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A CERTAIN CLASS OF IMMIGRATION SUPERPROCESSES AND ITS LIMIT THEOREM

(Dedicated to Professor Yasunori Okabe on the occasion of his retirement from University of Tokyo)

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Abstract

We consider a class of Dawson-Li superprocesses with deterministic immigration, and discuss a convergence problem for the rescaled processes. When such a superprocess associated with dependent spatial motion is given, its rescaled process becomes again an immigration superprocess of the same kind. Then we prove that under a suitable scaling, the rescaled immigration superprocesses converge to a superprocess with coalescing spatial motion in the sense of probability distribution on the space of measure-valued continuous paths.

1. Introduction

Let us consider, first of all, a super-Brownian motion (SBM for short), which is a typical example of measure-valued processes. Roughly speaking, starting from a family of branching Brownian motions, via renormalization procedure (which is also called *short time high density*

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limit, we refer to it later as HD *limit*), the super-Brownian motion can be obtained, indeed, as a measure-valued Markov process, cf. Watanabe [30]. It is often called a *Dawson-Watanabe superprocess*, too. Various kinds of superprocesses have been studied by many researchers, and in most cases those superprocesses are obtained as a limit of branching particle systems (BPS) under variety of settings. Recently, a new discovery has attracted us, that is to say, it is nothing but a new knowledge that SBM can be also obtained as a limit of distinct sorts of particle systems. In other words, under a suitable scaling, rescaled processes converge to an SBM. For example, rescaled contact processes converge to super-Brownian motion in two or more dimensions, which is due to Durrett-Perkins [15]. In [1], Cox et al. proved that rescaled voter models converge to super-Brownian motion, too. According to Hara-Slade [18], it can be found that a sort of percolation converges to super-Brownian motion, as a suitable scaling limit, in high-dimensions. Moreover, even in the theory of measure-valued processes, similar phenomena can be observed. For instance, a superprocess with dependent spatial motion (SDSM for short) is obtained by a HD limit from a family of interactive branching particle systems, whose branching density depends on its particle location. Such a $\{a, \rho, \sigma\}$-SDSM was first discussed and constructed by Wang [29]. There is a function $c(x)$, one of those parameters that play an important role in determining a SDSM. When $c(x)(\neq 0)$ is bounded, then under a suitable scaling SDSM converges to super-Brownian motion, see, e.g., Dawson et al. [3]. Here again it is recognized that SBM does appear universally as a suitable scaling limit. On the other hand, for the same SDSM the situation has changed drastically when $c(x) \equiv 0$. Under the same scaling as in the above example, SDSMs converge this time to a superprocess with coalescing spatial motion (SCSM). This remarkable occurrence was proved by Dawson et al. [4].

Let us consider a little bit complicated model with interaction, in which a notion of immigration is taken into account. For instance, such an immigration superprocess associated with SDSM was constructed in Dawson-Li [2]. The purpose of this paper is to discuss a convergence problem for rescaled processes of the above type. We observe that when such an immigration superprocess is given, then its rescaled process becomes again an immigration superprocess of the same kind. The generator of the rescaled immigration superprocesses $\{Y_t^{\theta};\ t \geq 0\}$, $(\theta \geq 1)$ is given by

$$\begin{aligned}\mathcal{I}_{\theta}F(\nu) = & \frac{1}{2}\rho(0)\int_{\mathbb{R}} \frac{d^2}{dx^2}\frac{\delta F(\nu)}{\delta\nu(x)}\nu(\mathrm{d}x) \\ & + \frac{1}{2}\iint_{\mathbb{R}^2}\rho_{\theta}(x-y)\frac{d^2}{dxdy}\frac{\delta^2 F(\nu)}{\delta\nu(x)\,\delta\nu(y)}\nu(\mathrm{d}x)\nu(\mathrm{d}y) \\ & + \frac{1}{2}\sigma_{\theta}\int_{\mathbb{R}}\frac{\delta^2 F(\nu)}{\delta\nu(x)^2}\nu(\mathrm{d}x) \\ & + q_{\theta}\int_{\mathbb{R}}\frac{\delta F(\nu)}{\delta\nu(x)}m(\mathrm{d}x) \quad \text{for } F(\nu) \in \mathrm{Dom}(\mathcal{I}_{\theta}), \end{aligned} \tag{0}$$

where θ is a scaling parameter, $F(\nu)$ is a function defined on the space of finite measure ν on $\mathbb{R}$, $\rho(0)$ is a positive constant, ρ_{θ} is some interaction parameter, $\sigma_{\theta} > 0$ denotes a branching rate and q_{θ} is an immigration rate (for the details, see Sections 2 and 3 below). Our goal is to prove that under a suitable scaling, the rescaled immigration superprocesses associated with SDSM converge to a Dawson-Li-Zhou superprocess with coalescing spatial motion in the sense of probability distribution on the space of measure-valued continuous paths.

This paper is organized as follows.

Section 1. Introduction: the running section

Section 2. Notation and Preliminaries

2.1. Superprocess with dependent spatial motion

2.2. Superprocess with coalescing spatial motion

2.3. Immigration superprocess

Section 3. A Limit Theorem

3.1. A class of immigration superprocesses

3.2. Scaling and rescaled processes

3.3. Main results

In Section 2 we introduce some notation which shall be commonly used in the succeeding sections. Then we make a quick review of key superprocesses in this paper, such as superprocess with dependent spatial motion (SDSM), superprocess with coalescing spatial motion (SCSM) and immigration superprocess (IMS). Those superprocesses live in the family of interacting measure-valued Markov processes. The main result is stated in Section 3. The following three sections are devoted to the proof of the limit theorem. In particular the key proposition describing a convergence result of the principal term shall be proved in Section 6. The proof of the key proposition is of course important, but is quite longsome as well. So it is divided into four steps, roughly. Each step will be explained in each subsection. Some concluding remarks are stated in the last section.

2. Notation and Preliminaries

Let $M_F(\mathbb{R})$ (resp. $M_a(\mathbb{R})$) be the space of all finite (resp. purely-atomic) measures on $\mathbb{R}$, respectively, and we denote by $C(\mathbb{R})$ the space of all bounded and continuous functions on $\mathbb{R}$. $C(\mathbb{R})^+$ is the totality of positive members in $C(\mathbb{R})$. We always consider the space $M_F(\mathbb{R})$ endowed with the topology of weak convergence. The symbol $\langle f, \mu \rangle$ denotes an integral $\int f d\mu$ of a measurable function f with respect to a measure μ. For $h \in C^1(\mathbb{R})$ and both $h, h' \in L^2(\mathbb{R})$, we define

$$\rho(x) = \int h(y - x)h(y)\mathrm{d}y, \quad x \in \mathbb{R}. \tag{1}$$

For a given topological space E, let $B(E)$ denote the totality of all bounded Borel functions on E. We denote by $\mathcal{P}(E)$ the space of all probability measures on E. For $F \in B(M_F(\mathbb{R}))$, we define the variational derivative of F with respect to $\mu \in M_F(\mathbb{R})$ as

$$\frac{\delta F(\mu)}{\delta\mu(x)} = \lim_{r \to 0+} \frac{1}{r}\{F(\mu + r\delta_x) - F(\mu)\}, \quad x \in E \tag{2}$$

if the limit exists. We can define $\delta^2 F(\mu)/\delta\mu(x)\delta\mu(y)$ in the same way with F replaced by $\delta F(\mu)/\delta\mu(y)$ on the right-hand side of (2). For simplicity, we put $C_M(\mathbb{R}_+) = C([0, \infty), M_F(\mathbb{R}))$ for the space of finite measure-valued continuous paths on $\mathbb{R}_+$. For the Skorokhod space, we use $D_M(\mathbb{R}_+) = D([0, \infty), M_F(\mathbb{R}))$. For the generator $\mathcal{A}$, we say that an $M_F(\mathbb{R})$-valued càdlàg process $X = (X_t)_{t \geq 0}$ is a solution of the $(\mathcal{A}, \mathrm{Dom}(\mathcal{A}))$-martingale problem, if there is a probability measure $\mathbb{P}_\mu \in \mathcal{P}(D_M(\mathbb{R}_+))$ on the space $D([0, \infty), M_F(\mathbb{R}))$ such that $\mathbb{P}_\mu(X_0 = \mu) = 1$ and

$$F(X_t) - F(X_0) - \int_0^t \mathcal{A}F(X_s)\mathrm{d}s, \quad t \geq 0 \tag{3}$$

is a martingale under $\mathbb{P}_\mu$ for each $F \in \mathrm{Dom}(\mathcal{A})$.

2.1. Superprocess with dependent spatial motion

In this subsection we shall introduce the superprocess with dependent spatial motion (SDSM) as a purely-atomic measure-valued diffusion. Let $\sigma \in C(\mathbb{R})^+$. We denote by $\mathcal{D}(\mathcal{L})$ the domain of the generator $\mathcal{L}$, which is a subset of the space $B(M_F(\mathbb{R}))$ of bounded measurable functions on $M_F(\mathbb{R})$. More precisely, let $\mathcal{D}(\mathcal{L})$ be the union of all functions $F(\mu)$ on $M_F(\mathbb{R})$ of the form

$$F(\mu) = F_{f,\{\phi_i\}}(\mu) = f(\langle\phi_1, \mu\rangle, \ldots, \langle\phi_n, \mu\rangle), \quad \mu \in M_F(\mathbb{R}) \tag{4}$$

with $f \in C^2(\mathbb{R}^n)$ and $\{\phi_i\} \subset C^2(\mathbb{R})$ and all functions of the form

$$F(\mu) = F_{m,f}(\mu) = \langle f, \mu^m \rangle, \quad \mu \in M_F(\mathbb{R}) \tag{5}$$

with $f \in C^2(\mathbb{R}^m)$, where μ^m is a tensor product of measures $\mu^{\otimes m}$. For any $F \in \mathcal{D}(\mathcal{L})$ we define

$$\mathcal{L}F(\mu) = \frac{1}{2}\int_{\mathbb{R}} \rho(0) \frac{d^2}{dx^2} \frac{\delta F(\mu)}{\delta \mu(x)} \mu(dx) + \frac{1}{2}\int_{\mathbb{R}} \sigma(x) \frac{\delta^2 F(\mu)}{\delta \mu(x)^2} \mu(dx)$$

$$+ \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} \rho(x-y) \frac{d^2}{dxdy} \frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)} \mu(dx)\mu(dy). \tag{6}$$

Here the function ρ in the second line of (6) expresses interaction, and the second term in the first line of (6) describes the branching mechanism. An $M_F(\mathbb{R})$-valued diffusion process $X = (X_t)$ is called a $\{\rho(0), \rho, \sigma\}$-*superprocess* with dependent spatial motion (or $\{\rho(0), \rho, \sigma\}$-SDSM) if X solves the $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$-martingale problem, cf. [3] (see also [29]). Actually it is proved that X lies in the space $M_a(\mathbb{R})$ (see Remark 2 below). Moreover, based on the results of Dawson et al. [3], we observe: for each $\mu \in M_F(\mathbb{R})$ there is a unique Borel probability measure $\mathbb{Q}_\mu$ on $C_M(\mathbb{R}_+)$ such that, for each $\varphi \in C^2(\mathbb{R})$,

$$M_t(\varphi) = \langle \varphi, X_t \rangle - \langle \varphi, \mu \rangle - \int_0^t \left\langle \frac{\rho(0)}{2}\varphi'', X_s \right\rangle ds, \quad t \geq 0, \tag{7}$$

is a continuous martingale under $\mathbb{Q}_\mu$ with quadratic variation process

$$\langle M(\varphi) \rangle_t = \int_0^t \langle \sigma\varphi^2, X_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z-\cdot)\varphi', X_s \rangle^2 dz. \tag{8}$$

Remark 1. The system $\{\mathbb{Q}_\mu;\, \mu \in M_F(\mathbb{R})\}$ defines a diffusion process named superprocess with dependent spatial motion. Here $\rho(0)$ is the migration rate and σ is the branching rate. The only difference between the SDSM and the SBM in $M_F(\mathbb{R})$ is the second term on the right-hand side of (8), which comes from the dependence of the spatial motion.

Remark 2. Because of the dependent spatial motion, the properties of SDSM are quite different from those of superprocess with independent spatial motion. For instance, it is well known (see Konno-Shiga [21]) that the super-Brownian motion started with an arbitrary initial state enters immediately the space of absolutely continuous measures and its density process satisfies a class of stochastic partial differential equations. On the contrary, the $\{\rho(0), \rho, \sigma\}$-SDSM lives in the space of purely atomic measures, cf. Wang [29]. Related to the above observation, the purely atomic version of the SDSM with a general initial state can only be constructed not by usual Feller branching diffusions, but by excursions (cf. [2]).

Lastly we shall introduce the remarkable result on the explicit representation of SDSM. Now let us consider a general initial state $\mu \in M_F(\mathbb{R})$ with $\langle 1, \mu \rangle > 0$. Suppose that there is a time-space white noise $W(\mathrm{d}s, \mathrm{d}y)$ on $[0, \infty) \times \mathbb{R}$ based on the Lebesgue measure $\mathrm{d}\ell$ and a Poisson random measure $N(\mathrm{d}a, \mathrm{d}w)$ on $\mathbb{R} \times W_0$ with intensity $\mu(\mathrm{d}a)$ $Q_k(\mathrm{d}w)$ on some complete standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Q_k is the excursion law of the β-branching diffusion, and W_0 is a subset of paths $w \in W = C([0, \infty), \mathbb{R}^+)$ such that $w(0) = w(t) = 0$ for $t \geq \tau_0(w)$ with $\tau_0(w) = \inf\{s > 0; w(s) = 0\}$ for $w \in W$. More precisely, Q_k is a unique σ-finite measure on $(W_0, \mathcal{B}(W_0))$ such that

$$Q_k\{w(t_1) \in \mathrm{d}y_1, \ldots, w(t_n) \in \mathrm{d}y_n\}$$

$$= k_{t_1}(\mathrm{d}y_1) Q^{\circ}_{t_2 - t_1}(y_1, \mathrm{d}y_2) \cdots Q^{\circ}_{t_n - t_{n-1}}(y_{n-1}, \mathrm{d}y_n) \tag{9}$$

$$\text{for } 0 < t_1 < \cdots < t_n \text{ and } y_1, \ldots, y_n \in (0, \infty),$$

where $k_t(dy) = 4(\beta t)^{-2} e^{-2y/\beta t} \mathrm{d}y$, $(t > 0,\ y > 0)$ and $Q^{\circ}_t(x, \cdot)$ denotes the restriction of the measure $Q_t(x, \cdot)$ to $(0, \infty)$ satisfying

$$\int_0^\infty e^{-zy} Q_t(x, \mathrm{d}y) = \exp\left\{-\frac{xz}{1 + (\beta t z/2)}\right\}, \quad t, x, z \geq 0, \quad \text{(cf. [17, p. 236])}.$$

It is well known that

$$\int_0^\infty (1 - e^{-zy}) k_{r+t}(\mathrm{d}y) = \int_0^\infty k_r(\mathrm{d}y) \int_0^\infty (1 - e^{-zy}) Q_t^\circ(x, \mathrm{d}y) \tag{10}$$

holds for $r,\ t > 0$ and $z \geq 0$. Then we have $k_r Q_t^\circ = k_{r+t}$ and hence $(k_t)_{t>0}$ is an entrance law for $(Q_t^\circ)_{t>0}$. For the details, see Section 2 of [2]. We also assume that $\{W(\mathrm{d}s, \mathrm{d}y)\}$ and $\{N(\mathrm{d}a, \mathrm{d}w)\}$ are independent. For any $a \in \mathbb{R}$, let $\{x(a, t);\ t \geq 0\}$ be a unique solution of the equation

$$x(t) = a + \int_0^t \int_{\mathbb{R}} h(y - x(s)) W(\mathrm{d}s, \mathrm{d}y), \quad t \geq 0, \tag{11}$$

cf. Lemma 1.3 of [29, p. 46] (see also Lemma 3.1 of [3, p. 11]). In addition, enumeration of the atoms of $N(\mathrm{d}a, \mathrm{d}w)$ into $\mathrm{supp}(N)$ is given by a sequence $\{(a_i, w_i);\ i = 1, 2, \ldots\}$ such that $\tau_0(w_{i+1}) < \tau_0(w_i)$ a.s. for all $i \geq 1$ and $\tau_0(w_i) \to 0$ as $i \to \infty$. For a fixed constant $\beta > 0$ let

$$\psi(a, t) = \beta^{-1} \int_0^t \sigma(x(a, s)) \mathrm{d}s, \quad t \geq 0,\ a \in \mathbb{R}, \tag{12}$$

and we define $w(a, t) = w(\psi(a, t))$ for $w \in W_0$. Then we have

Theorem 1 (Dawson-Li [2]). *Let* $\{X_t;\ t \geq 0\}$ *be defined by* $X_0 = \mu$ *and*

$$\begin{aligned} X_t &= \sum_{i=1}^\infty w_i(a_i, t) \delta_{x(a_i, t)} \\ &= \int_{\mathbb{R}} \int_{W_0} w(a, t) \delta_{x(a, t)} N(\mathrm{d}a, \mathrm{d}w), \quad t > 0. \end{aligned} \tag{13}$$

Then $\{X_t\}$ *relative to* $(\mathcal{G}_t)_{t \geq 0}$ *is an* $\{\rho(0), \rho, \sigma\}$*-SDSM, where* $\mathcal{G}_t$ *is the* σ*-algebra generated by all* $\mathbb{P}$*-null sets and the families of random variables*

$$\{W([0, s] \times B);\ 0 \leq s \leq t,\ B \in \mathcal{B}(\mathbb{R})\},\ \{w_i(a_i, s);\ 0 \leq s \leq t,\ i = 1, 2, \ldots\}, \tag{14}$$

for $t \geq 0$.

2.2. Superprocess with coalescing spatial motion

An n-dimensional continuous process $\{(y_1(t), \ldots, y_n(t)); t \geq 0\}$ is called an *n-system of coalescing Brownian motions* (n-SCBM) with speed $\widetilde{\rho} > 0$ if each $\{y_i(t); t \geq 0\}$ is a Brownian motion with speed $\widetilde{\rho}$ and, for $i \neq j$, $\{|y_i(t) - y_j(t)|; t \geq 0\}$ is a Brownian motion with speed $2\widetilde{\rho}$ stopped at the origin. The formal generator of the superprocess with coalescing spatial motion (SCSM) is given by

$$\widetilde{\mathcal{L}}F(\mu) = \frac{1}{2}\widetilde{\rho}\int_{\mathbb{R}} \frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(\mathrm{d}x) + \frac{1}{2}\int_{\mathbb{R}} \widetilde{\sigma}\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(\mathrm{d}x)$$

$$+ \frac{1}{2}\iint_{\Delta}\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(\mathrm{d}x)\mu(\mathrm{d}y), \tag{15}$$

where $\widetilde{\rho}$ is a positive constant, $\widetilde{\sigma} \in C(\mathbb{R})^+$ such that $\inf_x \widetilde{\sigma}(x) \geq \varepsilon$ for some $\varepsilon > 0$, and $\Delta = \{(x, x); x \in \mathbb{R}\}$.

In what follows we consider a superprocess with coalescing spatial motion (SCSM) with purely atomic initial state, namely, having a finite number of atoms, for instance, $\mu_0 = \sum_{i=1}^{n}\xi_i\delta_{a_i}$ just for simplicity. It goes almost similarly for the SCSM with a general initial state $\mu_0 \in M_F(\mathbb{R})$. For details, see Section 3 and Theorem 3.5 of Dawson et al. [4, pp. 686-688]. Let $\{(\xi_1(t), \ldots, \xi_n(t)); t \geq 0\}$ be a system of independent standard Feller branching diffusions with initial state $(\xi_1, \ldots, \xi_n) \in \mathbb{R}_+^n$. By setting

$$\psi_i^{\sigma}(t) = \int_0^t \sigma(y_i(s))\mathrm{d}s \text{ and } \xi_i^{\sigma}(t) = \xi_i(\psi_i^{\sigma}(t)), \tag{16}$$

we define

$$X_t = \sum_{i=1}^{n}\xi_i^{\sigma}(t)\delta_{y_i(t)}, \quad t \geq 0, \tag{17}$$

which gives a continuous $M_F(\mathbb{R})$-valued process. For a basic standard complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathcal{H}_t$ be the σ-algebra generated by the family of $\mathbb{P}$-null sets in $\mathcal{F}$ and the family of random variables $\{(y_1(s), \ldots, y_n(s), \xi_1^\sigma(s), \ldots, \xi_n^\sigma(s)); 0 \le s \le t\}$. We observe that the process $\{X_t; t \ge 0\}$ defined by (17) is a diffusion process relative to the filtration $(\mathcal{H}_t)_{t \ge 0}$ with state space $M_a(\mathbb{R})$, cf. Theorem 3.1 of [4, p. 682].

