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GENERALIZATION OF AN IMO INEQUALITY

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Abstract

We give an improvement of the inequality for four positive real numbers whose special case was given as a problem of the 42nd International Mathematical Olympiad.

The second problem of the 42nd International Mathematical Olympiad was the following interesting inequality.

Problem 2. Prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1$$

holds for all positive real numbers a, b and c.

The Problem 2 is the subject of several recent articles (see [1], [2], [3] and [4]). The official Web page http://imo.wolfram.com/ of the 42nd International Mathematical Olympiad (IMO) contains an elegant proof. There is also a mention of the second part of the following more general version of Problem 2.

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Generalization of Problem 2. Let a, b, c and λ be positive real numbers. Then

$$\frac{a}{\sqrt{a^2 + \lambda bc}} + \frac{b}{\sqrt{b^2 + \lambda ca}} + \frac{c}{\sqrt{c^2 + \lambda ab}}$$

is at least 1 for $0 < \lambda < 8$ and at least $\frac{3}{\sqrt{1 + \lambda}}$ for $\lambda \geq 8$.

Proof of the generalization of Problem 2. Let us introduce new variables x, y and z with formulas

$$x = \frac{a}{\sqrt{a^2 + \lambda bc}}, \quad y = \frac{b}{\sqrt{b^2 + \lambda ca}}, \quad z = \frac{c}{\sqrt{c^2 + \lambda ab}}.$$

The numbers x, y and z are positive real numbers strictly smaller than 1 and it holds

$$\frac{1}{x^2} - 1 = \frac{a^2 + \lambda bc}{a^2} - 1 = \frac{\lambda bc}{a^2}$$

and similarly for differences $\frac{1}{y^2} - 1$ and $\frac{1}{z^2} - 1$. By multiplying these three expressions and transferring the denominator on the other side, we get

$$(1 - x^2)(1 - y^2)(1 - z^2) = \lambda^3 (xyz)^2.$$

Now, we make the substitutions $x^2 \to x$ and $y^2 \to y$ and let $u = 1 - x$ and $v = 1 - y$. With these notations the above equality becomes

$$uv(1 - z^2) = \lambda^3 xyz^2.$$

When we solve this with respect to the variable z, we obtain $z = \frac{T}{S}$, where $S = uv + \lambda^3 xy$ and $T = \sqrt{uvS}$. Therefore, our problem is to find the minimum value of the function $f(x, y) = \sqrt{x} + \sqrt{y} + \frac{T}{S}$ for $0 < x < 1$ and $0 < y < 1$.

In order to get some idea about the function f, we can draw its graph for $\lambda = \frac{1}{2}$ and $\lambda = 15$.

