Far East J. Math. Sci. (FJMS) **22**(2) (2006), 175-184
This paper is available online at http://www.pphmj.com

α-FAVORABILITY OF $C(X)$ WITH A SET OPEN TOPOLOGY *II*

A. BOUCHAIR and S. KELAIAIA

(Received January 2, 2006)

Submitted by K. K. Azad

Abstract

As in our papers [Révue Sciences & Technologie, Univ. Mentouri Constantine A-No. 20 (2003), 17-20] and [Far East J. Math. Sci. (FJMS) 18(3) (2005), 305-312], we consider $C(X)$ the set of all continuous real-valued functions on a topological space X, when it is equipped with a set open topology defined with the help of a particular non-empty family γ of compact subsets of X. We give a necessary condition for $C(X)$ to be weakly-α-favorable and extend a result obtained of McCoy and Ntantu in the framework of the compact-open topology [Topology Appl. 2 (1986), 191-206]. Obtaining this result is essentially due to a hypothesis of admissibility of the family γ, introduced in [Topology Proc. 10 (1985), 329-345].

1. Introduction

Throughout this paper X denotes a completely regular Hausdorff space, $C(X)$ denotes the set of all continuous real-valued functions on X and γ denotes a non-empty family of compact subsets of X. The symbols $\mathbb{R}$ and $\mathbb{N}$ denote respectively the real numbers and the positive integers. $C(X)$ is equipped with a topology which has as subbase the collection:

2000 Mathematics Subject Classification: 54C35.

Keywords and phrases: set-open topology, admissible family, weakly-α-favorable, σ-compact.

$\{[A, V] : A \in \gamma, V \in \Im(\mathbb{R})\}$, where $[A, V] = \{f \in C(X) : f(A) \subset V\}$ and $\Im(\mathbb{R})$ denotes the collection of all open bounded intervals of $\mathbb{R}$. $C(X)$ equipped with this topology is denoted by $C_\gamma(X)$ and it is said to have a set-open topology [6].

A remark that we will need later is that if Y is a subset of X, then the set $C(Y)$ may be endowed with the set-open topology which has as subbase the collection $\{[A \cap Y, V] : A \in \gamma, V \in \Im(\mathbb{R})\}$, and it will be denoted by $C_{\gamma \cap Y}(Y)$.

Recall that the completeness properties of a topological space range from complete metrizability to the Baire space property. Most of the completeness properties of $C(X)$ when it is equipped with the compact-open topology were studied in [7].

In the beginning we give a definition and some results needed in the sequel.

A non-empty family γ of compact subsets of X is said to be *admissible* if for every $A \in \gamma$ and for every finite sequence $\mathcal{U}$ of open subsets of X which covers A, there exists a finite sequence $\mathcal{A}$ of members of γ which covers A such that for every $B \in \mathcal{A}$ there exists $U \in \mathcal{U}$ such that $B \subset U$.

A non-empty family γ of compact subsets of X is said to be *weakly-admissible* if for every $A \in \gamma$ and for every finite sequence $\mathcal{U}$ of open subsets of X disjoint pairwise which covers A, there exists a finite sequence $\mathcal{A}$ of members of γ which covers A such that for every $B \in \mathcal{A}$ there exists $U \in \mathcal{U}$ such that $B \subset U$.

If γ and β are families of compact subsets of X, then we say that β approximates γ or γ can be approximated by β provided that for every element A of γ and for every open U of X such that $A \subset U$, there exists $B_1, \ldots, B_n \in \beta$ with $A \subset \bigcup_{i=1}^{n} B_i \subset U$.

Remark that if Y is a closed subset of X and if γ is an admissible family of X, then so is the case for the family $\{A \cap Y : A \in \gamma\}$ in Y endowed with its subspace topology.

Remark also that when γ is admissible, then the set-open topology on $C(X)$ defined with the help of the family γ is the same with the topology of the uniform convergence on the elements of γ defined on $C(X)$. Recall that in the later topology the basic neighbourhoods of any point f in $C(X)$ have the form $\langle f, A, \varepsilon \rangle = \{g \in C_\gamma(X) : |g(x) - f(x)| \prec \varepsilon, \forall x \in A\}$, where $A \in \gamma$, and ε is a positive real number.