In order to understand the SCSM well, we consider martingale characterization of the process $X = (X_t)$. Let $\mathcal{D}(\widetilde{\mathcal{L}})$ be the set of all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m \rangle$ with $\mu \in M_F(\mathbb{R})$. We have an easy identity

$$\widetilde{\mathcal{L}} F_{m,f}(\mu) = F_{m, G_0^{(m)} f}(\mu) + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \ne j}}^{m} F_{m-1, \Phi_{ij} f}(\mu), \tag{18}$$

where $G_0^{(m)}$ is the generator of the m-system of coalescing Brownian motions with speed $\widetilde{\rho}$ and Φ_{ij} is the operator from $C(\mathbb{R}^m)$ to $C(\mathbb{R}^{m-1})$ defined by

$$\Phi_{ij} f(x_1, \ldots, x_{m-1}) = \widetilde{\sigma}(x_{m-1}) f(x_1, \ldots, \underset{\uparrow}{\overset{i-th}{x_{m-1}}}, \ldots, \underset{\uparrow}{\overset{j-th}{x_{m-1}}}, \ldots, x_{m-2}). \tag{19}$$

Then we have

Proposition 2 (cf. [4, p. 684]). *Let* $\{X_t; t \ge 0\}$ *be defined by* (17). *Then* $\{X_t; t \ge 0\}$ *solves the* $(\widetilde{\mathcal{L}}, \mathcal{D}(\widetilde{\mathcal{L}}))$*-martingale problem, namely, for each* $F_{m,f} \in \mathcal{D}(\widetilde{\mathcal{L}})$,

$$F_{m,f}(X_t) - F_{m,f}(X_0) - \int_0^t \widetilde{\mathcal{L}} F_{m,f}(X_s) \mathrm{d}s \tag{20}$$

is a $(\mathcal{H}_t)$*-martingale.*

The distribution of the process $\{X_t; t \ge 0\}$ can be characterized in terms of a dual process. Now let us consider a non-negative integer-valued càdlàg Markov process $\{M_t; t \ge 0\}$ with transition intensities

$\{q_{i,j}\}$ such that $q_{i,i-1} = -q_{i,i} = i(i-1)/2$ and $q_{i,j} = 0$ for all other pairs (i, j). In other words, this means that the process $\{M_t\}$ only has downward jumps which occur at rate $M_t(M_t - 1)/2$. Such a Markov process is well known as Kingman's coalescent process, cf. Kingman [20]. For $1 \le k \le M_0 - 1$, τ_k denotes the k-th jump time of $\{M_t; t \ge 0\}$ with $\tau_0 = 0$ and $\tau_{M_0} = \infty$. Let $\{\Gamma_k\}(1 \le k \le M_0 - 1)$ be a sequence of random operators from $C(\mathbb{R}^m)$ to $C(\mathbb{R}^{m-1})$, which is conditionally independent given $\{M_t; t \ge 0\}$, satisfying

$$\mathbb{P}\{\Gamma_k = \Phi_{ij} \mid M(\tau_k -) = \ell\} = \frac{1}{\ell(\ell-1)}, \quad 1 \le i \ne j \le \ell. \tag{21}$$

Let C^* denote the topological union of $\{C(\mathbb{R}^m); m = 1, 2, \ldots\}$, endowed with pointwise convergence on each $C(\mathbb{R}^m)$. By making use of the transition semigroup $(P_t^{(m)})_{t \ge 0}$ of the m-system of coalescing Brownian motions, another Markov process $\{Y_t; t \ge 0\}$ taking values from C^* is defined by

$$Y_t = P_{t-\tau_k}^{(M_{\tau_k})} \Gamma_k P_{\tau_k - \tau_{k-1}}^{(M_{\tau_{k-1}})} \Gamma_{k-1} \cdots P_{\tau_2 - \tau_1}^{(M_{\tau_1})} \Gamma_1 P_{\tau_1}^{(M_0)} Y_0, \tag{22}$$

for $\tau_k \le t < \tau_{k+1}$, $0 \le k \le M_0 - 1$.

Clearly, $\{(M_t, Y_t); t \ge 0\}$ is also a Markov process. We denote by $\mathbb{E}_{m,f}^{\sigma}$ the expectation related to the process (M_t, Y_t) given $M_0 = m$ and $Y_0 = f \in C(\mathbb{R}^m)$.

By Proposition 2, the process $\{X_t\}$ constructed by (17) is a diffusion process. Let $Q_t(\mu_0, d\nu)$ denote the distribution of X_t on $M_F(\mathbb{R})$ given $X_0 = \mu_0 \in M_a(\mathbb{R})$. Then we have

Theorem 3 (Dawson et al. [4]). *If $\{X_t; t \ge 0\}$ is a continuous $M_F(\mathbb{R})$-valued process such that $\mathbb{E}[\langle 1, X_t\rangle^m]$ is locally bounded in $t \ge 0$ for each $m \ge 1$ and $\{X_t\}$ solves the $(\tilde{\mathcal{L}}, \mathcal{D}(\tilde{\mathcal{L}}))$-martingale problem with $X_0 = \mu_0$,* *then*

$$\int_{M_F(\mathbb{R})} \langle f, \nu^m \rangle Q_t(\mu_0, d\nu) = \mathbb{E}^{\sigma}_{m,f}\left[\langle Y_t, \mu_0^{M_t} \rangle \exp\left\{\frac{1}{2}\int_0^t M_s(M_s - 1)ds\right\}\right] \quad (23)$$

for $t \geq 0$, $m \geq 1$ *and* $f \in C(\mathbb{R}^m)$.

A Markov process on $M_F(\mathbb{R})$ with transition semigroup $(Q_t)_{t \geq 0}$ given by (23) is called a *superprocess* with coalescing spatial motion with speed $\widetilde{\rho}$ and branching rate $\widetilde{\sigma}(\cdot)$ and with initial state $\mu_0 \in M_a(\mathbb{R})$, or shortly a $\{\widetilde{\rho}, \widetilde{\sigma}, \mu_0\}$-SCSM. Sometimes we use simply $\{\widetilde{\rho}, \widetilde{\sigma}\}$-SCSM unless any confusion occurs. The most important thing is here that the distribution of the SCSM can be determined uniquely via this formula (23).

Remark 3. It is obvious that $\{(y_1(t), \ldots, y_n(t)); t \geq 0\}$ is an n-system of coalescing Brownian motions with speed $\widetilde{\rho}$ if and only if

$$\langle y_i, y_j \rangle(t) = \widetilde{\rho} \cdot (t - t \wedge \tau_{ij}), \quad 1 \leq i, j \leq n, \quad (24)$$

where $\tau_{ij} = \inf\{t \geq 0; y_i(t) = y_j(t)\}$. Generally speaking, a branching particle system is said to have the *coalescence property* if the particle location processes are diffusion processes and, for any two particles, either they never separate or they never meet according as they start off from the same initial location or not.

Remark 4. The first two terms on the right-hand side of (15) just correspond to the generator of a usual super-Brownian motion, where the first term describes the spatial motion and the second term describes the branching. The last term on the right-hand side of (15) shows that interactions in the spatial motion only occur between particles located at the same positions.

Remark 5. The definition (17) means the following: the mass of the i-th particle is given by $\{\xi_i^{\sigma}(t); t \geq 0\}$, which is obtained from a standard Feller branching diffusion by a time change depending on the position of the i-th carrier. In this way a spatially dependent branching mechanism is involved with this process.

Remark 6. The Markov process $\{Y_t\}$ defined by (22) evolves in the time interval $[0, \tau_1)$ according to the linear semigroup $(P_t^{(M_0)})$, $t \geq 0$, and then it makes a jump given by Γ_1 at time τ_1. After that, it evolves in the interval $[\tau_1, \tau_2)$ according to $(P_t^{(M_{\tau_1})})$, $t \geq 0$, and then it makes another jump given by Γ_2 at time τ_2, and this sort of process is repeated almost in a similar way, and so on.

2.3. Immigration superprocess

In this subsection we shall consider a class of immigration diffusion processes associated with SDSM.

SDSM with interactive immigration

We first treat the case with a general interactive immigration rate. A key point consists in the fact that the construction of such an immigration process is done by solving a stochastic equation carried by a stochastic flow and driven by Poisson processes of excursions. The results below are originally discussed in Section 5 of [2].

Let σ be a positive constant, m be a σ-finite Borel measure on $\mathbb{R}$ (as a reference measure for immigration), and q be a Borel function on $M_F(\mathbb{R}) \times \mathbb{R}$, satisfying the conditions: there is a constant $K > 0$ such that

$$\langle q(\nu, \cdot), m\rangle \leq K(1 + \|\nu\|), \quad \nu \in M_F(\mathbb{R}), \tag{25}$$

and for each $R > 0$ there is a constant $K_R > 0$ such that

$$\langle | q(\nu, \cdot) - q(\gamma, \cdot)|, m\rangle \leq K_R \|\nu - \gamma\| \tag{26}$$

for any $\nu, \gamma \in M_F(\mathbb{R})$ satisfying $\langle 1, \nu\rangle \leq R$ and $\langle 1, \gamma\rangle \leq R$, where $\|\cdot\|$ denotes the total variation. This q is a function introduced in connection with the immigration.

Suppose that there are (i) a white noise $W(\mathrm{d}s, \mathrm{d}y)$ on $[0, \infty) \times \mathbb{R}$ based on the Lebesgue measure $\mathrm{d}\ell$; (ii) a sequence of independent σ-branching diffusions $\{\xi_i(t); t \geq 0\}$ with $\xi_i(0) \geq 0$ $(i = 1, 2, \ldots)$; (iii) a Poisson random measure $N(\mathrm{d}s, \mathrm{d}a, \mathrm{d}u, \mathrm{d}w)$ on $[0, \infty) \times \mathbb{R} \times [0, \infty) \times W_0$ with intensity

$\mathrm{d}sm(\mathrm{d}a)\mathrm{d}uQ_k(\mathrm{d}w)$, on a complete standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In addition we assume that $\sum_{i=1}^{\infty} \xi_i(0) < \infty$ and that $\{W\}$, $\{\xi_i\}$ and $\{N\}$ are independent of each other. For $t \geq 0$ let $\widetilde{\mathcal{G}}_t$ be the σ-algebra generated by all $\mathbb{P}$-null sets and the families of random variables

$$\{W([0, s] \times B), \xi_i(s);\ 0 \leq s \leq t,\ B \in \mathcal{B}(\mathbb{R}),\ i = 1, 2, \ldots\}, \tag{27}$$

and

$$\{N(J \times A);\ J \in \mathcal{B}([0, s] \times \mathbb{R} \times [0, \infty)),\ A \in \mathcal{B}_{t-s}(W_0),\ 0 \leq s \leq t\}. \tag{28}$$

A stochastic equation with purely atomic initial state is comparatively tractable. That is, for any sequence $\{a_i\} \subset \mathbb{R}$, we consider the stochastic equation

$$Y_t = \sum_{i=1}^{\infty} \xi_i(t)\delta_{x(0, a_i, t)}$$

$$+ \int_0^t \int_{\mathbb{R}} \int_0^{q(Y_s, a)} \int_{W_0} w(t-s)\delta_{x(s, a, t)} N(\mathrm{d}s, \mathrm{d}a, \mathrm{d}u, \mathrm{d}w), \quad t \geq 0. \tag{29}$$

Then it follows that the equation (29) has a unique continuous solution $\{Y_t;\ t \geq 0\}$, which is a diffusion process relative to $(\widetilde{\mathcal{G}}_t)$. Furthermore, there exists a Borel probability measure $\mathbb{Q}_\mu$ on $C_M(\mathbb{R}_+)$ such that for each $\varphi \in C^2(\mathbb{R})$,

$$M_t(\varphi) = \langle \varphi, Y_t \rangle - \langle \varphi, Y_0 \rangle - \int_0^t \left\langle \frac{\rho(0)}{2} \varphi'', Y_s \right\rangle \mathrm{d}s$$

$$- \int_0^t \langle q(Y_s, \cdot)\varphi, m \rangle \mathrm{d}s, \quad t \geq 0, \tag{30}$$

is a continuous martingale under $\mathbb{Q}_\mu$ with respect to the filtration $(\widetilde{\mathcal{G}}_t)_{t \geq 0}$ and its quadratic variation process is given by

$$\langle M(\varphi) \rangle_t = \int_0^t \langle \sigma\varphi^2, Y_s \rangle \mathrm{d}s + \int_0^t \mathrm{d}s \int_{\mathbb{R}} \langle h(z - \cdot)\varphi', Y_s \rangle^2 \mathrm{d}z, \quad t \geq 0. \tag{31}$$

Then the generator of the diffusion process $\{Y_t;\ t \geq 0\}$ is given by

$$\hat{\mathcal{L}}F(\nu) = \mathcal{L}F(\nu) + \int_{\mathbb{R}} q(\nu,\ x)\frac{\delta F(\nu)}{\delta\nu(x)}m(\mathrm{d}x), \quad \nu \in M_F(\mathbb{R}), \tag{32}$$

where $\mathcal{L}$ is defined by (6) and $q(\cdot,\ \cdot)$ is the interactive immigration rate. We call this process $\{Y_t\}$ an *immigration superprocess* associated with SDSM or more simply a $\{\rho(0),\ \rho,\ \sigma,\ q,\ m\}$-IMS.

Remark 7. The Markov property of $\{Y_t\}$ was obtained from the uniqueness of solution of (29). This application of the stochastic equation is essential since the uniqueness of solution of the martingale problem given by (30) and (31) still remains open (see [27]).

SDSM with deterministic immigration

In particular we consider here some immigration processes by one-dimensional excursions carried by stochastic flows. The construction of the process is due to Section 4 in [2]. Suppose that $m \in M_F(\mathbb{R})$ satisfies $\langle 1,\ m\rangle > 0,\ q(\cdot,\ \cdot) \equiv c$ and σ is a positive constant. We define

$$\hat{\mathcal{L}}F(\mu) = \mathcal{L}F(\mu) + \int_{\mathbb{R}} c\frac{\delta F(\mu)}{\delta\mu(x)}m(\mathrm{d}x), \quad \mu \in M_F(\mathbb{R}). \tag{33}$$

We put $\mathcal{D}(\hat{\mathcal{L}}) = \mathcal{D}(\mathcal{L})$. The $(\hat{\mathcal{L}},\ \mathcal{D}(\hat{\mathcal{L}}))$-martingale problem has a unique solution $\{Y_t\}$. The solution process is a diffusion, and this immigration SDSM started with any initial state actually lives in the space of purely atomic measures. Moreover, we have the following martingale characterization. A continuous $M_F(\mathbb{R})$-valued process $\{Y_t;\ t \geq 0\}$ is a solution of the $(\hat{\mathcal{L}},\ \mathcal{D}(\hat{\mathcal{L}}))$-martingale problem if and only if for each $\varphi \in C^2(\mathbb{R})$,

$$M_t(\varphi) = \langle\varphi,\ Y_t\rangle - \langle\varphi,\ Y_0\rangle - c\langle\varphi,\ m\rangle t - \int_0^t \left\langle\frac{\rho(0)}{2}\varphi'',\ Y_s\right\rangle \mathrm{d}s, \quad t \geq 0, \tag{34}$$

is a martingale with quadratic variation process

$$\langle M(\varphi)\rangle_t = \int_0^t \langle\sigma\varphi^2,\ Y_s\rangle \mathrm{d}s + \int_0^t \mathrm{d}s\int_{\mathbb{R}} \langle h(z - \cdot)\varphi',\ Y_s\rangle^2 \mathrm{d}z. \tag{35}$$

3. A Limit Theorem

3.1. A class of immigration superprocess

First of all we shall begin with introducing our IMS model $Y = \{Y_t;\, t \geq 0\}$. As for the limit theorem, we treat in this paper only the case of purely atomic initial state, namely, $Y_0 = \mu = \sum_{i=1}^{\infty} \xi_i(0)\delta_{a_i} \in M_a(\mathbb{R})$ for $\{a_i\} \subset \mathbb{R}$. Our subject process is a deterministic immigration superprocess associated with SDSM, see (33)-(35) in Subsection 2.3. Let m be a σ-finite Borel measure on $\mathbb{R}$ such that $0 < \langle 1, m\rangle < \infty$, $q(\cdot, \cdot) \equiv q \in \mathbb{R}$, and σ is a positive constant. As to the interaction parameter, let ρ and h be the functions just described as in (1) of Section 2. Let us now define

$$\mathcal{L} = \frac{1}{2}\rho(0)\int_{\mathbb{R}} \frac{d^2}{dx^2}\frac{\delta F(\nu)}{\delta\nu(x)}\nu(\mathrm{d}x) + \frac{1}{2}\sigma\int_{\mathbb{R}} \frac{\delta^2 F(\nu)}{\delta\nu(x)^2}\nu(\mathrm{d}x)$$

$$+ \frac{1}{2}\iint_{\mathbb{R}^2} \rho(x-y)\frac{d^2}{dxdy}\frac{\delta^2 F(\nu)}{\delta\nu(x)\delta\nu(y)}\nu(\mathrm{d}x)\nu(\mathrm{d}y), \tag{36}$$

$$\mathcal{A}F(\nu) = q\int_{\mathbb{R}} \frac{\delta F(\nu)}{\delta\nu(x)} m(\mathrm{d}x), \tag{37}$$

and

$$\mathcal{I}F(\nu) = \mathcal{L}F(\nu) + \mathcal{A}F(\nu). \tag{38}$$

Let $\mathcal{D}(\mathcal{L})$ be the same as in Subsection 2.1 with (4) and (5), and we define $\mathrm{Dom}(\mathcal{I}) = \mathcal{D}(\mathcal{L})$. Let $Y = \{Y_t;\, t \geq 0\}$ be a $\{\rho(0), \rho, \sigma, q, m\}$-IMS, and this Y solves the $(\mathcal{I}, \mathrm{Dom}(\mathcal{I}))$-martingale problem. Note that for the function $F(\nu) = f(\langle\phi, \nu\rangle) \in \mathrm{Dom}(\mathcal{I})$ with $f \in C^2(\mathbb{R})$, $\phi \in C^2(\mathbb{R})$ and $\nu \in M_F(\mathbb{R})$, the generator $\mathcal{I}$ has the form

$$\mathcal{I}F(\nu) = \mathcal{L}F(\nu) + \mathcal{A}F(\nu)$$

$$= \frac{1}{2}\rho(0)f'(\langle\phi, \nu\rangle)\langle\phi'', \nu\rangle + \frac{1}{2}f''(\langle\phi, \nu\rangle)\iint_{\mathbb{R}^2}\rho(x-y)\phi'(x)\phi'(y)\nu(\mathrm{d}x)\nu(\mathrm{d}y)$$

$$+ \frac{1}{2}\sigma f''(\langle\phi, \nu\rangle)\langle\phi^2, \nu\rangle + q\cdot f'(\langle\phi, \nu\rangle)\langle\phi, m\rangle. \tag{39}$$

3.2. Scaling and rescaled processes

Let θ be any positive number such that $\theta \geq 1$. The operator K_θ on $M_F(\mathbb{R})$ is given by $K_\theta \mu(B) = \mu(\{\theta x;\ x \in B\})$ for any Borel set B in $\mathbb{R}$. For a function $f = f(x)$ defined on $\mathbb{R}$, we put $f_\theta(x) = f(\theta x)$ as a scaled function. Take $F(\nu) = f(\langle \phi, \nu \rangle) \in \mathrm{Dom}(\mathcal{I})$ with $f, \phi \in C^2(\mathbb{R})$ and $\nu \in M_F(\mathbb{R})$. Then we have

$$F \circ K_\theta(\nu) = F(K_\theta \nu) = f(\langle \phi,\ K_\theta \nu \rangle) = f(\langle \phi_{1/\phi},\ \nu \rangle). \tag{40}$$