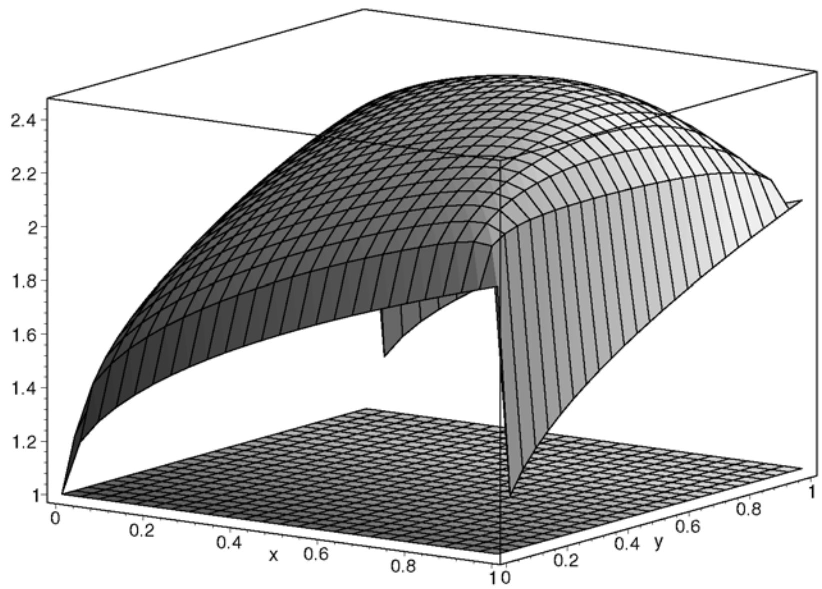

Figure 1. Graph of the function $f(x, y)$ for $\lambda = \frac{1}{2}$ and the plane $z = 1$ together.

The derivative of the function f by x is

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x}} - \frac{uy\lambda^3}{2ST}.$$

Since the situation is symmetrical in x and y, we automatically get

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}} - \frac{vx\lambda^3}{2ST}.$$

The derivatives of the second order are

$$J = \frac{\partial^2 f}{\partial x \partial y} = \frac{\lambda^3(\lambda^3 xy - 2uv)}{4S^2T},$$

$$H = \frac{\partial^2 f}{\partial^2 x} = \frac{vy\lambda^3(\lambda^3 y - 4S)}{4uS^2T} - \frac{1}{4\sqrt{x^3}},$$

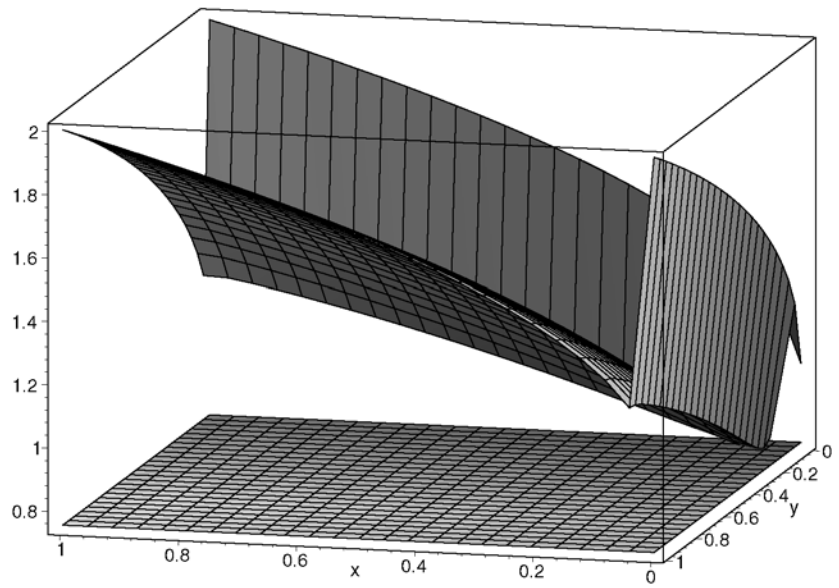

Figure 2. Graph of the function $f(x, y)$ for $\lambda = 15$ and the plane $z = \frac{3}{4} = \frac{3}{\sqrt{1+\lambda}}$ together.

and analogously

$$K = \frac{\partial^2 f}{\partial^2 y} = \frac{ux\lambda^3(\lambda^3 x - 4S)}{4vS^2 T} - \frac{1}{4\sqrt{y^3}}.$$

Let $\Delta = HK - J^2$.

In order to obtain the values of x and y when the (local) extremes are achieved we must solve the system $\left\{\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0\right\}$. This system is clearly equivalent to the following system of equations:

$$\{uS^3 - vxy^2\lambda^6 = 0, \ vS^3 - uyx^2\lambda^6 = 0\}.$$

Their left hand sides are

$$[u^2 - (\lambda^3 x^2 + u^2)y][(\lambda^3 y - v)^2 x^2 - (\lambda^3 y - v)(\lambda^3 y - 2v)x + v^2],$$

$$[v^2 - (\lambda^3 y^2 + v^2)x][(\lambda^3 x - u)^2 y^2 - (\lambda^3 x - u)(\lambda^3 x - 2u)y + u^2].$$

Let the square brackets in the above expressions be denoted by letters A, B, C and D. Let $\alpha = \lambda - 1$, $\beta = \lambda + 1$, $\gamma = \lambda - 2$, $\delta = \lambda^2 - \lambda + 1$ and $\varepsilon = \lambda^3$.

