Let S be the symmetric group on N and

$$F = \{\alpha \mid \alpha \in S, \text{ where; } \alpha(i) \neq i \text{ for finitely many; } i \in N\}.$$

Suppose that A_∞ is the subgroup of F generated by 3-cycles on X. Then A_∞ is infinite and simple (see [7]). So $A_\infty \notin P$.

For another example, let X be a set with at least two elements. Suppose F is the free group generated by X. Then $F \notin P$ (for, there is no non-trivial relation in F).

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Department of Mathematics
Faculty of Science
Guilan University
P. O. Box 451 Rasht, Iran
e-mail: m_hashemi@guilan.ac.ir