When $X = (X_t)$ is a $\{\rho(0), \rho, \sigma\}$-SDSM, then by the theory of transformation of Markov processes, the process $\{K_\theta X_t;\ t \geq 0\}$ has generator $\mathcal{L}^\theta$ defined by $\mathcal{L}^\theta F(\nu) = \mathcal{L}(F \circ K_\theta)(K_{1/\theta}\nu)$. While, note that

$$\frac{d}{dx}\phi_{1/\theta}(x) = \frac{1}{\theta}(\phi')_{1/\theta}(x)$$

and

$$\frac{d^2}{dx^2}\phi_{1/\theta}(x) = \frac{1}{\theta^2}(\phi'')_{1/\theta}(x).$$

Then we may see that $\{\theta^{-2}K_\theta Y_t;\ t \geq 0\}$ has generator

$$\begin{aligned} \mathcal{I}^\theta F(\nu) = {} & \frac{1}{2\theta^2}\rho(0) f'(\langle \phi,\ \nu \rangle)\langle \phi'',\ \nu \rangle \\ & + \frac{1}{2\theta^2} f''(\langle \phi,\ \nu \rangle)\iint_{\mathbb{R}^2} \rho_\theta(x - y)\phi'(x)\phi'(y)\nu(\mathrm{d}x)\nu(\mathrm{d}y) \\ & + \frac{1}{2\theta^2}\sigma f''(\langle \phi,\ \nu \rangle)\langle \phi^2,\ \nu \rangle + \frac{1}{\theta^2} q \cdot f'(\langle \phi,\ \nu \rangle)\langle \phi,\ K_\theta m \rangle. \end{aligned} \tag{41}$$

Suggested by the scaling argument of [3] and the discussion on rescaled limits of [4], we can easily see that $\{\theta^{-2}K_\theta Y_{\theta^2 t};\ t \geq 0\}$ has the right generator $\theta^2 \mathcal{I}^\theta$. Since $\theta^2 \mathcal{L}$ corresponds to $X_{\theta^2 t}$ when the generator of X_t is $\mathcal{L}$, putting $Y_t^\theta := \theta^{-2}K_\theta Y_{\theta^2 t}$ with $\forall \theta \geq 1$, the rescaled process

$\{Y_t^\theta;\ t \geq 0\}$ has generator

$$\mathcal{I}_\theta F(\nu) = \frac{1}{2}\rho(0) f'(\langle \phi, \nu \rangle)\langle \phi'', \nu \rangle$$

$$+ \frac{1}{2} f''(\langle \phi, \nu \rangle) \iint_{\mathbb{R}^2} \rho_\theta(x - y)\phi'(x)\phi'(y)\nu(dx)\nu(dy)$$

$$+ \frac{1}{2}\sigma_\theta f''(\langle \phi, \nu \rangle)\langle \phi^2, \nu \rangle + q_\theta \cdot f'(\langle \phi, \nu \rangle)\langle \phi, m \rangle \tag{42}$$

for $F(\nu) = f(\langle \phi, \nu \rangle) \in \mathrm{Dom}(\mathcal{I}_\theta) = \mathrm{Dom}(\mathcal{I})$, where $\{\sigma_\theta\}_\theta$ is a sequence of positive numbers and $\{q_\theta\}_\theta$ is a sequence of real numbers.

3.3. Main results

In [4], Dawson et al. showed an observation that the SCSM arises naturally as scaling limit of the purely atomic SDSM. In this paper we prove that the SCSM is also obtained as the scaling limit of the immigration superprocess associated with SDSM.

Taking (42) together with (36)-(39) into consideration, we can show that the rescaled processes $\{Y_t^\theta;\ t \geq 0\}$, $\theta \geq 1$, solve the $(\hat{\mathcal{L}}, \mathcal{D}(\hat{\mathcal{L}}))$-martingale problem for the $\{\rho(0), \rho_\theta, \sigma_\theta\}$-SDSM with deterministic immigration q_θ and the reference measure m. Hence under a proper scaling the rescaled processes prove to be the same type of immigration superprocess. Clearly we obtain:

Proposition 4. *Let $Y = \{Y_t;\ t \geq 0\}$ be a $\{\rho(0), \rho, \sigma, q, m\}$-IMS. For $\theta \geq 1$, set $Y_t^\theta := \theta^{-2} K_\theta Y_{\theta^2 t}$. Then the rescaled processes $\{Y_t^\theta;\ t \geq 0\}_\theta$ live in the family of $\{\rho(0), \rho_\theta, \sigma_\theta, q_\theta, m\}$-IMSs. Moreover, for each $\theta \geq 1$, $\{Y_t^\theta;\ t \geq 0\}$ solves the $(\mathcal{I}_\theta, \mathrm{Dom}(\mathcal{I}_\theta))$-martingale problem and this martingale problem is well-posed.*

Recall the stochastic equation (29) for the interactive immigration superprocess. For purely atomic initial state and non-negative predictable immigration rate $q(s, x, \omega)$ which is locally bounded in $t \geq 0$, by

Theorem 5.1 of [2] the continuous modification of

$$V_t = \sum_{i=1}^{\infty} \xi_i(t)\delta_{x(0, a_i, t)}$$
$$+ \int_0^t \int_{\mathbb{R}} \int_0^{q(s, a)} \int_{W_0} w(t - s)\delta_{x(s, a, t)} N(\mathrm{d}s, \mathrm{d}a, \mathrm{d}u, \mathrm{d}w) \tag{43}$$

satisfies the martingale characterization similar to (30) with (31), which is equivalent to the IMS-martingale problem. Suppose that $q(s, x, \omega) \equiv q(x) \in L^1(\mathbb{R}, m)$. Let $D_{q(x)}$ denote the set

$$\{(s, a, u, w); s \geq 0, a \in \mathbb{R}, 0 \leq u \leq q(a), w \in W_0\},$$

and set $N_{q(x)} := N \restriction D_{q(x)}$, namely, $N_{q(x)}$ is the restriction of the normal N to the set $D_{q(x)}$. Moreover, $\widetilde{N}_{q(x)}(\mathrm{d}s, \mathrm{d}a, \mathrm{d}w)$ denotes the image of $N_{q(x)}$ under the mapping : $(s, a, u, w) \mapsto (s, a, w)$. In other words, $\widetilde{N}_{q(x)}$ is a Poisson measure on $[0, \infty) \times \mathbb{R} \times W_0$ with intensity $\mathrm{d}s \cdot q(a) \cdot m(\mathrm{d}a) Q_k(\mathrm{d}w)$. Hence (43) can be rewritten as

$$V_t = \sum_{i=1}^{\infty} \xi_i(t)\delta_{x(0, a_i, t)} + \int_0^t \int_{\mathbb{R}} \int_{W_0} w(t - s)\delta_{x(s, a, t)} \widetilde{N}_{q(x)}(\mathrm{d}s, \mathrm{d}a, \mathrm{d}w). \tag{44}$$

While, when we replace $\rho(x)$ by $\rho_\theta(x)$, then by the definition (1) of ρ, the function h should be replaced by the scaled function $\sqrt{\theta} h_\theta$. On this account, the stochastic equation (11) which determines the interacting Brownian motion is changed into

$$x^\theta(t) = a + \int_0^t \int_{\mathbb{R}} \sqrt{\theta} h_\theta(y - x^\theta(s)) W(\mathrm{d}s, \mathrm{d}y), \quad t \geq 0. \tag{45}$$

So that, the interacting Brownian flow is changed simultaneously into $\{x^\theta(0, a_i^\theta, t)\}$ for a sequence $\{a_i^\theta\}_\theta$ of real numbers for each $i \in \mathbb{N}$. Under our previously adopted scaling, clearly $\{\rho(0), \rho_\theta, \sigma_\theta, q_\theta, m\}$-IMS has the following atomic representation.

Proposition 5. *Under the same scaling stated in Proposition* 4, *for each* $\theta \geq 1$,

$$Z_t^\theta := \sum_{i=1}^\infty \xi_i^{\sigma_\theta}(t)\delta_{x^\theta(0,a_i^\theta,t)} + \int_0^t \int_{\mathbb{R}} \int_{W_0} w(t-s)\delta_{x^\theta(s,a^\theta,t)} \widetilde{N}_{q_\theta}(\mathrm{d}s, \mathrm{d}a, \mathrm{d}w), \quad t \geq 0, \tag{46}$$

is a $\{\rho(0), \rho_\theta, \sigma_\theta\}$*-SDSM with deterministic immigration rate* q_θ *accompanied by the reference measure* m, *and for each* $\varphi \in C^2(\mathbb{R})$,

$$M_t^\theta(\varphi) := \langle \varphi, Z_t^\theta \rangle - \langle \varphi, Z_0^\theta \rangle - q_\theta \langle \varphi, m \rangle t - \frac{\rho(0)}{2}\int_0^t \langle \varphi'', Z_s^\theta \rangle \mathrm{d}s, \quad t \geq 0 \tag{47}$$

is a continuous martingale relative to the filtration $(\hat{\mathcal{G}}_t)_{t \geq 0}$ *with quadratic variation process*

$$\langle M^\theta(\varphi) \rangle_t = \int_0^t \langle \sigma_\theta \varphi^2, Z_s^\theta \rangle \mathrm{d}s + \theta \int_0^t \mathrm{d}s \int_{\mathbb{R}} \langle h_\theta(z - \cdot)\varphi', Z_s^\theta \rangle^2 \mathrm{d}z, \tag{48}$$

where $\xi_i^{\sigma_\theta}(t) = \xi_i(\sigma_\theta t)$ *for each* $i \in \mathbb{N}$ *and* $\hat{\mathcal{G}}_t$ *is the* σ*-algebra generated by all* $\mathbb{P}$*-null sets and the families of random variables* $\{W([0, s] \times B)\}$, $\{\xi_i(s)\}$ *and* $\{\widetilde{N}_{q_\theta}(J \times A)\}$ *of the forms* (27) *and* (28) *in Subsection* 2.3.

The purpose of this paper is to know what is the rescaled limit of immigration superprocesses. Our goal is to prove a limit theorem that under a suitable scaling, the rescaled SDSMs with deterministic immigration rate converge to SCSM in the distribution sense on a proper path space. Suppose that

(A.1) $\rho(x) \to 0$ (as $|x| \to \infty$);

(A.2) For a sequence $\{\sigma_\theta\}_{\theta \geq 1} \subset \mathbb{R}^+$, $\sigma_\theta \to (\exists)\sigma_0 \in \mathbb{R}^+$ (as $\theta \to \infty$);

(A.3) For a sequence $\{q_\theta\}_{\theta \geq 1} \subset \mathbb{R}^+$, $q_\theta \to 0$ (as $\theta \to \infty$);

(A.4) For the initial state

$$\mu_\theta = \sum_{i=1}^{\infty} \xi_i(0)\delta_{a_i^\theta} \in M_a(\mathbb{R}) \tag{49}$$

with a sequence $\{a_i^\theta\}_\theta \subset \mathbb{R}$ (for each $i \in \mathbb{N}$), there exists a sequence $\{b_i\} \subset \mathbb{R}$,

$$\mu_\theta \to \mu_0 = \sum_{i=1}^{\infty} \xi_i(0)\delta_{b_i} \in M_a(\mathbb{R}) \tag{50}$$

(as $\theta \to \infty$) (cf. (iv) in Section 7).

Remark 8. Usually we assume the boundedness of $\langle 1, \mu\rangle$ for the initial state μ of the process $\{Y_t;\, t \geq 0\}$. In fact, here $\langle 1, \mu_\theta\rangle$ is uniformly bounded in θ, which yields from the finiteness of $\{\xi_i(0)\}$ (see the assumption imposed in Subsection 2.3).

Now we are in a position to state the main theorem on rescaled limits in this paper.

Theorem 6 (Scaling Limit Theorem)**.** *Let* $0 < \langle 1, m\rangle < \infty$. *Assume* (A.1)-(A.4). *For* $\{\rho(0), \rho, \sigma, q, m\}$*-immigration superprocess* (*IMS*) $Y = \{Y_t;\, t \geq 0\}$ *defined in Subsection* 3.1, *put* $Y_t^\theta := \theta^{-2} K_\theta Y_{\theta^2 t}$ *for* $\theta \geq 1$. *Then the conditional distribution of* $\{\rho(0), \rho_\theta, \sigma_\theta, q_\theta, m\}$*-immigration superprocess* (*IMS*) $Y^\theta = \{Y_t^\theta;\, t \geq 0\}$ *given* $Y_0^\theta = \mu_\theta$ (*defined by* (49)) *converges as* $\theta \to \infty$ *to that of* $\{\rho(0), \sigma_0\}$*-superprocess with coalescing spatial motion* (*SCSM*) $X = \{X_t;\, t \geq 0\}$ *with initial state* μ_0 *defined by* (50).

The proof of the principal result (Theorem 6) will be given below all through the succeeding three sections (Sections 4, 5 and 6). It is interesting to note that the processes $\{Y_t^\theta;\, t \geq 0\}$, $\theta \geq 1$, constructed in Propositions 4 and 5 are $M_a(\mathbb{R})$-valued diffusion processes (cf. [2, p. 56]), and also that the limiting process $X = \{X_t;\, t \geq 0\}$, SCSM with speed $\rho(0)$, constant branching rate σ_0 and initial state μ_0 is an $M_a(\mathbb{R})$-valued diffusion process as well (cf. [4, p. 686]).

Next we shall give a brief description of the limiting SCSM $X = \{X_t;\ t \geq 0\}$. According to the discussion in Subsection 2.2, the limiting $\{\rho(0), \sigma_0\}$-SCSM has generator

$$\mathcal{L}_c F(\mu) = \frac{\sigma_0}{2}\int_{\mathbb{R}} \frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(\mathrm{d}x) + \frac{\rho(0)}{2}\int_{\mathbb{R}} \frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(\mathrm{d}x)$$

$$+ \frac{1}{2}\iint_{\Delta} \rho(0)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(\mathrm{d}x)\mu(\mathrm{d}y) \tag{51}$$

for $F(\mu) \in \mathrm{Dom}(\mathcal{L}_c) = \mathcal{D}(\widetilde{\mathcal{L}})$. Especially when $F(\mu) = f(\langle\phi, \mu\rangle)$ with f, $\phi \in C^2(\mathbb{R})$, (51) can be rewritten into a simpler one and indeed has the form

$$\mathcal{L}_c F(\mu) = \frac{1}{2}f''(\langle\phi, \mu\rangle)\langle\sigma_0\phi^2, \mu\rangle + \frac{1}{2}\rho(0)f'(\langle\phi, \mu\rangle)\langle\phi'', \mu\rangle$$

$$+ \frac{1}{2}f''(\langle\phi, \mu\rangle)\iint_{\Delta}\rho(0)\phi'(x)\phi'(y)\mu(\mathrm{d}x)\mu(\mathrm{d}y). \tag{52}$$

Moreover, $\{X_t;\ t \geq 0\}$ satisfies the local boundedness of $\mathbb{E}[\langle 1, X_t\rangle^m]$ in $t \geq 0$ for each $m \geq 1$, and $\{X_t, t \geq 0\}$ solves the $(\mathcal{L}_c, \mathrm{Dom}(\mathcal{L}_c))$-martingale problem relative to $(\mathcal{H}_t)_{t\geq 0}$, and for each $\phi \in C^2(\mathbb{R})$,

$$M_t(\phi) = \langle\phi, X_t\rangle - \langle\phi, X_0\rangle - \frac{\rho(0)}{2}\int_0^t \langle\phi'', X_s\rangle \mathrm{d}s, \quad t \geq 0 \tag{53}$$

is a continuous martingale with respect to the filtration $(\mathcal{H}_t)_{t\geq 0}$ with quadratic variation process

$$\langle M(\phi)\rangle_t = \int_0^t \langle\sigma_0\phi^2, X_s\rangle \mathrm{d}s + \int_0^t \mathrm{d}s\iint_{\Delta}\rho(0)\phi'(x)\phi'(y)X_s(\mathrm{d}x)X_s(\mathrm{d}y). \tag{54}$$

4. Tightness Argument

The proof of the main result (Theorem 6) begins at this section. The purpose of this section is to show the tightness of the rescaled processes. As a matter of fact, we can prove the following assertion.

Proposition 7. *The family* $\{Y_t^{\theta}; t \geq 0\}$, $\theta \geq 1$ *is tight in the continuous path space* $C_M(\mathbb{R}_+) = C([0, \infty), M_F(\mathbb{R}))$.

Before we state the proof of Proposition 7, we need the following lemma for a simple estimate of the total mass process $\langle 1, Y_t^{\theta} \rangle$.

Lemma 8. *For any* η *given, for each* $T > 0$, *we have the estimate*

$$\mathbb{P}(\sup_{0 \leq t \leq T} \langle 1, Y_t^{\theta} \rangle > \eta) \leq \frac{C_0 \{\langle 1, m \rangle + \langle 1, \mu_{\theta} \rangle\}}{\eta} < \infty, \tag{55}$$

where C_0 *is some positive constant depending only on* T *and the parameter* σ_{θ}.

Proof. By the discussion similar to Lemma 4.1 of Dawson-Li [2, p. 50], when we denote by $N_q^*(\mathrm{d}s, \mathrm{d}w)$ the image of $\tilde{N}_q(\mathrm{d}s, \mathrm{d}a, \mathrm{d}w)$ under the mapping : $(s, a, w) \mapsto (s, w)$, then $N_q^*(\mathrm{d}s, \mathrm{d}w)$ is a Poisson random measure on $[0, \infty) \times W_0$ with intensity $\langle 1, m \rangle \mathrm{d}s Q_k(\mathrm{d}w)$ and is independent of Feller branching diffusions $\{\xi_i(t); t \geq 0\}$, $i \in \mathbb{N}$. We may employ the pathwise expression (46) in Proposition 5 to obtain

$$\langle 1, Y_t^{\theta} \rangle = \sum_{i=1}^{\infty} \xi_i^{\sigma_{\theta}}(t) + \int_0^t \int_{W_0} w(t-s) N_{q_{\theta}}^*(\mathrm{d}s, \mathrm{d}w), \quad t \geq 0. \tag{56}$$

Moreover, by Theorem 4.1 of Pitman-Yor [26, p. 442], $\{\langle 1, Y_t^{\theta} \rangle; t \geq 0\}$ is a diffusion process with generator

$$\frac{1}{2} \sigma_{\theta} x \frac{d^2}{dx^2} + \langle 1, m \rangle \frac{d}{dx}. \tag{57}$$

For simplicity, put $U_t^{\theta} := \langle 1, Y_t^{\theta} \rangle$. By the standard theory of diffusion processes [19], U_t^{θ} satisfies a stochastic differential equation (SDE) corresponding to the generator (57), i.e.,

$$\begin{cases} \mathrm{d}U_t^{\theta} = \sqrt{\sigma_{\theta} U_t^{\theta}} \mathrm{d}B_t + \langle 1, m \rangle \mathrm{d}t \\ U_0^{\theta} = \langle 1, Y_0^{\theta} \rangle = \langle 1, \mu_{\theta} \rangle < \infty \end{cases} \tag{58}$$

with $\mu_\theta \in M_a(\mathbb{R})$ (see Remark 8), where B_t is a one-dimensional standard Brownian motion. The theory of SDEs [19] guarantees that the stochastic equation (58) has a unique solution, so that, let us derive an estimate of the solution $\{U_t^\theta;\, t \geq 0\}$. Let η be any positive number given. For each $T > 0$, we can easily get

$$\mathbb{P}(\sup_{0\leq t\leq T}|U_t^\theta| > \eta) \leq \mathbb{P}\left(\sup_t|U_0^\theta + \langle 1,\, m\rangle t| > \frac{\eta}{2}\right) + \mathbb{P}\left(\sup_t\left|\int_0^t \sqrt{\sigma_\theta U_s^\theta}\,\mathrm{d}B_s\right| > \frac{\eta}{2}\right)$$

$$\leq \frac{2}{\eta}\,\mathbb{E}[\sup_t|U_0^\theta + \langle 1,\, m\rangle t|] + \frac{4}{\eta^2}\,\mathbb{E}\left|\int_0^T \sqrt{\sigma_\theta U_s^\theta}\,\mathrm{d}B_s\right|^2, \tag{59}$$

where we made use of Markov's inequality (resp., Doob's martingale inequality with the case of $p = 2$) for the first term (resp., the second term) on the right-hand side of (59), respectively. Furthermore, the first term of (59) can be estimated majorantly by

$$\frac{2}{\eta}(\langle 1,\, m\rangle T + \langle 1,\, \mu_\theta\rangle). \tag{60}$$

By Itô's isometry for stochastic integrals, the second term of (59) can be rewritten as $(4\sigma_\theta/\eta^2)\mathbb{E}\int_0^T U_s^\theta \mathrm{d}s.$ Here we need to make an estimate of the integral term $\mathbb{E}\int U_s^\theta \mathrm{d}s.$ Substituting the integral form of SDE (58) for U_s^θ, it is easy to see that

$$\mathbb{E}\int_0^T U_s^\theta \mathrm{d}s \leq \langle 1,\, \mu_\theta\rangle T + \frac{T^2}{2}\langle 1,\, m\rangle, \tag{61}$$

where we employed the Fubini theorem and some properties in elementary stochastic calculus for the stochastic integral term. Consequently, we may combine (60) and (61) together with (59) to obtain the required result (55).