When $A = 0$ and $C = 0$, then from $A = 0$, we have $y = \dfrac{u^2}{u^2 + \varepsilon x^2}$. By substituting this back into C will give the quotient

$$\frac{\varepsilon ux(u - \lambda x)\left[(u - \lambda x)^2 + \lambda ux\right]}{(\varepsilon x^2 + u^2)^2}.$$

This can be zero only for $x = \dfrac{1}{\beta}$. Returning this back into the above expression for y, we obtain $y = \dfrac{1}{\beta}$.

When $A = 0$ and $D = 0$, then from $A = 0$, we have $y = \dfrac{u^2}{u^2 + \varepsilon x^2}$. By substituting this now into D will produce the quotient

$$\frac{\varepsilon u^2 x[1 - \delta x]\left[(2\beta\delta x + \gamma)^2 + 3\lambda^2\right]}{4\delta(\varepsilon x^2 + u^2)^2}.$$

This will be zero only for $x = \dfrac{1}{\delta}$. Returning this value back into the above expression for y, we get $y = \dfrac{\alpha^2}{\delta}$.

When $B = 0$ and $C = 0$, then from $B = 0$ for $y \geq \dfrac{4}{4 + \varepsilon}$, we get

$$x = \frac{\varepsilon y - 2v \pm U}{2M},$$

where $M = \varepsilon y - v$ and $U = \sqrt{\varepsilon y(\varepsilon y - 4v)}$. Substituting this into C will give the quotient

$$\frac{\pm UV + \varepsilon yW}{2M},$$

where $V = \beta\delta y^2 + 2v - 1$ and $W = \beta\delta y^2 - 1$. The numerator of this quotient will be zero if and only if $U^2V^2 - \varepsilon^2 y^2 W^2$ is equal to zero. It is easy to see that this difference is the quotient

$$\frac{\varepsilon y v^2(\delta y - 1)\left[(2\varepsilon y + 2y + \gamma)^2 + 3\lambda^2\right]}{\delta}.$$

It will be zero only for $y = \frac{1}{\delta}$. Notice that this value of y is always at least $\frac{4}{4+\varepsilon}$. Substituting this into the above expression for x, we get $x = \frac{\alpha^2}{\delta}$ and $x = \frac{1}{\delta}$.

Finally, in the last case, when $B = 0$ and $D = 0$, then the above value of x from $B = 0$ and for $y \geq \frac{4}{4+\varepsilon}$ is substituted into D we shall get the quotient

$$\frac{-\varepsilon\beta\delta v y[\pm UV + W]}{2M}$$

with $V = \beta\delta y - 2$ and $W = \beta\delta(\varepsilon + 2)y^2 - 4\beta\delta y + 2$. The numerator of this quotient will be zero if and only if $U^2V^2 - W^2$ is equal to zero. However, this difference is the quotient

$$\frac{v(\delta y - 1)\left[(2\varepsilon y + 2y + \gamma)^2 + 3\lambda^2\right]}{\delta}.$$

It will vanish only for $y = \frac{1}{\delta}$. Observe that this value of y is always at least $\frac{4}{4+\varepsilon}$. Returning this value back into the above expression for x will produce again $x = \frac{\alpha^2}{\delta}$ and $x = \frac{1}{\delta}$.