A useful concept for studying topological properties of function spaces is the *induced function*. Every continuous function $\Phi : Y \to X$ induces a function $\Phi^* : C(X) \to C(Y)$ defined by $\Phi^*(f) = f \circ \Phi$, for each $f \in C(X)$. Most of the properties of Φ^* were studied by McCoy and Ntantu in [6], and Kelaiaia in [5].

Theorem 1 [6]**.** *Let* $\Phi : Y \to X$ *be a continuous function, and let* γ *and* β *be families of compact subsets of* X *and* Y, *respectively. Then* $\Phi^* : C_\gamma(X) \to C_\beta(Y)$ *is continuous if and only if* $\Phi(\beta)$ *is approximated by* γ.

Theorem 2 [5]**.** *Let* X *be a topological space,* Y *be a sub-space of* X, β *be an admissible family of compact subsets of* Y *and* γ *be a weakly-admissible family of compact subsets of* X. *Let* $i : Y \to X$ *be inclusion mapping. Then* $i^* : C_\gamma(X) \to C_\beta(Y)$ *is open onto its image if and only if* β *approximates* $\gamma \cap \overline{Y}$.

2. The Banach-Mazur and the R. A. McCoy Games

Let X be a topological space. Then the Banach-Mazur game $\Gamma_{BM}(X)$ on X is played as follows: Two players *I* and *II* take turns choosing open subsets U_n and V_n, respectively.

The player *I* first chooses an open subset U_1.

When the player I has chosen the open subset U_n, the player II chooses an open subset V_n included in U_n.

When the player II has chosen the open subset V_n, the player I chooses an open subset U_{n+1} included in V_n.

The player II wins the game if $\bigcap\{U_n : n \in \mathbb{N}\} \neq \emptyset$.

A strategy in the game $\Gamma_{BM}(X)$ is a function $\sigma : S(\tau^*) \to \tau^*$, where τ^* denotes the collection of all non-empty open subsets of X and $S(\tau^*)$ denotes the collection of the finite sequences of members of τ^*, such that $\sigma(U_1, U_2, ..., U_n) \subset U_n$ for all $n \geq 1$. A strategy σ is said to be *winning* for the player II in $\Gamma_{BM}(X)$ if for each play $U_1, V_1, U_2, V_2, \ldots$ such that $U_1 \supset V_1 = \sigma(U_1) \supset U_2 \supset V_2 = \sigma(U_1, U_2) \supset \cdots$ we have $\bigcap\{U_n : n \in \mathbb{N}\} \neq \emptyset$.

Usually strategies depend on all previous plays, a strategy σ which depends only on the previous play is a strategy such that $\sigma(U_1, U_2, ..., U_n) = \sigma(U_n)$.

A topological space X is called *weakly* α-*favorable space* if the player II has a winning strategy in $\Gamma_{BM}(X)$ which depends on all previous games. The space X is called α-*favorable* if the player II has a winning strategy in $\Gamma_{BM}(X)$ which depends only on the previous play. We say also that X is *weakly* β-*defavorable* (resp. β-*defavorable*) if player I has no winning strategy in $\Gamma_{BM}(X)$.

A topological space X is called a *Baire space* if the intersection of any sequence of dense open subsets of X is dense in X. In his paper Oxtoby [8] proved that a topological space X is a Baire space if and only if it is weakly β-defavorable.

Now let us introduce two games due to McCoy and Ntantu [7] played on a topological space X using a non-empty family γ of compact subsets of X including the empty set. These two games which are denoted by $\Gamma_\gamma^1(X)$ and $\Gamma_\gamma^2(X)$, will allow us to answer when $C_\gamma(X)$ is weakly α-favorable.

The game $\Gamma_\gamma^1(X)$ is played as follows: Two players *I* and *II* take turns choosing elements of the family γ. On the nth play, player *I* chooses $A_n \in \gamma$ and player *II* chooses $B_n \in \gamma$. The only restriction is on player *I*, who must choose A_n disjoint from $B_1 \cup \cdots \cup B_{n-1}$ for $n \succ 1$. Player *II* wins if $\{A_n : n \in \mathbb{N}\}$ is a discrete family in *X*; otherwise player *I* wins.