Now let us go back to the proof of Proposition 7 and prove the tightness of the rescaled processes.

Proof of Proposition 7. To prove the tightness of the family $\{Y_t^\theta\}_\theta$, we shall adopt the ordinary reduction program. So that, we resort to an orthodox tightness criterion. Note that $M_F(\mathbb{R})$ is a complete separable metric space by a metric that induces the weak topology. For instance, by Theorem 9.1 of Ethier-Kurtz [16, p. 142], if $\{Y_t^\theta;\, t \geq 0\}$ satisfies the compact containment condition, i.e., for $\eta > 0$ and $T > 0$, there exists a compact set $\Gamma = \Gamma_{\eta, T} \subset M_F(\mathbb{R})$ such that

$$\inf_\theta \mathbb{P}(Y_t^\theta \in \Gamma_{\eta, T} \text{ for } 0 \leq t \leq T) \geq 1 - \eta,$$

then the relative compactness of distributions of $\{Y_t^\theta\}_\theta$ is attributed to that of $\{F \circ Y_t^\theta\}_\theta$ as a family of processes with sample paths in $C([0, \infty), \mathbb{R})$ for each F in the dense subset H of $C(M_F(\mathbb{R}))$ in the topology of uniform convergence on compact sets. Usually, instead of direct check of the criterion by $\{Y_t^\theta\}$, it is attributed to an easier check of the compact containment condition by $\{\langle 1, Y_t^\theta \rangle\}_\theta$. However, we encounter the problem, namely, the set of the type $\{\mu \in M_F(\mathbb{R});\, \langle 1, \mu \rangle \leq K\}$ is not compact because of the non-compactness of $\mathbb{R}$. To avoid this difficulty we take advantage of the one-point compactification like $\hat{\mathbb{R}} := \mathbb{R} \cup \{\partial\}$, and we shall check its convergence in a wider space $M_F(\hat{\mathbb{R}})$. Recall the estimate (55) in Lemma 8. It is easy to see from (55) that for $\eta > 0$, for each $T > 0$,

$$\inf_\theta \mathbb{P}\{\sup_{0 \leq t \leq T} \langle 1, Y_t^\theta \rangle \leq \eta\} \geq 1 - \frac{C_0}{\eta} \{\langle 1, m \rangle + \langle 1, \mu_\theta \rangle\} \tag{62}$$

holds. Hence the tightness of distributions of $\{Y_t^\theta;\, t \geq 0\}$ in $C([0, \infty), M_F(\hat{\mathbb{R}}))$ has been attributed to that of $\{F \circ Y_t^\theta\}$. Next, according to Theorem 9.4 of Ethier-Kurtz [16, p. 145], in order to verify the tightness of $\{F \circ Y_t^\theta\}$, we have only to verify that there exists (X^α, Z^α) such that $X_t^\alpha - \int_0^t Z_s^\alpha \mathrm{d}s$ is a $(\mathcal{F}_t^\alpha)$-martingale for the parameter α and

$$\sup_\alpha \mathbb{E}\{\sup_{0 \leq t \leq T} |X_t^\alpha - F \circ Y_t^\alpha|\} < \varepsilon.$$

Take $F(\nu) = F_{f,\{\phi_i\}}(\nu) \in \mathrm{Dom}(\mathcal{I}_\theta)$ with $f \in C^2(\mathbb{R}^n)$ and $\{\phi_i\} \subset C^2(\mathbb{R})$ (see (4) in Subsection 2.1). Paying attention to simple computation

$$\frac{\delta}{\delta\nu(x)} F_{f,\{\phi_i\}}(\nu) = \sum_{i=1}^{n} f_i'(\langle\phi_1, \nu\rangle, \ldots, \langle\phi_n, \nu\rangle) \cdot \phi_i(x)$$

and

$$\frac{\delta^2 F_{f,\{\phi_i\}}(\nu)}{\delta\nu(x)\delta\nu(y)} = \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}''(\langle\phi_1, \nu\rangle, \ldots, \langle\phi_n, \nu\rangle) \cdot \phi_i(x)\phi_j(y),$$

we readily obtain

$$\begin{aligned}
&\mathcal{I}_\theta F_{f,\{\phi_i\}}(\nu) \\
&= \frac{\rho(0)}{2}\sum_{i=1}^{n} f_i'(\langle\phi_1, \nu\rangle, \ldots, \langle\phi_n, \nu\rangle) \cdot \langle\phi_i'', \nu\rangle \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{n} f_{ij}''(\langle\phi_1, \nu\rangle, \ldots, \langle\phi_n, \nu\rangle) \iint_{\mathbb{R}^2} \rho_\theta(x-y)\phi_i'(x)\phi_j'(y)\nu(dx)\nu(dy) \\
&\quad + \frac{\sigma_\theta}{2}\sum_{i,j=1}^{n} f_{ij}''(\langle\phi_1, \nu\rangle, \ldots, \langle\phi_n, \nu\rangle) \cdot \langle\phi_i\phi_j, \nu\rangle \\
&\quad + q_\theta\sum_{i=1}^{n} f_i'(\langle\phi_1, \nu\rangle, \ldots, \langle\phi_n, \nu\rangle) \cdot \langle\phi_i, m\rangle. \qquad (63)
\end{aligned}$$

On the other hand, with Itô process

$$d\langle\phi, Y_t^\theta\rangle = \left\{\frac{\rho(0)}{2}\langle\phi'', Y_t^\theta\rangle + q_\theta\langle\phi, m\rangle\right\}dt + dM_t^\theta(\phi), \qquad (64)$$

an application of Itô's formula to the function $F(Y_t^\theta) = F_{f,\{\phi_i\}}(Y_t^\theta) = f(\langle\phi_1, Y_t^\theta\rangle, \ldots, \langle\phi_n, Y_t^\theta\rangle)$ allows us to have

$$\begin{aligned}
F_{f\{\phi_i\}}(Y_t^\theta) &= F_{f,\{\phi_i\}}(\mu_\theta) + \int_0^t \mathcal{I}_\theta F_{f,\{\phi_i\}}(Y_s^\theta)ds \\
&\quad + \sum_{i=1}^{n}\int_0^t f_i'(\langle\phi_1, Y_s^\theta\rangle, \ldots, \langle\phi_n, Y_s^\theta\rangle)dM_s(\phi_i), \qquad (65)
\end{aligned}$$

where we employed the relation

$$d\langle M(\phi_i),\, M(\phi_j)\rangle_t$$

$$= \langle \sigma_\theta \phi_i \phi_j,\, Y_t^\theta\rangle dt + \theta\left(\int_{\mathbb{R}} \langle h_\theta(z-\cdot)\phi_i',\, Y_t^\theta\rangle \langle h_\theta(z-\cdot)\phi_j',\, Y_t^\theta\rangle dz\right)dt. \tag{66}$$

This implies immediately that

$$F_{f,\{\phi_i\}}(Y_t^\theta) - F_{f,\{\phi_i\}}(Y_0^\theta) - \int_0^t \mathcal{I}_\theta F_{f,\{\phi_i\}}(Y_s^\theta)ds$$

is a $(\hat{\mathcal{G}}_t)$-martingale under the probability measure $\mathbb{P}_{\mu_\theta}$ for which the martingale characterization (47) and (48) of $\{\rho(0),\, \rho_\theta,\, \sigma_\theta\}$-SDSM is valid with deterministic immigration q_θ and reference measure m. Therefore it follows from Ethier-Kurtz' criterion that $\{F \circ Y^\theta\}_\theta = \{F_{f,\{\phi_i\}}(Y_t^\theta)\}_\theta$ is relatively compact for each $F_{f,\{\phi_i\}} \in \mathrm{Dom}(\mathcal{I}_\theta)$. After all, the tightness of $\{Y_t^\theta;\, t \geq 0\}$ in $C([0, \infty),\, M_F(\hat{\mathbb{R}}))$ is derived. Let $\mathbb{Q}_\theta$ denote the distribution of $\{Y_t^\theta\}$ on $C([0, \infty),\, M_F(\hat{\mathbb{R}}))$. By the same discussion of Theorem 4.1 of Dawson et al. [3], it can be shown that

$$\mathbb{Q}_\theta\{\sup_{0\leq t\leq u} Y_t^\theta(\{\partial\}) > \delta\} < \varepsilon, \tag{67}$$

hence $\mathbb{Q}_\theta(Y_t^\theta(\{\partial\}) = 0, \forall t \in [0, u]) = 1$ holds for $u > 0$. Clearly this implies that any limit point of $\mathbb{Q}_\theta$ is supported by the space $C([0, \infty),\, M_F(\mathbb{R}))$. Thus we attain that $\{\mathbb{Q}_\theta\}$ is tight as a probability measure on $C([0, \infty),\, M_F(\mathbb{R}))$, and equivalently it proves to be that $\{Y_t^\theta\}$ is tight in $C([0, \infty),\, M_F(\mathbb{R}))$. This completes the proof of Proposition 7.

5. Convergence Argument

The purpose of this section is to prove the main theorem (Theorem 6). We have already proved that the family of rescaled processes $\{Y_t^\theta;\, t \geq 0\}_\theta$ is tight in $C_M(\mathbb{R}_+) = C([0, \infty),\, M_F(\mathbb{R}))$ (see Proposition 7). Then we can extract a convergent subsequence of distributions of $\{Y_t^\theta\}$. Choose any sequence $\{\theta_k\}_k \subset \{\theta \geq 1\}$ such that the distributions of $\{Y_t^{\theta_k};\, t \geq 0\}_k$ converge as $k \to \infty$ to some probability measure $\mathbb{Q}_{\mu_0}$ on the continuous path space, namely, $\mathbb{Q}_{\mu_0} \in \mathcal{P}(C_M(\mathbb{R}_+))$. We shall show that the above limit measure $\mathbb{Q}_{\mu_0}$ is a solution of the $(\mathcal{L}_c, \mathrm{Dom}(\mathcal{L}_c))$-martingale problem of the target process SCSM. As a matter of fact, as explained in Subsection 2.2, the distribution of the SCSM is uniquely determined by the transition semigroup $Q(\mu_0, d\nu)$ via the duality method (see Theorem 3). Therefore the distribution of $\{Y_t^\theta;\, t \geq 0\}$ itself actually converges to $\mathbb{Q}_{\mu_0}$ as $\theta \to \infty$. Roughly speaking, this completes the proof. In the level of convergence discussion we need to employ the useful and important key proposition (Proposition 12), which guarantees the convergence of the principal term of the generator $\mathcal{I}_\theta$. As described before in Section 1, the proof of key proposition is quite longsome, so that we suppress here the proof of the key proposition. We would rather admit the result and dare to prove the main theorem on ahead in this section. The full proof of key proposition will be given in the succeeding section.

When the distribution of $\{Y_t^{\theta_k};\, t \geq 0\}$ converges as $k \to \infty$ to $\mathbb{Q}_{\mu_0} \in \mathcal{P}(C_M(\mathbb{R}_+))$ on some complete standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then by virtue of Skorokhod's representation theorem (e.g., see Theorem 1.4 of [14, p. 274]), we can construct processes $\{Y_t^{(k)};\, t \geq 0\}$ and $\{Y_t^{(0)};\, t \geq 0\}$ on a new proper probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ in such a way that (i) $\{Y_t^{\theta_k}\}$ and $\{Y_t^{(k)}\}$ are identically distributed, that is, $\mathcal{L}(\{Y_t^{\theta_k};\, t \geq 0\}) = \mathcal{L}(\{Y_t^{(k)};\, t \geq 0\})$ holds, where the symbol $\mathcal{L}(X)$ means the law of random variable X; (ii) the new limiting process $\{Y_t^{(0)};\, t \geq 0\}$ has the distribution $\mathbb{Q}_{\mu_0}$; and (iii) $\{Y_t^{(k)};\, t \geq 0\}$ converges almost surely (a.s.) as $k \to \infty$ to $\{Y_t^{(0)};\, t \geq 0\}$ in the space $C([0, \infty), M_F(\mathbb{R}))$. Notice that $\{Y_t^{\theta_k};\, t \geq 0\}$ is a $\{\rho(0), \rho_{\theta_k}, \sigma_{\theta_k}, q_{\theta_k}, m\}$-IMS, and is a solution of the $(\mathcal{I}_{\theta_k}, \mathrm{Dom}(\mathcal{I}_{\theta_k}))$-martingale problem. It is known that this martingale problem is permitted to possess a unique solution. Since $\{Y_t^{\theta_k}\}$ has the same distribution as $\{Y_t^{(k)}\}$, clearly $\{Y_t^{(k)};\, t \geq 0\}$ solves the $(\mathcal{I}_{\theta_k}, \mathrm{Dom}(\mathcal{I}_{\theta_k}))$-martingale problem. Consequently, for each k,

$$F(Y_t^{(k)}) - F(Y_0^{(k)}) - \int_0^t \mathcal{I}_{\theta_k} F(Y_s^{(k)})\mathrm{d}s, \quad t > 0 \tag{68}$$

is a continuous martingale relative to $(\hat{\mathcal{G}}_t)_{t \geq 0}$.

Our main concern here is to show, roughly speaking, that the generator $\mathcal{I}_{\theta_k}$ of the form (42) in Subsection 3.2 converges as $k \to \infty$ to the generator $\mathcal{L}_c$ of the form (52) in Subsection 3.3 under the setting described in Theorem 6. In other words, for $F(\mu) = f(\langle \phi, \mu \rangle)$ with $f \in C^2(\mathbb{R})$, $\phi \in C^2(\mathbb{R})$ and $\mu \in M_F(\mathbb{R})$, we are interested in the convergence of the generator

$$\begin{aligned}\mathcal{I}_{\theta_k} F(\mu) = {} & \frac{\rho(0)}{2} f'(\langle \phi, \mu \rangle) \cdot \langle \phi'', \mu \rangle \\ & + \frac{1}{2} f''(\langle \phi, \mu \rangle) \iint_{\mathbb{R}^2} \rho_{\theta_k}(x - y)\phi'(x)\phi'(y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \\ & + \frac{\sigma_{\theta_k}}{2} f''(\langle \phi, \mu \rangle) \cdot \langle \phi^2, \mu \rangle + q_{\theta_k} \cdot f'(\langle \phi, \mu \rangle) \cdot \langle \phi, m \rangle. \end{aligned} \tag{69}$$

Of course, the generator of the limiting process $\{Y_t^{(0)};\, t \geq 0\}$ should have the form

$$\begin{aligned}\mathcal{L}_c F(\mu) = {} & \frac{\rho(0)}{2} f'(\langle \phi, \mu \rangle) \cdot \langle \phi'', \mu \rangle \\ & + \frac{1}{2} f''(\langle \phi, \mu \rangle) \iint_{\Delta} \rho(0)\phi'(x)\phi'(y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \\ & + \frac{\sigma_0}{2} f''(\langle \phi, \mu \rangle) \cdot \langle \phi^2, \mu \rangle. \end{aligned} \tag{70}$$

By Skorokhod's representation, for $\phi \in C^2(\mathbb{R})$ and for each $T > 0$,

$$\|\langle \phi, Y_\cdot^{(k)} \rangle - \langle \phi, Y_\cdot^{(0)} \rangle\|_\infty$$

$$= \sup_{0 \le t \le T} |\langle \phi, Y_t^{(k)} \rangle - \langle \phi, Y_t^{(0)} \rangle| \to 0 \quad \text{a.s.} \quad (\text{as } k \to \infty). \tag{71}$$

Hence it is obvious that

$$F(Y_t^{(k)}) \to F(Y_t^{(0)}) \quad \text{a.s.} \quad (\text{as } k \to \infty) \tag{72}$$

uniformly in t on compact sets for any $F \in \mathrm{Dom}(\mathcal{I}_\theta) = \mathrm{Dom}(\mathcal{I})$. Similarly,

$$F(Y_0^{(k)}) \to F(Y_0^{(0)}) \quad \text{a.s.} \quad (\text{as } k \to \infty). \tag{73}$$

Lemma 9. *For any* $t > 0$, *we have*

$$\lim_{k \to \infty} \int_0^t \mathbb{E}|q_{\theta_k} f'(\langle \phi, Y_s^{(k)} \rangle) \cdot \langle \phi, m \rangle| \mathrm{d}s = 0. \tag{74}$$

Proof. Paying attention to a simple inequality $|\langle \phi, m \rangle| \le \| \phi \| \cdot |\langle 1, m \rangle| < \infty$ together with the assumption: $0 < \langle 1, m \rangle < \infty$ (see Section 3), we may apply the Lebesgue bounded convergence theorem to obtain

$$\int_0^t \mathbb{E}|q_{\theta_k} f'(\langle \phi, Y_s^{(k)} \rangle) \cdot \langle \phi, m \rangle| \mathrm{d}s$$

$$\le \int_0^t \mathbb{E}|q_{\theta_k} \{f'(\langle \phi, Y_s^{(k)} \rangle) - f'(\langle \phi, Y_s^{(0)} \rangle)\} \cdot \langle \phi, m \rangle| \mathrm{d}s$$

$$+ \int_0^t \mathbb{E}|q_{\theta_k} f'(\langle \phi, Y_s^{(0)} \rangle) \cdot \langle \phi, m \rangle| \mathrm{d}s$$

$$\le t|q_{\theta_k}| \cdot |\langle \phi, m \rangle| \cdot \mathbb{E}\|f'(\langle \phi, Y_\cdot^{(k)} \rangle) - f'(\langle \phi, Y_\cdot^{(0)} \rangle)\|_\infty$$

$$+ |\langle \phi, m \rangle| \cdot |q_{\theta_k}| \int_0^t \mathbb{E}|f'(\langle \phi, Y_s^{(0)} \rangle)| \mathrm{d}s$$

$$\to 0 \quad (\text{as } k \to \infty),$$

where we made use of (71), (72) and the assumption (A.3): $q_{\theta_k} \to 0$ (as $k \to \infty$).