In this way we showed that the function f can have local extremes only in the pairs $p = \left(\frac{1}{\beta}, \frac{1}{\beta}\right)$, $q = \left(\frac{1}{\delta}, \frac{1}{\delta}\right)$, $r = \left(\frac{\alpha^2}{\delta}, \frac{1}{\delta}\right)$, and

$s = \left(\frac{1}{\delta}, \frac{\alpha^2}{\delta}\right)$. In order to simplify our notation, for a pair t and a function g of two variables, let g_t denote the value $g(t)$.

We shall use now the well-known theorem from the mathematical analysis for functions in two real variables which says that such a function f will have in the pair t for which $\left(\frac{\partial f}{\partial x}\right)_t = 0 = \left(\frac{\partial f}{\partial y}\right)_t$ the local minimum if $\Delta_t > 0$ and $H_t > 0$ (the alternative A_1), local maximum if $\Delta_t > 0$ and $H_t < 0$ (the alternative A_2), may or may not have in t the local extreme if $\Delta_t = 0$ (the alternative A_3), and definitely does not have the local extreme in the pair t if $\Delta_t < 0$ (the alternative A_4).

Since $\Delta_p = \frac{3\beta^3\gamma^2}{16\lambda^2}$, $H_p = \frac{\beta^{\frac{3}{2}}\gamma}{2\lambda}$ and $f_p = \frac{3}{\sqrt{\beta}}$, it follows that for $\lambda > 2$ the A_1 holds, for $\lambda < 2$ the A_2 is true, and for $\lambda = 2$ the alternative A_3 is valid.

Since Δ_q, H_q and f_q are respectively equal to $\frac{\delta^3(6-\lambda)(3\delta-5)}{16\alpha\lambda^2}$, $\frac{-\delta^{\frac{3}{2}}(\lambda^2+2\gamma)}{4\lambda\alpha}$ and $\frac{3-\lambda}{\sqrt{\delta}}$ for $0 < \lambda < 1$, it follows that in the range $0 < \lambda < 1$ the alternative A_2 holds. On the other hand, for $\lambda > 1$ we see that $\Delta_q = \frac{-3\beta\gamma^2\delta^3}{16\alpha\lambda^2}$ is clearly negative so that the A_4 holds.

Analogously, since for $\lambda > 0$ the terms Δ_r, H_r and f_r are respectively equal to $\frac{-3\delta^3\beta\gamma^2}{16\alpha^3\lambda^2}$, $\frac{-\gamma^2\delta^{\frac{3}{2}}}{4\lambda\alpha^3}$ and $\frac{\beta}{\sqrt{\delta}}$, it follows that for $0 < \lambda < 1$ the A_1 holds while for $\lambda > 1$ the A_4 is true.

At last, since Δ_s is equal to $\frac{-3\beta\gamma^2\delta^3}{16\alpha^3\lambda^2}$ for $\lambda > 1$ we conclude that in this range it is negative so that A_4 holds. On the other hand, for $0 < \lambda < 1$, we have $\Delta_s = \frac{-(3+5\lambda)\gamma^2\delta^3}{16\alpha^3\lambda^2}$, $H_s = \frac{-(\lambda^2-\beta)\delta^{\frac{3}{2}}}{\alpha\lambda}$ and $f_s = \frac{-\alpha}{\sqrt{\delta}}$ so that the alternative A_2 is true.

In this way we showed that the function f can have the least value $\frac{3}{\sqrt{\beta}}$ in the pair p for $\lambda > 2$ and the least value $\frac{\beta}{\sqrt{\delta}}$ in the pair r for $0 < \lambda < 1$. Since $f(0, 0) = 1$, the number $\frac{\beta}{\sqrt{\delta}}$ is always greater than 1 for $0 < \lambda < 1$, and the number $\frac{3}{\sqrt{\beta}}$ is less or equal to the number 1 only for $\lambda \geq 8$ we conclude that $f \geq 1$ for $0 < \lambda < 8$ and that $f \geq \frac{3}{\sqrt{\beta}}$ for $\lambda \geq 8$. This concludes the proof.

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