In the second game $\Gamma_\gamma^2(X)$, players *I* and *II* also take turns choosing elements of the family γ, this time with no restriction. On the nth play, player *I* chooses $A_n \in \gamma$ and player *II* chooses $B_n \in \gamma$. Let $R_1 = A_1$, and for each $n \succ 1$, let $R_n = A_n \setminus (B_1 \cup \cdots \cup B_{n-1})$. Player *II* wins if $\{R_n : n \in \mathbb{N}\}$ is a discrete family in *X*; otherwise player *I* wins.

The winning strategies for the games $\Gamma_\gamma^1(X)$ and $\Gamma_\gamma^2(X)$ are defined in the same way as for the Banach-Mazur game.

Obviously, if player *II* has a winning strategy in $\Gamma_\gamma^2(X)$, then player *II* has also a winning strategy in $\Gamma_\gamma^1(X)$.

Theorem 3 [2]**.** *Let X be a normal space*, γ *be an admissible family including the empty set and which is stable by finite unions and verifying the property that any point of X admits a member of* γ *as a neighborhood. Then the following are equivalent*:

1. $C_\gamma(X)$ *is weakly* α*-favorable.*

2. *Player II has a winning strategy in* $\Gamma_\gamma^1(X)$.

3. *Player II has a winning strategy in* $\Gamma_\gamma^2(X)$.

3. More on Weak-α–favorability and σ–compacity

A space is a σ-*compact* space provided that it is a countable union of compact subspaces. The first main result relates to the σ-*compact* property with the game $\Gamma_\gamma^2(X)$.

Proposition 4. *Let X be a topological space, γ be a family of compact subsets of X contains the empty set. Let γ_0 be a countably subfamily of γ such that $X = \overline{\bigcup\{K : K \in \gamma_0\}}$. If player II has a winning strategy in $\Gamma_\gamma^2(X)$, then X is a countably union of members of γ (then, σ-compact).*

Before giving the proof of this proposition we need to recall a lemma and a definition.

Lemma 5 [7]**.** *Let X be a topological space and γ be a family of compact subsets of X. If player II has a winning strategy σ in $\Gamma_\gamma^2(X)$ and (C_n) is a sequence of elements of γ, then the strategy σ' defined by $\sigma'(A_1, \ldots, A_n) = \sigma(A_1, \ldots, A_n) \cup C_n$ is also a winning strategy for player II in $\Gamma_\gamma^2(X)$.*

Definition 6. A strategy σ for player II in $\Gamma_\gamma^2(X)$ is called a *spanning one* if for each positive integer n, we have $A_n \subset \sigma(A_1, \ldots, A_n)$.

Proof. Let $X = \overline{\bigcup\{K_i : i \in \mathbb{N}^*\}}$, $K_i \in \gamma_0$ and σ be a winning strategy for player II in $\Gamma_\gamma^2(X)$. Then the strategy σ' as defined in Lemma 5 by taking $C_n = K_n$ for all n, is also a winning spanning strategy for player II in $\Gamma_\gamma^2(X)$. Let $B_0 = \phi$, as the player II has a winning strategy σ' in the game $\Gamma_\gamma^2(X)$. Then whatever is the play of the player I in this game the player II will win using this strategy. So let the player I use the same strategy σ' and begin the play by choosing $A_1 = \sigma'(B_0)$. The player II will answer by $B_1 = \sigma'(A_1)$. For the nth move, define A_n and B_n inductively by:

$$A_n = \sigma'(B_0, B_1, \ldots, B_{n-1}) \text{ and } B_n = \sigma'(A_1, A_2, \ldots, A_n).$$

For each n, we put

$$R_n = A_n \setminus (B_0 \cup \cdots \cup B_{n-1}) \text{ and } S_n = B_n \setminus (A_1 \cup \cdots \cup A_n).$$

Then both $\{R_n : n \geq 1\}$ and $\{S_n : n \geq 1\}$ are discrete families in X, because σ' is a winning strategy in $\Gamma_\gamma^2(X)$. We prove that

$$(\bigcup\{R_n : n \geq 1\}) \cup (\bigcup\{S_n : n \geq 1\}) = (\bigcup\{A_n : n \geq 1\}) \cup (\bigcup\{B_n : n \geq 1\}).$$