Lemma 10. *For any* $t > 0$, *we have*

$$\lim_{k\to\infty}\int_0^t \mathbb{E}\Big|\frac{1}{2}\sigma_{\theta_k} f''(\langle\phi, Y_s^{(k)}\rangle)\cdot\langle\phi^2, Y_s^{(k)}\rangle - \frac{1}{2}\sigma_0 f''(\langle\phi, Y_s^{(0)}\rangle)\cdot\langle\phi^2, Y_s^{(0)}\rangle\Big| ds = 0. \tag{75}$$

Proof. The constant $\frac{1}{2}$ is disinfluential for estimation, so we discuss the matter omitting it. We readily get

$$\begin{aligned}
&|\sigma_{\theta_k} f''(\langle\phi, Y_t^{(k)}\rangle)\langle\phi^2, Y_t^{(k)}\rangle - \sigma_0 f''(\langle\phi, Y_t^{(0)}\rangle)\langle\phi^2, Y_t^{(0)}\rangle| \\
&\le \sigma_{\theta_k}|f''(\langle\phi, Y_t^{(k)}\rangle)\langle\phi^2, Y_t^{(k)}\rangle - f''(\langle\phi, Y_t^{(0)}\rangle)\langle\phi^2, Y_t^{(0)}\rangle| \\
&\quad + |\sigma_{\theta_k} - \sigma_0|\cdot|f''(\langle\phi, Y_t^{(0)}\rangle)\cdot\langle\phi^2, Y_t^{(0)}\rangle| \\
&\le \sigma_{\theta_k}|f''(\langle\phi, Y_t^{(k)}\rangle) - f''(\langle\phi, Y_t^{(0)}\rangle)|\cdot|\langle\phi^2, Y_t^{(k)}\rangle| \\
&\quad + \sigma_{\theta_k}|f''(\langle\phi, Y_t^{(0)}\rangle)|\cdot|\langle\phi^2, Y_t^{(k)}\rangle - \langle\phi^2, Y_t^{(0)}\rangle| \\
&\quad + |\sigma_{\theta_k} - \sigma_0|\cdot|f''(\langle\phi, Y_t^{(0)}\rangle)\cdot\langle\phi^2, Y_t^{(0)}\rangle| \\
&=: A_1 + A_2 + A_3 \quad (\leftarrow \text{we, put this way}).
\end{aligned}$$

As to the first term, because of almost sure convergence of $Y_t^{(k)}$ towards $Y_t^{(0)}$, we can deduce from (71), (72) and continuity of the function that

$$\int_0^t \mathbb{E}\{A_1\}ds \le t\sigma_{\theta_k}\mathbb{E}\{\|f''(\langle\phi, Y_\cdot^{(k)}\rangle) - f''(\langle\phi, Y_\cdot^{(0)}\rangle)\|_\infty\cdot\|\langle\phi^2, Y_\cdot^{(k)}\rangle\|_\infty\}$$
$$\to 0 \quad (\text{as } k\to\infty)$$

by employing the Fubini theorem and the Lebesgue convergence theorem. As for the second and third terms, it goes almost similarly by the same reasons. Indeed, by the Fubini theorem and the Lebesgue theorem we can get easily

$$\int_0^t \mathbb{E}\{A_2\}ds \le t\sigma_{\theta_k}\|f''\|\cdot\mathbb{E}\|\langle\phi^2, Y_\cdot^{(k)} - Y_\cdot^{(0)}\rangle\|_\infty \to 0 \quad (\text{as } k\to\infty),$$

and

$$\int_0^t \mathbb{E}\{A_3\}ds \le t|\sigma_{\theta_k} - \sigma_0|\cdot\|f''\|\cdot\mathbb{E}\|\langle\phi^2, Y_\cdot^{(0)}\rangle\|_\infty \to 0 \quad (\text{as } k\to\infty).$$

Summing up, we finally obtain the required result (75).

Proposition 11. *For $t > 0$ we have*

$$\lim_{k\to\infty}\int_0^t \mathbb{E}\left| \frac{1}{2} f''(\langle \phi, Y_s^{(k)}\rangle) \iint_{\mathbb{R}^2} \rho_{\theta_k}(x-y)\phi'(x)\phi'(y) Y_s^{(k)}(dx) Y_s^{(k)}(dy) \right.$$

$$\left. - \frac{\rho(0)}{2} f''(\langle \phi, Y_s^{(0)}\rangle) \iint_{\Delta} \phi'(x)\phi'(y) Y_s^{(0)}(dx) Y_s^{(0)}(dy) \right| ds = 0. \quad (76)$$

If the above-mentioned proposition is proved, then by combining Lemmas 9 and 10 we are able to show that $\int_0^t \mathcal{I}_{\theta_k} F(Y_s^{(k)}) ds$ converges to $\int_0^t \mathcal{L}_c F(Y_s^{(0)}) ds$. If that is the case, we can step forward and it is possible to assert from (72) and (73) that the martingale term (68) can be rewritten into

$$F(Y_t^{(0)}) - F(Y_0^{(0)}) - \int_0^t \mathcal{L}_c F(Y_s^{(0)}) ds, \quad t > 0 \quad (77)$$

when k tends to infinity. In fact, instead of (76), equivalently it suffices to prove the following key proposition.

Proposition 12 (Key proposition). *For $t > 0$ we have*

$$\lim_{k\to\infty} \mathbb{E}\left| \int_0^t ds f''(\langle \phi, Y_s^{(k)}\rangle) \iint_{\mathbb{R}^2} \rho_{\theta_k}(x-y)\phi'(x)\phi'(y) Y_s^{(k)}(dx) Y_s^{(k)}(dy) \right.$$

$$\left. - \int_0^t ds f''(\langle \phi, Y_s^{(0)}\rangle) \iint_{\Delta} \rho(0)\phi'(x)\phi'(y) Y_s^{(0)}(dx) Y_s^{(0)}(dy) \right| = 0. \quad (78)$$

We shall postpone giving the proof of Proposition 12 until we move to Section 6. Here we admit the result of key proposition for a while, and we concentrate our attention on the limiting subject of rescaled processes and first proceed with the proof of the main result namely Theorem 6.

Lemma 13. *For $t > 0$,*

$$\lim_{k\to\infty}\int_0^t \mathbb{E}|\mathcal{I}_{\theta_k} F(Y_s^{(k)}) - \mathcal{L}_c F(Y_s^{(0)})| ds = 0. \quad (79)$$

Proof. The assertion (79) yields directly from Lemmas 9 and 10 and Propositions 11 and 12.

Proposition 14. *For* $F \in \mathrm{Dom}(\mathcal{L}_c)$

$$F(Y_t^{(0)}) - F(Y_0^{(0)}) - \int_0^t \mathcal{L}_c F(Y_s^{(0)}) \mathrm{d}s, \quad t > 0 \tag{80}$$

is a martingale.

Proof. By approximation procedure we may assume without loss of generality that $F(\mu) = f(\langle \phi, \mu \rangle)$ with $f \in C^2(\mathbb{R})$ and $\phi \in C^2(\mathbb{R})$. Suppose that a collection of functions $\{\Phi_i\}_{i=1}^n$ forms a subset of $C(M_F(\mathbb{R}))$. Let $\widetilde{\Delta} = \{t_k\}_k$ be a partition of times such that $0 \le t_1 < t_2 < \cdots < t_n < t_{n+1}$. Then by using the Fubini theorem and the Lebesgue theorem it is easy to see from (72), (73) and (77) together with Lemma 13 that

$$\mathbb{E}\left\{\left(F(Y_{t_{n+1}}^{(0)}) - F(Y_{t_n}^{(0)}) - \int_{t_n}^{t_{n+1}} \mathcal{L}_c F(Y_s^{(0)}) \mathrm{d}s\right) \cdot \prod_{i=1}^n \Phi_i(Y_{t_i}^{(0)})\right\}$$

$$= \mathbb{E}\left\{F(Y_{t_{n+1}}^{(0)}) \prod_{i=1}^n \Phi_i(Y_{t_i}^{(0)})\right\} - \mathbb{E}\left\{F(Y_{t_n}^{(0)}) \prod_{i=1}^n \Phi_i(Y_{t_i}^{(0)})\right\}$$

$$- \int_{t_n}^{t_{n+1}} \mathbb{E}\left\{\mathcal{L}_c F(Y_s^{(0)}) \cdot \prod_{i=1}^n \Phi_i(Y_{t_i}^{(0)})\right\} \mathrm{d}s$$

$$= \lim_{k\to\infty} \mathbb{E}\left\{F(Y_{t_{n+1}}^{(k)}) \prod_{i=1}^n \Phi_i(Y_{t_i}^{(k)})\right\} - \lim_{k\to\infty} \mathbb{E}\left\{F(Y_{t_n}^{(k)}) \prod_{i=1}^n \Phi_i(Y_{t_i}^{(k)})\right\}$$

$$- \lim_{k\to\infty} \int_{t_n}^{t_{n+1}} \mathbb{E}\left\{\mathcal{I}_{\theta_k} F(Y_s^{(k)}) \cdot \prod_{i=1}^n \Phi_i(Y_{t_i}^{(k)})\right\} \mathrm{d}s$$

$$= \lim_{k\to\infty} \mathbb{E}\left\{\left(F(Y_{t_{n+1}}^{(k)}) - F(Y_{t_n}^{(k)}) - \int_{t_n}^{t_{n+1}} \mathcal{I}_{\theta_k} F(Y_s^{(k)}) \mathrm{d}s\right) \cdot \prod_{i=1}^n \Phi_i(Y_{t_i}^{(k)})\right\}$$

$$= 0, \tag{81}$$

because $\{Y_t^{(k)}; t \ge 0\}$ is a $\{\rho(0), \rho_{\theta_k}, \sigma_{\theta_k}, q_{\theta_k}, m\}$-IMS and solves the $(\mathcal{I}_{\theta_k}, \mathrm{Dom}(\mathcal{I}_{\theta_k}))$-martingale problem. Moreover, by means of repetition of similar discussions, it follows from (81) that for any collection of $\{\Phi_i\}_{i=1}^n$ with any $n \in \mathbb{N}$, and for any time partition $\widetilde{\Delta}$,

$$\mathbb{E}\left\{\left(F(Y_{t_{n+1}}^{(0)}) - F(Y_{t_n}^{(0)}) - \int_{t_n}^{t_{n+1}} \mathcal{L}_c F(Y_s^{(0)}) \mathrm{d}s\right) \cdot \prod_{i=1}^{n} \Phi_i(Y_{t_i}^{(0)})\right\} = 0. \qquad (82)$$

That is to say, this obviously implies that $F(Y_t^{(0)}) - F(Y_0^{(0)}) - \int_0^t \mathcal{L}_c F(Y_s^{(0)}) \mathrm{d}s$ is a martingale for $t > 0$.

Recall that the operator $\mathcal{L}_c$ is definitely given by (70) here. Clearly it turns out to be that this $\{Y_t^{(0)}; t \geq 0\}$ becomes a solution of the $(\mathcal{L}_c, \mathrm{Dom}(\mathcal{L}_c))$-martingale problem for the SCSM. Under the purely atomic initial state $\mu_0 \in M_a(\mathbb{R})$, the distribution of SCSM is unique in the sense of duality formalism. By virtue of the above mentioned discussion on the rescaled limit of $\{\rho(0), \rho_\theta, \sigma_\theta, q_\theta, m\}$-IMS itself with initial state μ_θ, the $(\mathcal{I}_\theta, \mathrm{Dom}(\mathcal{I}_\theta))$-martingale problem induces the $(\mathcal{L}_c, \mathrm{Dom}(\mathcal{L}_c))$-martingale problem (cf. Proposition 14), which is nothing but the $\{\rho(0), \sigma_0\}$-SCSM martingale problem with the initial state $Y_0^{(0)} = \mu_0$. Furthermore, this also indicates that the limiting process

$$Y_t^{(0)} = \sum_{i=1}^{\infty} \xi_i^{\sigma_0}(t) \delta_{y_i(0, b_i, t)}$$

is a $\{\rho(0), \sigma_0\}$-SCSM. In other words, the limit $\mathbb{Q}_{\mu_0}$ of distributions of $\{Y_t^\theta\}$ is a solution of the martingale problem of the $\{\rho(0), \sigma_0\}$-SCSM. Thus we attain that the distribution of $(\mathcal{I}_\theta, \mathrm{Dom}(\mathcal{I}_\theta))$-IMS with $Y_0^\theta = \mu_\theta$ defined by (49) converges as $\theta \to \infty$ to that of the $(\mathcal{L}_c, \mathrm{Dom}(\mathcal{L}_c))$-SCSM with $Y_0^{(0)} = \mu_0$ defined by (50). We finally realize that $\{\rho(0), \sigma_0\}$-SCSM naturally arises in the rescaled limits of $\{\rho(0), \rho, \sigma, q, m\}$-IMS under the setting (A.1)-(A.4) with the scaling $Y_t^\theta := \theta^{-2} K_\theta Y_{\theta^2 t}$, $\theta \geq 1$. This completes the proof of Theorem 6 which is the main result in this paper.

6. Proof of Key Proposition

The aim of this section is to prove the key proposition (Proposition 12), the proof of which was postponed in the previous section because of its lengthy story. To avoid trivial notational redundancy and also for brevity's sake, in what follows we shall not hesitate to use some abbreviated signs and symbols occasionally as far as no confusion occurs.

6.1. Simple reduction

A simple triangular inequality gives

$$\begin{aligned}
&\Bigg| \int_0^t \mathrm{d}s f''(\langle \phi, Y_s^{(k)}\rangle) \iint_{\mathbb{R}^2} \rho_{\theta_k}(x-y)\phi'(x)\phi'(y) Y_s^{(k)}(\mathrm{d}x) Y_s^{(k)}(\mathrm{d}y) \\
&\quad - \int_0^t \mathrm{d}s f''(\langle \phi, Y_s^{(0)}\rangle) \iint_{\Delta} \rho(0)\phi'(x)\phi'(y) Y_s^{(0)}(\mathrm{d}x) Y_s^{(0)}(\mathrm{d}y) \Bigg| \\
&\le \Bigg| \int_0^t \mathrm{d}s f''[Y_s^k] \iint \rho_k(x-y)\phi'(x)\phi'(y)\mathrm{d}Y_x^k \mathrm{d}Y_y^k \\
&\quad - \int_0^t \mathrm{d}s f''[Y_s^0] \iint \rho_k(x-y)\phi'(x)\phi'(y)\mathrm{d}Y_x^k \mathrm{d}Y_y^k \Bigg| \\
&\quad + \Bigg| \int_0^t \mathrm{d}s f''[Y_s^0] \iint \rho_k(x-y)\phi'(x)\phi'(y)\mathrm{d}Y_x^k \mathrm{d}Y_y^k \\
&\quad - \int_0^t \mathrm{d}s f''[Y_s^0] \iint_{\Delta} \rho(0)\phi'(x)\phi'(y)\mathrm{d}Y_x^0 \mathrm{d}Y_y^0 \Bigg| \\
&=: J_1[83] + J_2[83],
\end{aligned} \tag{83}$$

where we put $\rho_k := \rho_{\theta_k}$, $Y_s^* := Y_s^{(*)}$, $f''[Y_s^*] := f''(\langle \phi, Y_s^{(*)}\rangle)$ and $\mathrm{d}Y_x^* \equiv Y_s^*(\mathrm{d}x) := Y_s^{(*)}(\mathrm{d}x)$ with $* = k$ or 0. Moreover,

$$\begin{aligned}
J_1[83] &\le \| f''(\langle \phi, Y_\cdot^{(k)}\rangle) - f''(\langle \phi, Y_\cdot^{(0)}\rangle)\|_\infty \\
&\quad \times \int_0^t \Bigg| \iint_{\mathbb{R}^2} \rho_k(x-y)\phi'(x)\phi'(y)\mathrm{d}Y_x^k \mathrm{d}Y_y^k \Bigg| \mathrm{d}s \\
&\to 0 \quad (\text{as } k \to \infty),
\end{aligned} \tag{84}$$

because $Y_t^{(k)}$ converges a.s. to $Y_t^{(0)}$, hence $f''[Y_t^k]$ converges a.s. to $f''[Y_t^0]$ uniformly in t on compact sets as k approaches to infinity. Combining (83) with (84), in order to prove Proposition 12 it is sufficient to show that $\lim_{k\to\infty}\mathbb{E}\{J_2[83]\}=0$. In addition, thanks to an easy estimate

$$J_2[83] \le \| f'' \| \cdot \left| \int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} \rho_k(x-y)\phi'(x)\phi'(y)\mathrm{d}Y_x^k \mathrm{d}Y_y^k \right.$$
$$\left. - \int_0^t \mathrm{d}s \iint_{\Delta} \rho(0)\phi'(x)\phi'(y)\mathrm{d}Y_x^0 \mathrm{d}Y_y^0 \right|, \tag{85}$$

it suffices indeed to verify the following:

Lemma 15. *For* $t > 0$

$$\lim_{k\to\infty} \mathbb{E}\left| \int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} \rho_{\theta_k}(x-y)\phi'(x)\phi'(y)Y_s^{(k)}(\mathrm{d}x)Y_s^{(k)}(\mathrm{d}y) \right.$$
$$\left. - \int_0^t \mathrm{d}s \iint_{\Delta} \rho(0)\phi'(x)\phi'(y)Y_s^{(0)}(\mathrm{d}x)Y_s^{(0)}(\mathrm{d}y) \right| = 0. \tag{86}$$

6.2. Purely atomic representation

Recall a useful representation of the superprocess in terms of excursions, which has been derived recently in Dawson-Li [2]. Since we have $\mathcal{L}(Y_\cdot^{(k)}) = \mathcal{L}(Y_\cdot^{\theta_k})$ and $\{Y_t^{\theta_k}\}$ is a unique solution of deterministic IMS-martingale problem, we may reconstruct $Y_t^{\theta_k}$ on the probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ if necessary. In fact we have an explicit representation (46). We can realize

$$Z_t^{\theta_k} = \sum_{i=1}^{\infty} \xi_i^{\sigma_{\theta_k}}(t)\delta_{x^{\theta_k}(0, a_i^{\theta_k}, t)} + \int_0^t \int_{\mathbb{R}} \int_{W_0} w(t-s)\delta_{x(s,a,t)}\widetilde{N}_{q_{\theta_k}}(\mathrm{d}s, \mathrm{d}a, \mathrm{d}w)$$
$$=: Z_t^{1,\theta_k} + Z_t^{2,\theta_k} \tag{87}$$

in the same way as above as deterministic $\{\rho(0), \rho_{\theta_k}, \sigma_{\theta_k}, q_{\theta_k}, m\}$-IMS. Then by Proposition 5 $Z_t^{\theta_k}$ solves the $(\mathcal{I}_{\theta_k}, \mathrm{Dom}(\mathcal{I}_{\theta_k}))$-martingale problem relative to $\widetilde{\mathbb{P}}$ (see Section 7), while $Y_t^{(k)}$ is also a solution of the same type of martingale problem. By the uniqueness we may regard $Z_t^{\theta_k}$ of (87) as purely atomic representation of $Y_t^{(k)}$. Since $\theta_k \to \infty$ and $q_{\theta_k} \to 0$, too, as $k \to \infty$ by the assumption (A.3), notice that

$$Z_t^{2,\theta_k} = \int_0^t \int_{\mathbb{R}} \int_{W_0} w(t-s)\delta_{x(s,a,t)}\widetilde{N}_{q_{\theta_k}}(\mathrm{d}s, \mathrm{d}a, \mathrm{d}w) \quad (\to 0) \tag{88}$$

vanishes $\widetilde{\mathbb{P}}$-a.s. as k tends towards infinity. On the other hand, from (A.1), (A.2) and (A.4) (see also Section 7) it is expected that $Z_t^{\theta_k}$ converges almost surely as $k \to \infty$ to some limit process $Z_t^{\infty} = \sum_{i=1}^{\infty} \xi_i^{\sigma_0}(t)\delta_{y_i(0,b_i,t)}$, where $\{y_i(0, b_i, t)\}$ denotes the coalescing Brownian flow. So that, by the almost sure convergence of $\{Y_t^{(k)}\}$ and the uniqueness of limit, it should be that

$$Y_t^{(0)} = \sum_{i=1}^{\infty} \xi_i^{\sigma_0}(t)\delta_{y_i(0,b_i,t)}. \tag{89}$$

Anyway let us consider the first term of (86) in Lemma 15. From (87) we have

$$\int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} \rho_{\theta_k}(x-y)\phi'(x)\phi'(y)Y_0^{(k)}(\mathrm{d}x)Y_s^{(k)}(\mathrm{d}y)$$

$$= \int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} \rho_{\theta_k}(x-y)\phi'(x)\phi'(y)(Z_s^{1,\theta_k} + Z_s^{2,\theta_k})(\mathrm{d}x)\cdot(Z_s^{1,\theta_k} + Z_s^{2,\theta_k})(\mathrm{d}y)$$

$$= \int_0^t \mathrm{d}s \iint \rho_k(x-y)\phi'(x)\phi'(y)Z_s^{1,\theta_k}(\mathrm{d}x)Z_s^{1,\theta_k}(\mathrm{d}y)$$

$$+ \int_0^t \mathrm{d}s \iint \rho_k(x-y)\phi'(x)\phi'(y)Z_s^{2,\theta_k}(\mathrm{d}x)Z_s^{1,\theta_k}(\mathrm{d}y)$$

$$+ \int_0^t \mathrm{d}s \iint \rho_k(x-y)\phi'(x)\phi'(y)Z_s^{1,\theta_k}(\mathrm{d}x)Z_s^{2,\theta_k}(\mathrm{d}y)$$

$$+ \int_0^t \mathrm{d}s \iint \rho_k(x-y)\phi'(x)\phi'(y)Z_s^{2,\theta_k}(\mathrm{d}x)Z_s^{2,\theta_k}(\mathrm{d}y)$$

$$=: I_1[90] + I_2[90] + I_3[90] + I_4[90]. \tag{90}$$

Since $Z_s^{2,\theta_k}(B) \to 0$, $\widetilde{\mathbb{P}}$-a.s. (as $k \to \infty$) for $B \in \mathcal{B}(\mathbb{R})$, it is not hard to see that

$$\lim_{k\to\infty} \mathbb{E}\{I_2[90]\} = 0, \quad \lim_{k\to\infty} \mathbb{E}\{I_3[90]\} = 0,$$

and

$$\lim_{k\to\infty} \mathbb{E}\{I_4[90]\} = 0. \tag{91}$$

Therefore, (86) of Lemma 15 can be reduced into a simpler form, and what really we have to show is now as follows.

Proposition 16. *For* $t > 0$

$$\lim_{k\to\infty} \mathbb{E}\left| \int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} \rho_{\theta_k}(x-y)\phi'(x)\phi'(y) Z_s^{1,\theta_k}(\mathrm{d}x) Z_s^{1,\theta_k}(\mathrm{d}y) \right.$$
$$\left. - \int_0^t \mathrm{d}s \iint_{\Delta} \rho(0)\phi'(x)\phi'(y) Y_s^{(0)}(\mathrm{d}x) Y_s^{(0)}(\mathrm{d}y) \right| = 0. \tag{92}$$

Now we are going to discuss the expression (89). Recall that $\{\xi_i(t)\}$ is a one-dimensional standard Feller branching diffusion. Hence it is obvious that for each $i \in \mathbb{N}$, $\xi_i^{\sigma_{\theta_k}}(t)$ converges a.s. to $\xi_i^{\sigma_0}(t)$ as $k \to \infty$ if $\sigma_\theta \to \sigma_0$ (as $\theta \to \infty$), see (A.2). According to Theorem 2.3 of Dawson et al. [4], under the assumption (A.1), for any $N \in \mathbb{N}$ given, if $a_i^{\theta_k} \to b_i$ $(k \to \infty)$ for each $i = 1, 2, \ldots, N$ (see also Section 7 (iv)), then the law of N-system of interacting Brownian motions $\{x_i(0, a_i^{\theta_k}, t)\}$ with initial state $\{a_i^{\theta_k}\}$ converges to that of N-system of coalescing Brownian motions $\{y_i(0, b_i, t)\}$ with speed $\rho(0)$ starting from $\{b_i\}$, where $\{x_i(t)\}$ is a unique solution of the stochastic equation of the type (45). Therefore, by virtue of the similar method described in the proof of Theorem 4.2 of [4], it is easy to show that $Z_t^{1,\theta_k} = \sum_i \xi_i^{\sigma_{\theta_k}}(t)\delta_{x_i(0,a_i^{\theta_k},t)}, t \geq 0$ converges in distribution to $\sum_i \xi_i^{\sigma_0}\delta_{y_i(0,b_i,t)}$. By Skorokhod's representation a selection of proper version provides with almost sure convergence of $\{\widetilde{x}_i(0, a_i^{\theta_k}, t)\}$ towards

$\{\widetilde{y}_i(0, b_i, t)\}$, whereby guaranteed is the $\widetilde{\mathbb{P}}$-a.s. convergence

$$\widetilde{Z}_t^{1, \theta_k} = \sum_i \xi_i^{\sigma_{\theta_k}}(t)\delta_{\widetilde{x}_i(0, a_i^{\theta_k}, t)} \to \sum_i \xi_i^{\sigma_0}(t)\delta_{\widetilde{y}_i(0, b_i, t)}. \tag{93}$$

While, by taking a.s. convergence of $Y_t^{(k)} = Z_t^{1, \theta_k}$ to $Y_t^{(0)}$ into account, we can deduce that

$$\widetilde{Y}_t^{(k)} = \widetilde{Z}_t^{1, \theta_k} \to \widetilde{Y}_t^{(0)}, \text{ a.s.} \tag{94}$$

on a version basis (if necessary), and also that the identity

$$\widetilde{Y}_t^{(0)} = \sum_i \xi_i^{\sigma_0}(t)\delta_{\widetilde{y}_i(0, b_i, t)} \tag{95}$$

holds. After that, let us write it simply as $Y_t^{(0)} = \sum_i \xi_i^{\sigma_0}(t)\delta_{y_i(0, b_i, t)}$ by notational abuse.

Lemma 17. *For any* $t > 0$, *we have*

$$\lim_{k\to\infty} \mathbb{E}\left| \int_0^t \mathrm{d}s \iint_{\mathbb{R}^2\setminus\Delta} \rho_{\theta_k}(x-y)\phi'(x)\phi'(y)Z_s^{1, \theta_k}(\mathrm{d}x)Z_s^{1, \theta_k}(\mathrm{d}y) \right| = 0. \tag{96}$$

Proof. For simplicity we put $F_k(x, y) = \rho_{\theta_k}(x-y)\phi'(x)\phi'(y)$. Noting that $\phi \in C^2(\mathbb{R})$ and $\phi' \in C^1(\mathbb{R})$, it follows that $F_k(x, y)$ vanishes as $k \to \infty$ for $(x, y) \in D := \mathbb{R}^2 \setminus \Delta$, because $\rho_{\theta_k}(x-y) \to 0$ $(k \to \infty)$ on D (where $x \neq y$) by assumption (A.1). From the aforementioned discussion (93)-(95), Z_t^{1, θ_k} converges a.s. to $Y_t^{(0)}$. Set $Z_t^k(\mathrm{d}x \times \mathrm{d}y) = Z_t^{1, \theta_k}(\mathrm{d}x)Z_t^{1, \theta_k}(\mathrm{d}y)$ and $Z_t^0(\mathrm{d}x \times \mathrm{d}y) = Y_t^{(0)}(\mathrm{d}x)Y_t^{(0)}(\mathrm{d}y)$. Then $Z_t^k \to Z_t^0$ a.s. Consider

$$\begin{aligned}
&\left| \iint_{\mathbb{R}^2\setminus\Delta} \rho_{\theta_k}(x-y)\phi'(x)\phi'(y)Z_s^{1, \theta_k}(\mathrm{d}x)Z_s^{1, \theta_k}(\mathrm{d}y) \right| \\
&= \left| \int_D F_k(x, y)Z_s^k(\mathrm{d}x \times \mathrm{d}y) \right| \\
&\leq \left| \int_D F_k(x, y)Z_s^k(\mathrm{d}x \times \mathrm{d}y) - \int_D F_k(x, y)Z_s^0(\mathrm{d}x \times \mathrm{d}y) \right| \\
&\quad + \left| \int_D F_k(x, y)Z_s^0(\mathrm{d}x \times \mathrm{d}y) \right| \\
&=: J_1[97] + J_2[97].
\end{aligned} \tag{97}$$

The ordinary Lebesgue type convergence theorem is applicable to $J_2[97]$, and

$$\lim_{k\to\infty}\int_D F_k \mathrm{d}Z_s^0 = \int_D \{\lim_{k\to\infty} F_k\}\mathrm{d}Z_s^0 = 0$$

holds for each $s > 0$ and ω. Hence it follows immediately that

$$\lim_{k\to\infty} \mathbb{E}\left\{\int_0^t \mathrm{d}s \int_D F_k(x,\ y) Z_s^0(\mathrm{d}x \times \mathrm{d}y)\right\} = 0. \tag{98}$$

In other words, for $\varepsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that if $k \geq N_0$,

$$J_2[97] = \left|\int_D F_k \mathrm{d}Z_s^0\right| < \frac{\varepsilon}{3}, \quad \widetilde{\mathbb{P}}\text{-a.s.} \tag{99}$$

uniformly in $s \in [0, t]$. Since $F_k \in C(D)$, F_k can be approximated by simple functions. In fact, there exists a family $\{\varphi_l^{(k)}\}_l$ of simple functions such that

$$\lim_{l\to\infty} \varphi_l^{(k)}(x,\ y) = F_k(x,\ y), \quad \text{a.e.-}(x,\ y) \tag{100}$$

with $\varphi_l^{(k)}(x, y) = \sum_{i=1}^{N(k,l)} a_i^{(k),l} 1_{A_i^{(k),l}}(x, y)$. For each l, for $\varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ such that if $k \geq N_1$, we have

$$\left|\int_D \varphi_l^{(k)} \mathrm{d}(Z_s^k - Z_s^0)\right| \leq \sum_i^{N(k,l)} |a_i^{(k),l}| \cdot \|Z_s^k - Z_s^0\| < \frac{\varepsilon}{3} \tag{101}$$

because of convergence $Z_t^k \to Z_t^0$ a.s. Similarly, from (100), for each k, for $\varepsilon > 0$, $\exists N_2 \in \mathbb{N}$ such that if $l \geq N_2$, we get

$$\left|\int_D (F_k - \varphi_l^{(k)})\mathrm{d}(Z_s^k - Z_s^0)\right| \leq \lim_N \sum_j |F_k - \varphi_l^{(k)}| \cdot \|Z_s^k - Z_s^0\| < \frac{\varepsilon}{3}. \tag{102}$$

Then combining (101) and (102), we readily obtain

$$\begin{aligned} J_1[97] &= \left|\int_D F_k(x,\ y)\mathrm{d}(Z_s^k - Z_s^0)\right| \\ &\leq \left|\int_D (F_k - \varphi_l^{(k)})\mathrm{d}(Z_s^k - Z_s^0)\right| + \left|\int_D \varphi_l^{(k)}\mathrm{d}(Z_s^k - Z_s^0)\right| < \frac{2\varepsilon}{3}, \text{ a.s.} \end{aligned} \tag{103}$$

uniformly in $s \in [0, t]$, for sufficiently large k, $l \geq N' = \max\{N_1, N_2\}$. In consequence, we can deduce from (97), (99) and (103) that for $\varepsilon > 0$, $\exists N^* = \max\{N, N'\}$ such that if $k \geq N^*$, then

$$\left| \iint_{\mathbb{R}^2 \setminus \Delta} \rho_{\theta_k}(x - y)\phi'(x)\phi'(y)Z_s^{1, \theta_k}(\mathrm{d}x)Z_s^{1, \theta_k}(\mathrm{d}y) \right| < \varepsilon \quad \text{a.s.}$$

uniformly in $s \in [0, t]$. That is why the Lebesgue type convergence theorem for the integral with respect to $\mathrm{d}s \times \mathrm{d}\widetilde{\mathbb{P}}$ over $[0, t] \times \widetilde{\Omega}$ leads to the assertion (96) immediately.

Dividing the integral region into two parts $\iint_{\mathbb{R}^2} = \iint_{\mathbb{R}^2 \setminus \Delta} + \iint_{\Delta}$, we have from (92) in Proposition 16

$$\begin{aligned}
&\mathbb{E}\left| \int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} \rho_{\theta_k}(x - y)\phi'(x)\phi'(y)Z_s^{1, \theta_k}(\mathrm{d}x)Z_s^{1, \theta_k}(\mathrm{d}y) \right. \\
&\quad \left. - \int_0^t \mathrm{d}s \iint_{\Delta} \rho(0)\phi'(x)\phi'(y)Y_s^{(0)}(\mathrm{d}x)Y_s^{(0)}(\mathrm{d}y) \right| \\
&\leq \mathbb{E}\left| \int_0^t \mathrm{d}s \iint_{\mathbb{R}^2 \setminus \Delta} \rho_{\theta_k}(x - y)\phi'(x)\phi'(y)Z_s^{1, \theta_k}(\mathrm{d}x)Z_s^{1, \theta_k}(\mathrm{d}y) \right| \\
&\quad + \mathbb{E}\left| \int_0^t \mathrm{d}s \iint_{\Delta} \rho(0)\phi'(x)\phi'(y)Z_s^{1, \theta_k}(\mathrm{d}x)Z_s^{1, \theta_k}(\mathrm{d}y) \right. \\
&\quad \left. - \int_0^t \mathrm{d}s \iint_{\Delta} \rho(0)\phi'(x)\phi'(y)Y_s^{(0)}(\mathrm{d}x)Y_s^{(0)}(\mathrm{d}y) \right| \\
&=: J_1[104] + J_2[104].
\end{aligned} \tag{104}$$

Then $\lim_{k \to \infty} J_1[104] = 0$ yields immediately from Lemma 17. Therefore, in order to finish the proof of key proposition we have only to show the following.

Lemma 18. *For* $t > 0$

$$\begin{aligned}
\lim_{k \to \infty} \mathbb{E}&\left| \int_0^t \mathrm{d}s \iint_{\Delta} \rho(0)\phi'(x)\phi'(y)Z_s^{1, \theta_k}(\mathrm{d}x)Z_s^{1, \theta_k}(\mathrm{d}y) \right. \\
&\quad \left. - \int_0^t \mathrm{d}s \iint_{\Delta} \rho(0)\phi'(x)\phi'(y)Y_s^{(0)}(\mathrm{d}x)Y_s^{(0)}(\mathrm{d}y) \right| = 0.
\end{aligned} \tag{105}$$

6.3. Approximation procedure

This subsection is devoted to an establishment of another explicit representation for the principal terms in question in connection with the proof of the key proposition. We shall first pick up the second term of (105) in Lemma 18 and begin with rewriting it into another useful form. When we try to do the same thing for the first term of (105), we encounter a difficulty on a sudden. To overcome it we need some approximation technique.

Let us consider now the second term of (105) in Lemma 18. We put

$$I_2 := \int_0^t \mathrm{d}s \iint_\Delta \rho(0)\phi'(x)\phi'(y)Y_s^{(0)}(\mathrm{d}x)Y_s^{(0)}(\mathrm{d}y), \tag{106}$$

where $\Delta = \{(x, x);\ x \in \mathbb{R}\}$.

Lemma 19. *For $t > 0$ we have the following identity*

$$I_2 = \int_0^t \mathrm{d}s \iint_\Delta \rho(0)\phi'(x)\phi'(y)Y_s^{(0)}(\mathrm{d}x)Y_s^{(0)}(\mathrm{d}y)$$

$$= \sum_{i,\,j=1}^{\infty}\int_{\tau_{ij}}^{t} \mathrm{d}s\xi_i^{\sigma_0}(s)\xi_j^{\sigma_0}(s)\rho(0)\phi'(y_i(0, b_i, s))\phi'(y_j(0, b_j, s)). \tag{107}$$

Proof. By using (89) in Subsection 6.2 we may rewrite (106) as

$$I_2 = \int_0^t \mathrm{d}s \iint_\Delta \rho(0)\phi'(x)\phi'(y)\left\{\sum_{i=1}^{\infty} \xi_i^{\sigma_0}(s)\delta_{y_i(0, b_i, s)}(\mathrm{d}x)\right\}$$

$$\left\{\sum_{j=1}^{\infty} \xi_j^{\sigma_0}(s)\delta_{y_j(0, b_j, s)}(\mathrm{d}y)\right\}$$

$$= \int_0^t \mathrm{d}s \iint_\Delta \rho(0)\phi'(x)\phi'(y)\left\{\sum_{i=1}^{\infty} \xi_i^{0}(s)\delta_{y_i(s)}(\mathrm{d}x)\right\}\left\{\sum_{j=1}^{\infty} \xi_j^{0}(s)\delta_{y_j(s)}(\mathrm{d}y)\right\}, \tag{108}$$

where we used some abbreviated notations for the time being, namely, $\xi_i^0(s) := \xi_i^{\sigma_0}(s)$ and $\delta_{y_i(s)} := \delta_{y_i(0, b_i, s)}$. By the characterization (24) of the coalescing Brownian motions described in Remark 3 in Subsection 2.2, we have

$$\langle y_i,\, y_j \rangle_t = \rho(0)\cdot(t - t \wedge \tau_{ij}) \tag{109}$$

for $1 \le i,\, j \le m$ and for each $m \in \mathbb{N}$ given, where τ_{ij} is a stopping time called the *first hitting time* between two coalescing Brownian particles $y_i(t)$ and $y_j(t)$, that is,

$$\tau_{ij} = \inf\{t > 0;\, y_i(t) = y_j(t)\}. \tag{110}$$

Paying attention to this coalescing property we may make use of (109) and (110) to obtain

$$\begin{aligned}
(108) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^t \mathrm{d}s\,\xi_i^0(s)\xi_j^0(s)\iint_{\Delta}\rho(0)\phi'(x)\phi'(y)\delta_{y_i(s)}(\mathrm{d}x)\delta_{y_j(s)}(\mathrm{d}y)\\
&= \sum_{i,j}\int_{\tau_{ij}}^t \mathrm{d}s\,\xi_i^0(s)\xi_j^0(s)\rho(0)\iint_{\Delta}\phi'(x)\phi'(y)\delta_{y_i(s)}(\mathrm{d}x)\delta_{y_j(s)}(\mathrm{d}y)\\
&= \sum_{i,j}\int_{\tau_{ij}}^t \mathrm{d}s\,\xi_i^0(s)\xi_j^0(s)\rho(0)\iint_{\mathbb{R}^2}\phi'(x)\phi'(y)\delta_{y_i(s)}(\mathrm{d}x)\delta_{y_j(s)}(\mathrm{d}y)\\
&= \sum_{i,j}\int_{\tau_{ij}} \mathrm{d}s\,\xi_i^0(s)\xi_j^0(s)\rho(0)\left(\int_{\mathbb{R}}\phi'(x)\delta_{y_i(s)}(\mathrm{d}x)\right)\cdot\left(\int_{\mathbb{R}}\phi'(y)\delta_{y_j(s)}(\mathrm{d}x)\right)\\
&= \sum_{i,j}\int_{\tau_{ij}}^t \mathrm{d}s\,\xi_i^0(s)\xi_j^0(s)\rho(0)\phi'(y_i(s))\phi'(y_j(s)).
\end{aligned} \tag{111}$$

This finishes the proof of Lemma 19.

Next let us consider the first term of (105) in Lemma 18. We put

$$I_1 := \int_0^t \mathrm{d}s\iint_{\Delta}\rho(0)\phi'(x)\phi'(y)Z_s^{1,\theta_k}(\mathrm{d}x)Z_s^{1,\theta_k}(\mathrm{d}y). \tag{112}$$

By the approach similar to discussions in the proof of Lemma 19, we are going to rewrite I_1 into another explicit representation. In so doing, the expression that we have to prove can be converted into a more tractable one. As a matter of fact, as we have seen in another explicit representation (107) of Lemma 19, the mathematical structure of the subject integral form I_2 is certainly changed into a much simpler one in the convergence topological sense, because purely atomic measure terms disappear and are replaced by functions superficially.

Lemma 20. *There exists a family* $\{D_l\}_l$ *of monotone proper sets such that* $D_l \supset \Delta$, $D_l \supset D_{l+1}$ $(\forall l)$ *and* $D_l \to \Delta$ *as* $l \to \infty$, *and for* $t > 0$, *we have the following identity*

$$I_1 = \int_0^t \mathrm{d}s \iint_\Delta \rho(0)\phi'(x)\phi'(y) Z_s^{1,\theta_k}(\mathrm{d}x) Z_s^{1,\theta_k}(\mathrm{d}y)$$

$$= \sum_{i,j=1}^{\infty} \int_0^t \mathrm{d}s \xi_i^{\sigma\theta_k}(s) \xi_j^{\sigma\theta_k}(s) \rho(0) \{\lim_{l\to\infty} 1_{D_l}(x_i^{\theta_k}(s), x_j^{\theta_k}(s))\}$$

$$\times \phi'(x_i^{\theta_k}(0, a_i^{\theta_k}, s)) \phi'(x_j^{\theta_k}(0, a_j^{\theta_k}, s)). \tag{113}$$

Proof. Recall the purely atomic representation of Z_t^{1,θ_k} (see (87) in Subsection 6.2):

$$Z_t^{1,\theta_k} = \sum_{i=1}^{\infty} \xi_i^{\sigma\theta_k}(t) \delta_{x^{\theta_k}(0,\, a_i^{\theta_k},\, t)}. \tag{114}$$

As the first step to the expression (113), let us consider, for instance, the following setting. For the set $\Delta = \{(x, x); x \in \mathbb{R}\}$, we first define a sequence of inclusively approximate subsets as

$$D_l = \left\{(x', y') \in \mathbb{R}^2; \inf_{(x', y') \in \mathbb{R}^2} \|(x, x) - (x', y')\| \le \frac{1}{l}\right\}, \quad \text{for } l \ge 1, \tag{115}$$

where $\|\cdot\|$ denotes the Euclidean norm in $E^2 = \mathbb{R}^2$. Then we observe that $D_l \supset \Delta$, $D_l \supset D_{l+1}$ and $D_l \to \Delta$ as $l \to \infty$ as set inclusion order. Clearly $1_{D_l}(x, y) \to 1_\Delta(x, y)$ a.e. (as $l \to \infty$) with respect to $\ell^2(\mathrm{d}x \times \mathrm{d}y)$ $= (\ell \otimes \ell)(\mathrm{d}x, \mathrm{d}y)$. So that, application of the Lebesgue type convergence theorem gives an observation that

$$\iint_{\Delta} \phi'(x)\phi'(y)\delta_a(\mathrm{d}x)\delta_b(\mathrm{d}y)$$

$$= \iint_{\mathbb{R}^2} 1_{\Delta}(x,\ y)\phi'(x)\phi'(y)\delta_a(\mathrm{d}x)\delta_b(\mathrm{d}y)$$

$$= \iint_{\mathbb{R}^2} \{\lim_{l\to\infty} 1_{D_l}(x,\ y)\}\phi'(x)\phi'(y)\delta_a(\mathrm{d}x)\delta_b(\mathrm{d}y)$$

$$= \lim_{l\to\infty}\iint_{\mathbb{R}^2} 1_{D_l}(x,\ y)\phi'(x)\phi'(y)\delta_a(\mathrm{d}x)\delta_b(\mathrm{d}y). \tag{116}$$

Note that the discussion of the integral in (116) does not depend on the choice of approximate sequence as far as it keeps monotone property and a kind of uniformity in convergence.

As the second step to this approximation procedure, we shall think of choosing an appropriate approximate sequence of more specific form. For each $l \geq 1$ given, we define $D_l^{[\varepsilon(M)]}$ as follows. Let $M > 0$ be a positive integer, i.e., $M \in \mathbb{Z}^+$, and let $Sq[M,\ l]$ denote a square with a length of side $(\sqrt{2}Ml)^{-1}$, called the *unit square*. The unit strip of $D_l^{[\varepsilon(M)]}$ along the X-axis in the Euclidean XY-plane $\mathbb{R}^2$ is given as follows. This unit strip just corresponds to a neighborhood of level 0 (meaning $x = 0$ with $(x,\ x) \in \Delta$). The horizontal strip form of neighborhood along the X-axis consists of $(8M + 2)$ pieces of squares in such a way that just $(4M + 1)$ pieces of unit squares $Sq[M,\ l]$ are laid exactly on the strip region

$$\left[-\frac{2\sqrt{2}}{l},\ \frac{\sqrt{2}(4M+1)}{2Ml}\right] \times \left[0,\ \frac{\sqrt{2}}{2Ml}\right]$$

and simultaneously other $(4M + 1)$ pieces of unit squares $Sq[M,\ l]$ are laid exactly on the other strip region

$$\left[-\frac{\sqrt{2}(4M+1)}{2Ml},\ \frac{2\sqrt{2}}{l}\right] \times \left[-\frac{\sqrt{2}}{2Ml},\ 0\right].$$

Likewise, for each level $x = 2k/Ml$ with $k \in \mathbb{Z}(x = 2k/Ml, (x, x) \in \Delta)$, we repeat the same procedure of laying the similar type of strip region of the same area (which is transferred from level 0 in a parallel way) with $(8M + 2)$ pieces of unit squares $Sq[M, l]$ similarly arranged. $D_l^{[\varepsilon(M)]}$ is an aggregate of squares $Sq[M, l]$, which is obtained by laying the whole region

$$\bigcup_{x=\frac{2k}{Ml}, k\in\mathbb{Z}} \{\text{level } x \text{ region}\}$$

with a collection of the bulk of squares for the unit strip. Clearly this construction of $D_l^{[\varepsilon(M)]}$ gives an observation that, for each l and M,

$$D_l \subset D_l^{[\varepsilon(M)]} \text{ and } 0 \leq \{\text{dist}(x, D_l^{[\varepsilon(M)]});\ x \in D_l\} \leq \frac{1}{2Ml} \text{ at most;}$$

moreover,

$$\lim_{M\to\infty} D_l^{[\varepsilon(M)]} = D_l \text{ and } \lim_{l\to\infty}(\lim_{M\to\infty} D_l^{[\varepsilon(M)]}) = \Delta$$

in a monotone way, where $\text{dist}(x, A)$ is a distance from a point x to a set A. On this account, we can approximate the indicator 1_Δ by using this sequence $\{D_l^{[\varepsilon(M)]}\}_{M,l}$. Furthermore we can verify with ease that there exists a collection of disjoint intervals $\{I_p\}_p = \{I_p^{M,l}\}_p$ in x-direction and a collection of disjoint intervals $\{J_p\}_p = \{J_p^{M,l}\}_p$ in y-direction such that

$$D_l^{[\varepsilon(M)]} = \bigcup_{i=1}^{\infty} I_p \times J_p \text{ with mutually disjoint } \{I_p \times J_p\}_p$$

and for $(x, y) \in D_l^{[\varepsilon(M)]}$, $\exists p \in \mathbb{N}$ such that $I_p \times J_p \ni (x, y)$. In addition, it follows that

$$1_\Delta(x, y) = \lim_{l\to\infty}(\lim_{M\to\infty} 1_{D_l^{[\varepsilon(M)]}}(x, y)), \tag{117}$$

$$1_{D_l^{[\varepsilon(M)]}}(x, y) = \sum_{p=1}^{\infty} 1_{\{I_p \times J_p\}}(x, y) \tag{118}$$

and

$$1_{\{I_p \times J_p\}}(x, y) = 1_{I_p}(x) \times 1_{J_p}(y). \tag{119}$$

After all, by the independence of the choice of approximating sequences, the diagonal method and renumbering procedure, we may rewrite it simply as $\{D_l\}$ anew, instead of $\{D_l^{[\varepsilon(M)]}\}_{M,l}$, from the begining.

Based upon the above-mentioned approximation argument, by employing the Fubini theorem and the Lebesgue theorem we can deduce with an easy computation that

$$\begin{aligned}
I_1 &= \int_0^t \mathrm{d}s \iint_\Delta \rho(0)\phi'(x)\phi'(y) Z_s^{1,\theta_k}(\mathrm{d}x) Z_s^{1,\theta_k}(\mathrm{d}y) \\
&= \int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} \rho(0) 1_\Delta(x, y)\phi'(x)\phi'(y) Z_s^{1,\theta_k}(\mathrm{d}x) Z_s^{1,\theta_k}(\mathrm{d}y) \\
&= \int_0^t \mathrm{d}s\rho(0) \iint_{\mathbb{R}^2} \{\lim_{l\to\infty} 1_{D_l}(x, y)\}\phi'(x)\phi'(y) Z_s^{1,\theta_k}(\mathrm{d}x) Z_s^{1,\theta_k}(\mathrm{d}y) \\
&= \int_0^t \mathrm{d}s\rho(0) \left\{\lim_{l\to\infty} \iint_\Delta \left\{\sum_{p=1}^\infty 1_{I_p^l \times J_p^l}(x, y)\right\}\phi'(x)\phi'(y) Z_s^{1,\theta_k}(\mathrm{d}x) Z_s^{1,\theta_k}(\mathrm{d}y)\right\} \\
&= \int_0^t \mathrm{d}s\rho(0) \lim_{l\to\infty} \sum_{p=1}^\infty \left(\int_{\mathbb{R}} 1_{I_p^l}(x)\phi'(x)\mathrm{d}Z_s^{1,\theta_k}\right)\left(\int_{\mathbb{R}} 1_{J_p^l}(y)\phi'(y)\mathrm{d}Z_s^{1,\theta_k}\right).
\end{aligned} \tag{120}$$

As is well known, the indicator can be approximated by C^∞-functions. Hence for each l, p, there exist $\{\psi_m\}_m = \{\psi_m^{l,p}\}_m \subset C_0^\infty(\mathbb{R})$ and $\{\eta_{m'}\}_{m'} = \{\eta_{m'}^{l,p}\}_{m'} \subset C_0^\infty(\mathbb{R})$ such that $\psi_m(x) \to 1_{I_p^l}(x)$ pointwise as $m \to \infty$, and $\eta_{m'}(y) \to 1_{J_p^l}(y)$ pointwise as $m' \to \infty$. Therefore, an easy computation with application of the Fubini theorem and the Lebesgue theorem together with the representation (114) yields to the assertion that

$$I_1$$

$$= \int_0^t \mathrm{d}s \iint_\Delta \rho(0)\phi'(x)\phi'(y) Z_s^{1,\theta_k}(\mathrm{d}x) Z_s^{1,\theta_k}(\mathrm{d}y)$$

$$= \sum_{i,j=1}^{\infty} \int_0^t \mathrm{d}s \xi_i^k(s)\xi_j^k(s) \iint_\Delta \rho(0)\phi'(x)\phi'(y)\delta_{x_i^k(s)}(\mathrm{d}x)\delta_{x_j^k(s)}(\mathrm{d}y)$$

$$= \sum_{i,j=1}^{\infty} \int_0^t \mathrm{d}s \xi_i^k(s)\xi_j^k(s)\rho(0) \iint_{\mathbb{R}^2} \{\lim_{l\to\infty} 1_{D_l}\}\phi'(x)\phi'(y)\delta_{x_i^k(s)}(\mathrm{d}x)\delta_{x_j^k(s)}(\mathrm{d}y)$$

$$= \sum_{i,j=1}^{\infty} \int_0^t \mathrm{d}s \xi_i^k \xi_j^k \rho(0) \lim_{l\to\infty} \sum_{p=1}^{\infty} \iint_{\mathbb{R}^2} 1_{I_p^l \times J_p^l}(x, y)\phi'(x)\phi'(y)\delta_{x_i^k(s)}(\mathrm{d}x)\delta_{x_j^k(s)}(\mathrm{d}y)$$

$$= \sum_{i,j} \int_0^t \mathrm{d}s \xi_i^k \xi_j^k \rho(0) \lim_l \sum_p \left\{ \int_{\mathbb{R}} 1_{I_p^l} \phi'(x)\delta_{x_i^k}(\mathrm{d}x) \right\} \left\{ \int_{\mathbb{R}} 1_{J_p^l} \phi'(y)\delta_{x_j^k}(\mathrm{d}y) \right\}, \quad (121)$$

where we used some notational abuse: $\xi_i^k(s) := \xi_i^{\sigma\theta_k}(s)$ and $x_i^k(s) := x^{\theta_k}(0, a_i^{\theta_k}, s)$ and so on. Here consider the integral $\int 1_{I_p^l} \phi'\delta_{x_i^k}(\mathrm{d}x)$. Under the same setting as in (121) and with the same mathematical tools, it is easy to see that

$$\int_{\mathbb{R}} 1_{I_p^l}(x)\phi'(x)\delta_{x_i^k(s)}(\mathrm{d}x)$$

$$= \lim_{m\to\infty} \int_{\mathbb{R}} \psi_m^{l,p}(x)\phi'(x)\delta_{x_i^k(s)}(\mathrm{d}x) = \lim_{m\to\infty} \psi_m(x_i^k(s))\phi'(x_i^k(s)). \quad (122)$$

For the integral $\int 1_{J_p^l} \phi'\delta_{x_j^k}(\mathrm{d}y)$, the same thing. In fact, we have

$$\int_{\mathbb{R}} 1_{J_p^l}(y)\phi'(y)\delta_{x_j^k(s)}(\mathrm{d}y)$$

$$= \lim_{m'\to\infty} \int_{\mathbb{R}} \eta_{m'}^{l,p}(y)\phi'(y)\delta_{x_j^k(s)}(\mathrm{d}y) = \lim_{m'\to\infty} \eta_{m'}(x_j^k(s))\phi'(x_j^k(s)). \quad (123)$$

Therefore we may combine (122) and (123) with (121) to obtain

$$I_1 = \sum_{i,j}\int_0^t ds\xi_i^k\xi_j^k\rho(0)\lim_l\sum_p\{\lim_m\psi_m(x_i^k(s))\phi'(x_i^k(s))$$

$$\cdot\lim_{m'}\eta_{m'}(x_j^k(s))\phi'(x_j^k(s))\}$$

$$= \sum_{i,j}\int_0^t ds\xi_i^k\xi_j^k\rho(0)\lim_l\sum_p\{1_{I_p^l}(x_i^k(s))\phi'(x_i^k(s))\cdot 1_{J_p^l}(x_j^k(s))\phi'(x_j^k(s))\}$$

$$= \sum_{i,j}\int_0^t ds\xi_i^k(s)\xi_j^k(s)\rho(0)\lim_l\left\{\sum_p 1_{\{I_p^l\times J_p^l\}}(x_i^k,\, x_j^k)\phi'(x_i^k(s))\phi'(x_j^k(s))\right\}$$

$$= \sum_{i,j=1}^{\infty}\int_0^t ds\xi_i^k(s)\xi_j^k(s)\rho(0)\{\lim_l 1_{D_l}(x_i^k(s),\, x_j^k(s))\}\phi'(x_i^k(s))\phi'(x_j^k(s)). \quad (124)$$

Finally we obtain the assertion (113). This completes the proof of Lemma 20.

6.4. Convergence in law and interchangeability argument

According to the discussion in Subsection 6.2, in order to prove the key proposition (Proposition 12), we have only to show Lemma 18. However, if we take the results obtained in Subsection 6.3, then Lemma 18 is even attributed to another assertion besides. In fact, from Lemmas 19 and 20 we recognize immediately that, to verify Lemma 18 it suffices to prove the following.

Lemma 21. *For* $t > 0$,

$$\lim_{k\to\infty}\mathbb{E}\Bigg|\sum_{i,j=1}^{\infty}\int_0^t ds\xi_i^{\sigma_{\theta_k}}(s)\xi_j^{\sigma_{\theta_k}}(s)\rho(0)\{\lim_{l\to\infty}1_{D_l}(x_i^{\theta_k}(s),\, x_j^{\theta_k}(s))\}$$

$$\times\phi'(x_i^{\theta_k}(0,\, a_i^{\theta_k},\, s))\phi'(x_j^{\theta_k}(0,\, a_j^{\theta_k},\, s))$$

$$-\sum_{i,j=1}^{\infty}\int_{\tau_{ij}}^t ds\xi_i^{\sigma_0}(s)\xi_j^{\sigma_0}(s)\rho(0)\phi'(y_i(0,\, b_i,\, s))\phi'(y_j(0,\, b_j,\, s))\Bigg| = 0. \quad (125)$$

On the other hand, for each ω and i, j we have

$$\left| \int_0^t \xi_i^k(s)\xi_j^k(s)\rho(0)(\lim_l 1_{D_l})\phi'(x_i^k)\phi'(x_j^k)\mathrm{d}s - \int_0^t \xi_i^0(s)\xi_j^0(s)\rho(0)\phi'(y_i)\phi'(y_j)\mathrm{d}s \right|$$

$$\leq \left| \int_0^t \xi_i^k(s)\xi_j^k(s)\rho(0)(\lim_l 1_{D_l})\phi'(x_i^k)\phi'(x_j^k)\mathrm{d}s - \int_0^t \xi_i^0(s)\xi_j^0(s)\rho(0)(\lim_l 1_{D_l})\phi'(x_i^k)\phi'(x_j^k)\mathrm{d}s \right|$$

$$+ \left| \int_0^t \xi_i^0(s)\xi_j^0(s)\rho(0)(\lim_l 1_{D_l})\phi'(x_i^k)\phi'(x_j^k)\mathrm{d}s - \int_0^t \xi_i^0(s)\xi_j^0(s)\rho(0)\phi'(y_i)\phi'(y_j)\mathrm{d}s \right|$$

$$=: J_1[126] + J_2[126]. \tag{126}$$

As to $J_1[126]$, we can get easily

$$J_1[126] = \left| \int_0^t \{\xi_i^k(s)\xi_j^k(s) - \xi_i^0(s)\xi_j^0(s)\}(\lim_l 1_{D_l})\phi'(x_i^k)\phi'(x_j^k)\mathrm{d}s \right|$$

$$\leq \int_0^t |\xi_i^k(s)\xi_j^k(s) - \xi_i^0(s)\xi_j^0(s)| \cdot |\lim_l 1_{D_l} \cdot \phi'(x_i^k)\phi'(x_j^k)|\mathrm{d}s. \tag{127}$$

Recall that $\{\xi_i(t)\}$ is a standard Feller branching diffusion, and also that $\xi_i^{\sigma_{\theta_k}}(s)$ converges a.s. as $k \to \infty$ to $\xi_i^{\sigma_0}(s)$ uniformly in $s \in [0, t]$. Hence it follows that

$$(127) \leq \int_0^t \sup_{0\leq s\leq t} |\xi_i^k(s)\xi_j^k(s) - \xi_i^0(s)\xi_j^0(s)| \cdot |\lim_l 1_{D_l} \cdot \phi'(x_i^k)\phi'(x_j^k)|\mathrm{d}s$$

$$\leq \|\xi_i^k\xi_j^k - \xi_i^0\xi_j^0\|_\infty \int_0^t |\lim_l 1_{D_l} \cdot \phi'(x_i^k)\phi'(x_j^k)|\mathrm{d}s$$

$$\leq \{\|\xi_i^k - \xi_i^0\|_\infty \|\xi_j^k\|_\infty + \|\xi_i^0\|_\infty \|\xi_j^k - \xi_j^0\|_\infty\} \int_0^t |\lim_l 1_{D_l} \cdot \phi'(x_i^k)\phi'(x_j^k)|\mathrm{d}s$$

$$\to 0 \quad \text{a.s.} \quad (\text{as } k \to \infty). \tag{128}$$

Consequently, from (126)-(128), to show (125) it suffices to prove the following lemma. For brevity's sake we put $x_i^k(s) := x_i^{\theta_k}(0, a_i^{\theta_k}, s)$.

Lemma 22. *For* $t > 0$

$$\lim_{k\to\infty} \mathbb{E}\left| \sum_{i,j=1}^{\infty} \int_0^t ds \xi_i^{\sigma_0}(s) \xi_j^{\sigma_0}(s) \rho(0) \{\lim_{l\to\infty} 1_{D_l}(x_i^k(s), x_j^k(s))\} \phi'(x_i^k(s)) \phi'(x_j^k(s)) \right.$$
$$\left. - \sum_{i,j=1}^{\infty} \int_{\tau_{ij}}^t ds \xi_i^{\sigma_0}(s) \xi_j^{\sigma_0}(s) \rho(0) \phi'(y_i(0, b_i, s)) \phi'(y_j(0, b_j, s)) \right| = 0. \quad (129)$$

Next we will think of a prestep to another reduction of (129) into a simpler one.

Lemma 23. *For any* $s \in [0, t]$ *we have*

$$\lim_{k\to\infty} \mathbb{E}\left| \sum_{i,j=1}^{\infty} \xi_i^0(s) \xi_j^0(s) \rho(0) 1_{D_l}(x_i^k(s), x_j^k(s)) \phi'(x_i^k(s)) \phi'(x_j^k(s)) \right.$$
$$\left. - \sum_{i,j=1}^{\infty} \xi_i^0(s) \xi_j^0(s) \rho(0) 1_{D_l}(y_i(s), y_j(s)) \phi'(y_i(s)) \phi'(y_j(s)) \right| = 0, \quad (130)$$

where we put $\xi_i^0 := \xi_i^{\sigma_0}$, $x_i^k(s) := x_i^{\sigma_0 \theta_k}(0, a_i^{\theta_k}, s)$ *and* $y_i(s) := y_i(0, b_i, s)$ *for brevity's sake.*

Proof. First of all we assume that

$$\sum_{i,j=1}^{\infty} \mathbb{E}|\xi_i^0(s) \xi_j^0(s) \rho(0) 1_{D_l}(x_i^k(s), x_j^k(s)) \phi'(x_i^k(s)) \phi'(x_j^k(s))| < \infty. \quad (131)$$

By the standard theory of integration this is nothing but the criterion for interchangeability between summation and integration. That is to say, under (131) we have the identity

$$\mathbb{E}\left\{ \sum_{i,j=1}^{\infty} \xi_i^0(s) \xi_j^0(s) \rho(0) 1_{D_l} \phi'(x_i^k(s)) \phi'(x_j^k(s)) \right\}$$
$$= \sum_{i,j=1}^{\infty} \mathbb{E}\{\xi_i^0(s) \xi_j^0(s) \rho(0) 1_{D_l} \phi'(x_i^k(s)) \phi'(x_j^k(s))\}. \quad (132)$$

This interchangeability proves to be trivially correct. In fact, the boundedness of the test functions yields to an easy estimate

$$\text{the left-hand side of} \quad (131) \le \rho(0) \| \phi' \|^2 \sum_{i, j=1}^{\infty} \mathbb{E}|\xi_i^0(s)\xi_j^0(s)|. \tag{133}$$

Since $\{\xi_i(t)\}_i$ are mutually independent Feller branching diffusions, for each $i \in \mathbb{N}$ the process $\xi_i(t)$ satisfies a stochastic equation

$$\xi_i(t) = \xi_i(0) + \int_0^t \sqrt{\beta \xi_i(s)}\, \mathrm{d}B_s. \tag{134}$$

Hence for some constant $K > 0$, the inequality

$$\begin{aligned}(133) &\le \rho(0) \| \phi' \| \sum_{i, j=1}^{\infty} \mathbb{E}\{\xi_i^0(s)\} \cdot \mathbb{E}\{\xi_j^0(s)\} \\ &\le K \left(\sum_{i=1}^{\infty} \mathbb{E}\{\xi_i^0(0)\} \right)^2 < \infty \end{aligned} \tag{135}$$

follows from (134) and the condition for the initial state of $\{\xi_i(t)\}$ imposed in Subsection 2.3, where we made use of independence and the property of $\mathbb{E}\{\text{martingale}\} = 0$.

Then the establishment of this interchangeable equality (132) enables us to approximate the infinite sum by a finite sum simply through the limiting procedure $\lim_{N \to \infty}$, namely,

$$\begin{aligned} &\mathbb{E}\left\{ \sum_{i, j=1}^{\infty} \xi_i^0(s)\xi_j^0(s)\rho(0)1_{D_l}\phi'(x_i^k(s))\phi'(x_j^k(s)) \right\} \\ &= \lim_{N \to \infty} \mathbb{E}\left\{ \sum_{i, j=1}^{N} \xi_i^0(s)\xi_j^0(s)\rho(0)1_{D_l}\phi'(x_i^k(s))\phi'(x_j^k(s)) \right\} \\ &=: \lim_{N \to \infty} \mathbb{E}\left\{ \sum_{i, j=1}^{N} \Phi_{ij}^s[x_i^k, x_j^k; \xi] \right\}, \end{aligned} \tag{136}$$

where for later citation we employed a simple sign in the last line that is equal to the previous line up.

By virtue of Theorem 2.3 of [4] we know that under the assumption (A.1), for each $N \in \mathbb{N}$, the law $\mathcal{L}(\{x_i^{\theta_k}(\cdot)\})$ of N-system of interacting Brownian motions (N-SIBM) with interaction parameter ρ_{θ_k} converges as $k \to \infty$ to the law $\mathcal{L}(\{y_i(\cdot)\})$ of N-system of coalescing Brownian motions (N-SCBM) with speed $\rho(0)$, if the initial state $\{a_i^{\theta_k}\}$ converges to the starting points $\{b_i\}$. Hence for each t fixed, for $\{x_i^{\theta_k}(t)\} \in \mathbb{R}^N$ and $\{y_i(t)\} \in \mathbb{R}^N$, the same convergence assertion for those laws as probability measures $(\in \mathcal{P}(\mathbb{R}^N))$ on $\mathbb{R}^N$ is also valid. Let us now consider random variables $X^k = (X_1^k, \ldots, X_N^k) \in \mathbb{R}^N$ and $Y^0 = (Y_1^0, \ldots, Y_N^0) \in \mathbb{R}^N$, and let $Q_N^k,\ Q_N^0 \in \mathcal{P}(\mathbb{R}^N)$ denote the laws of X^k and Y^0, respectively. Since we have a trivial identity

$$\mathbb{E}\{f(X)\} \equiv \int_\Omega f(X(\omega))\mathbb{P}(d\omega) = \int_{\mathbb{R}^n} f(x)(\mathbb{P} \circ X^{-1})(dx) \tag{137}$$

for any $\mathbb{R}^n$-valued random variable X and any bounded continuous function f on $\mathbb{R}^n$, we note that the weak convergence $Q_N^k \Rightarrow Q_N^0$ as measure $\in \mathcal{P}(\mathbb{R}^N)$ is equivalent to the convergence in law of X^k to Y^0. Hence, for any bounded continuous $H \in C(\mathbb{R}^N)$, the convergence $\mathbb{E}\{H(X^k)\} \to \mathbb{E}\{H(X^0)\}$ yields from the convergence

$$\int_{\mathbb{R}^N} H(x)Q_N^k(dx) \to \int_{\mathbb{R}^N} H(x)Q_N^0(dx). \tag{138}$$

Therefore, when for $H \in C(\mathbb{R}^N \times \mathbb{R}^N)$ we define

$$H(\{x_i^{\theta_k}(s);\ i = 1, \ldots, N\}, \{x_j^{\theta_k}(s);\ j = 1, \ldots, N\}) := \sum_{i, j=1}^{N} \Phi_{ij}^s[x_i^k, x_j^k; \xi] \tag{139}$$

by using the function Φ_{ij}^s in (136), then the above-mentioned convergence of N-SIBM to N-SCBM provides with the convergence: $\mathbb{E}[H(\{x_i^k(s)\},$

$\{x_j^k(s)\})] \to \mathbb{E}[H(\{y_i(s)\}, \{y_j(s)\})]$, namely,

$$\lim_{k\to\infty} \mathbb{E}\left\{\sum_{i,j=1}^{N} \Phi_{ij}^s[x_i^k, x_j^k; \xi]\right\} = \mathbb{E}\left\{\sum_{i,j=1}^{N} \Phi_{ij}^s[y_i, y_j; \xi]\right\}. \tag{140}$$

Notice that there happens no problem in the above convergence because we may approximate 1_{D_l} by a smooth function in the same way as done in Subsection 6.3. For the details, we shall leave it to the reader as an easy exercise. Therefore, paying attention to the identity

$$\lim_{k\to\infty}\left\{\lim_{N\to\infty} \mathbb{E}\left\{\sum_{i,j=1}^{N} \Phi_{ij}^s[x_i^k, x_j^k; \xi]\right\}\right\} = \lim_{N\to\infty} \mathbb{E}\left\{\sum_{i,j=1}^{N} \Phi_{ij}^s[y_i, y_j; \xi]\right\}, \tag{141}$$

we can easily verify from (136) and (140) that

$$\lim_{k} \mathbb{E}\left\{\sum_{i,j=1}^{\infty} \xi_i^0(s)\xi_j^0(s)\rho(0)1_{D_l}(x_i^k(s), x_j^k(s))\phi'(x_i^k(s))\phi'(x_j^k(s))\right\}$$

$$= \lim_{k}\left\{\lim_{N} \mathbb{E}\left\{\sum_{i,j=1}^{N} \Phi_{ij}^s[x_i^k, x_j^k; \xi]\right\}\right\}$$

$$= \lim_{N}\left\{\lim_{k} \mathbb{E}\left\{\sum_{i,j=1}^{N} \Phi_{ij}^s[x_i^k, x_j^k; \xi]\right\}\right\}$$

$$= \lim_{N} \mathbb{E}\left\{\sum_{i,j=1}^{N} \Phi_{ij}^s[y_i, y_j; \xi]\right\} = \mathbb{E}\left\{\sum_{i,j=1}^{N} \Phi_{ij}^s[y_i, y_j; \xi]\right\}. \tag{142}$$

The assertion (130) follows immediately from (142).

Lemma 24. *For $t > 0$ we have*

$$\lim_{k\to\infty} \mathbb{E}\Bigg| \sum_{i,j=1}^{\infty} \int_0^t \xi_i^0(s)\xi_j^0(s)\rho(0)\{\lim_{l} 1_{D_l}(x_i^k, x_j^k)\}\phi'(x_i^k(s))\phi'(x_j^k(s))ds$$

$$- \sum_{i,j=1}^{\infty} \int_0^t \xi_i^0(s)\xi_j^0(s)\rho(0)\{\lim_{l} 1_{D_l}(y_i, y_j)\}\phi'(y_i(s))\phi'(y_j(s))ds\Bigg| = 0. \tag{143}$$

Note that we used the same abbreviated notations as in Lemma 23.

Proof. We shall use the same notations as in the proof of Lemma 23. We have proved

$$\lim_k \mathbb{E}\left\{\sum_{i,j}^{\infty} \xi_i^0(s)\xi_j^0(s)\rho(0)1_{D_l}\phi'(x_i^k(s))\phi'(x_j^k(s))\right\}$$

$$= \lim_k \mathbb{E}\left\{\sum_{i,j}^{\infty} \Phi_{ij}^s[x_i^k,\ x_j^k;\ \xi]\right\} = \mathbb{E}\left\{\sum_{i,j}^{\infty} \Phi_{ij}^s[y_i,\ y_j;\ \xi]\right\}. \tag{144}$$

By employing the same approximation procedure in Subsection 6.3 we can replace 1_{D_l} by a smooth function $\lambda_m \equiv \lambda_m(x,\ y)$ for each $l \in \mathbb{N}$, namely, we may put $\lambda_m = \sum_p \psi_m^p(x)\eta_m^p(y)$ for instance. Let $\Phi_{ij}^s[x_i,\ x_j;\ \xi,\ \lambda_m]$ denote the function $\Phi_{ij}^s[x_i,\ x_j;\ \xi]$ with 1_{D_l} replaced by λ_m. Then the integral of $\Phi_{ij}^s[x_i,\ x_j;\ \xi,\ \lambda_m]$ with respect to ds over $[0,\ t]$ can be approximated by a finite sum

$$\sum_{q=1}^{N'} \Phi_{ij}^{\tau_q}[x_i,\ x_j;\ \xi,\ \lambda_m]\Delta s_q,$$

where a partition $\widetilde{\Delta} : s_0 = 0 < s_1 < \cdots < s_{N'} = t$ of the time interval $[0,\ t]$ is given, we put $\Delta s_q = s_q - s_{q-1}$ $(1 \le q \le N')$, τ_q is a point arbitrarily taken from the subinterval $[s_{q-1},\ s_q]$ for each q, and $|\widetilde{\Delta}| = \max_q \Delta s_q$, because the integrand is continuous in this occasion. Hence from (144) we have

$$\lim_k \mathbb{E}\left\{\sum_{i,j}\sum_{q}^{N'} \Phi_{ij}^{\tau_q}[x_i^k,\ x_j^k;\ \xi,\ \lambda_m]\Delta s_q\right\}$$

$$= \mathbb{E}\left\{\sum_{i,j}\sum_{q}^{N'} \Phi_{ij}^{\tau_q}[y_i,\ y_j;\ \xi,\ \lambda_m]\Delta s_q\right\}. \tag{145}$$

Moreover, by virtue of the condition (131), we can deduce at once with passage to the limit $N' \to \infty$ or equivalently $|\Delta| \to 0$ together with the reverse operation of approximation procedure for 1_{D_l} that

$$\lim_k \mathbb{E}\left\{\sum_{i,j}\int_0^t \Phi_{ij}^s[x_i^k,\ x_j^k;\ \xi]ds\right\} = \mathbb{E}\left\{\sum_{i,j}\int_0^t \Phi_{ij}^s[y_i,\ y_j;\ \xi]ds\right\}. \tag{146}$$

By virtue of (131) again, the Lebesgue type convergence theorem will take care of the interchange between integration and limit $l \to \infty$ in (146). On this account we finally establish

$$\lim_k \mathbb{E}\left\{\sum_{i,j}^{\infty}\int_0^t \xi_i^0(s)\xi_j^0(s)\rho(0)\{\lim_l 1_{D_l}(x_i^k,\ x_j^k)\}\phi'(x_i^k(s))\phi'(x_j^k(s))ds\right\}$$

$$= \mathbb{E}\left\{\sum_{i,j}^{\infty}\int_0^t \xi_i^0(s)\xi_j^0(s)\rho(0)\{\lim_l 1_{D_l}(y_i,\ y_j)\}\phi'(y_i(s))\phi'(y_j(s))ds\right\}. \tag{147}$$

The assertion (143) yields immediately from (147).

Proof of Lemma 22. By a triangular inequality we get

$$\left|\sum_{i,j}\int_0^t ds\xi_i^0(s)\xi_j^0(s)\rho(0)\{\lim_l 1_{D_l}(x_i^k,\ x_j^k)\}\phi'(x_i^k)\phi'(x_j^k)\right.$$

$$\left.-\sum_{i,j}\int_{\tau_{ij}}^t ds\xi_i^0(s)\xi_j^0(s)\rho(0)\phi'(y_i)\phi'(y_j)\right|$$

$$\leq \left|\sum_{i,j}\int_0^t ds\xi_i^0(s)\xi_j^0(s)\rho(0)\{\lim_l 1_{D_l}(x_i^k,\ x_j^k)\}\phi'(x_i^k)\phi'(x_j^k)\right.$$

$$\left.-\sum_{i,j}\int_0^t ds\xi_i^0(s)\xi_j^0(s)\rho(0)\{\lim_l 1_{D_l}(y_i,\ y_j)\}\phi'(y_i)\phi'(y_j)\right|$$

$$+\left|\sum_{i,j}\int_0^t ds\xi_i^0(s)\xi_j^0(s)\rho(0)\{\lim_l 1_{D_l}(y_i,\ y_j)\}\phi'(y_i)\phi'(y_j)\right.$$

$$\left.-\sum_{i,j}\int_{\tau_{ij}}^t ds\xi_i^0(s)\xi_j^0(s)\rho(0)\phi'(y_i)\phi'(y_j)\right|$$

$$=: J_1[148] + J_2[148]. \tag{148}$$

Since an application of (143) in Lemma 24 verifies $\lim_k \mathbb{E}\{J_1[148]\} = 0$, the assertion of Lemma 22 is attributed to showing that

$$\lim_{k\to\infty} \mathbb{E}\{J_2[148]\} = 0. \tag{149}$$

However, the term $J_2[148]$ is free from the parameter k-dependence, so that, what really we have to show is

$$\mathbb{E}\left| \sum_{i,j} \int_0^t \Xi_{ij}^s(\xi, \rho)\{\lim_l 1_{D_l}(y_i, y_j)\}\phi'(y_i(s))\phi'(y_j(s))\mathrm{d}s \right.$$

$$\left. - \sum_{i,j} \int_{\tau_{ij}}^t \Xi_{ij}^s(\xi, \rho)\phi'(y_i(s))\phi'(y_j(s))\mathrm{d}s \right| = 0, \tag{150}$$

where we put $\Xi_{ij}^s(\xi, \rho) := \xi_i^0(s)\xi_j^0(s)\rho(0)$ for simplicity. Recall the discussion on the approximation procedure in Subsection 6.3. Since we have $\lim_l 1_{D_l}(x, y) = 1_\Delta(x, y)$, $\ell(\mathrm{d}x) \otimes \ell(\mathrm{d}y)$-a.e. from (117), it is obvious that

$$\lim_{l\to\infty} 1_{D_l}(y_i(s), y_j(s)) = 1_\Delta(y_i(s), y_j(s)), \quad \mathbb{P}\text{-a.s.} \tag{151}$$

Therefore it follows that the first term in (150) becomes

$$\sum_{i,j=1}^{\infty} \int_0^t \Xi_{ij}^s(\xi, \rho)1_\Delta(y_i(s), y_j(s))\phi'(y_i(s))\phi'(y_j(s))\mathrm{d}s. \tag{152}$$

Clearly $1_\Delta(y_i, y_j)$ becomes 1 when $(y_i, y_j) \in \Delta$ and its value becomes null if $(y_i, y_j) \in \Delta^c$. In other words the case of $(y_i, y_j) \in \Delta^c$ has no contribution to the integral (152) in its value. Then what is the situation in the integral (152) if $(y_i, y_j) \in \Delta$? For each $i \in \mathbb{N}$, the coalescing Brownian path $y_i(0, b_i, t)$ starts at the point b_i, and generally speaking, $b_i \neq b_j$ for distinct pair (i, j), $i \neq j$. In addition, by the coalescing property of particles, during the time interval $[0, t]$, the phenomenon $(y_i, y_j) \in \Delta$ can be observed only in $[\tau_{ij}, t]$, where τ_{ij} is the first hitting time when $y_i(s) = y_j(s)$ for $s > 0$. Under this consideration the expression

(152) is reasonably rewritten as

$$\sum_{i,\,j=1}^{\infty}\int_{\tau_{ij}}^{t}\Xi_{ij}^{s}(\xi,\,\rho)1_{\Delta}(y_i(s),\,y_j(s))\phi'(y_i(s))\phi'(y_j(s))\mathrm{d}s, \tag{153}$$

implying that (150) proves to be true. This completes the proof of Lemma 22.

The discussion through Subsections 6.1-6.4 completes the proof of the key proposition, namely Proposition 12.

7. Concluding Remarks

(i) In the preliminaries of Section 2 we made a quick review of several superprocesses as prerequisite knowledge to read this paper. The content of Subsection 2.1 is partly due to Dawson et al. [3] and also partly due to the path wise construction of SDSM in Dawson-Li [2], and the content of Subsection 2.2 is chiefly due to Dawson et al. [4]. While the content of Subsection 2.3 is mainly based upon Dawson-Li [2].

(ii) In [26], Pitman-Yor constructed a certain class of one-dimensional immigration diffusion processes as sums of excursions selected by Poisson point processes. Similar types of constructions under infinite-dimensional setting can be found, for example, in Fu-Li [17], Li [22], Li-Shiga [25] and Shiga [27]. Especially, $M_a(\mathbb{R})$-valued immigration branching diffusions without spatial motion were constructed in Shiga [27], where the subject diffusion is obtained as the unique solution of stochastic equation similar to (29), but with the term $\delta_{x(s,a,t)}$ replaced by simple δ_a. In [17], Fu-Li succeeded in derivation of non-trivial extension of Shiga's result to a more general case of independent spatial motion, by considering the notion of measure-valued excursions.

(iii) In connection with the formalism underlying the proof of the main theorem, the martingale characterization for the rescaled processes $\{Z_t^{\theta};\, t \geq 0\}$ described in Proposition 5 of Subsection 3.3 is equivalent to the assertion that $\{Z_t^{\theta}\}$ is a solution of the $(\mathcal{I}_{\theta},\, \mathrm{Dom}(\mathcal{I}_{\theta}))$-martingale problem.

(iv) To assert the main theorem, some assumption on the convergence of initial data is needed: for instance, for the original IMS $Y_0 = \mu \in M_a(\mathbb{R})$, the rescaled process $Y_0^\theta = \mu_\theta$ converges to a certain purely atomic measure $Y_0^\infty = \mu_0$. In our model, related to the purely atomic representation, for the initial data $\mu = \sum_{i=1}^\infty \xi_i(0)\delta_{a_i}$, automatically the convergence: $\mu_\theta = \sum_{i=1}^\infty \xi_i(0)\delta_{a_i^\theta} \to \mu_0 = \sum_{i=1} \xi_i(0)\delta_{b_i}$ follows for some sequence $\{b_i\} \subset \mathbb{R}$. However, this is completely equivalent to the convergence: $\sum_i \xi_i^{\sigma_\theta}(0)\delta_{a_i^\theta} \to \sum_i \xi_i^{\sigma_0}(0)\delta_{b_i}$, whereby derived is $a_i^\theta \to b_i$ as $\theta \to \infty$ for each $i \in \mathbb{N}$. This is nothing but one of the conditions for the limiting result that the interacting Brownian motions with starting points $\{a_i^\theta\}$ converge in distribution sense to the coalescing Brownian motions with initial state $\{b_i\}$.

(v) In this paper we treat the case of vanishing deterministic immigration at infinity: $q_\theta \to 0$ as $\theta \to \infty$. However, if the deterministic immigration rate does not vanish, then another type of limit theorem is obtained. This new result is explained in the companion paper [12]. Moreover, we can consider the rescaled convergence problem for the case of function-type immigration rate. This challenging limit theorem for superprocesses with non-trivial immigration shall be discussed in the forthcoming paper [13].

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