Obviously, we have

$$(\bigcup\{R_n : n \geq 1\}) \cup (\bigcup\{S_n : n \geq 1\}) \subset (\bigcup\{A_n : n \geq 1\}) \cup (\bigcup\{B_n : n \geq 1\}).$$

For the converse case, let $x \in (\bigcup\{A_n : n \geq 1\}) \cup (\bigcup\{B_n : n \geq 1\})$ and let n_1 respectively n_2 be the smallest integers such that $x \in A_{n_1}$ and $x \in B_{n_2}$ with $n_1 = \infty$ if $x \notin \bigcup\{A_n : n \geq 1\}$ and $n_2 = \infty$ if $x \notin \bigcup\{B_n : n \geq 1\}$. If $n_1 \leq n_2$, then $x \in R_{n_1}$ and if $n_1 \succ n_2$, then we have $x \in S_{n_2}$.

Now the set $D = (\bigcup\{A_n : n \geq 1\}) \cup (\bigcup\{B_n : n \geq 1\})$ is a dense σ-compact subset of *X*. To see that $D = X$. Let $x \in X$. Then there exists a neighborhood *U* of *x* which intersects at most one R_n and at most one S_n. Since *D* is dense in *X*, there exist n_0 and n_1 such that $x \in \overline{R_{n_0}}$ or $x \in \overline{S_{n_1}}$. This means that $x \in A_{n_0}$ or $x \in B_{n_1}$ and then $D = X$. So that *X* is σ-compact.

Before giving the next main result, we need to give the following lemmas.

Lemma 7 [5]. *Let X be a topological space, K be compact subset of X, F be closed subset of X and let* $f : X \to \mathbb{R}$ *be a continuous function such that* $f(K \cap F) \subset V$, *where V is a bounded open interval. Then there exists a continuous function* $f_1 : X \to \mathbb{R}$ *such that* $f_1|_F = f|_F$ *and* $f_1(K) \subset V$.

Lemma 8. *Let X be a topological space, Y be a sub-space of X,* γ *be a family of compact subsets of X with* $B \cap Y = B \cap \overline{Y}$ *for each* $B \in \gamma$ *and let* $g \in C(Y)$ *be a function extendable to a continuous function over X. Let* $B_1, \ldots, B_n \in \gamma$ *and* $V_1, \ldots, V_n$ *be a bounded open intervals in* $\mathbb{R}$ *such that* $g(B_i \cap \overline{Y}) \subset V_i$ *for each* $i = 1, \ldots, n$. *Then there exists* $g' \in C(X)$ *an extension of g such that* $g'(B_i) \subset V_i$ *for each* $i = 1, \ldots, n$.

Proof. Using the reasoning by recurrence. Let $g \in C(Y)$, $B \in \gamma$ and V be a bounded open interval such that $g(B \cap \overline{Y}) \subset V$. Then by Lemma 7 with $F = \overline{Y}$ and $K = B$, we obtain a function $g' \in C(X)$ prolonge g such that $g'(B) \subset V$.

Suppose that the property is true up to n. Let $B_1, \ldots, B_{n+1} \in \gamma$ and $V_1, \ldots, V_{n+1}$ be a bounded open intervals in $\mathbb{R}$ such that $g(B_i \cap \overline{Y}) \subset V_i$ for each $i = 1, \ldots, n+1$. By hypothesis we have $g((B_i \cap B_{n+1}) \cap \overline{Y}) \subset V_i \cap V_{n+1}$ for each $i = 1, \ldots, n$. The family $\{B_1 \cap B_{n+1}, \ldots, B_n \cap B_{n+1}\}$ verifies $(B_i \cap B_{n+1}) \cap Y = (B_i \cap B_{n+1}) \cap \overline{Y}$ for each $i = 1, \ldots, n$. Then there exists $g_1' \in C(X)$ prolonge g such that $g_1'(B_i \cap B_{n+1}) \subset V_i \cap V_{n+1}$, for each $i = 1, \ldots, n$. We put $Y_1 = Y \cup \left(\bigcup_{i=1}^{n} (B_i \cap B_{n+1})\right)$. It is easy to see that $B_i \cap Y_1 = B_i \cap \overline{Y_1}$ for each $i = 1, \ldots, n+1$, and we have $g_1'(B_i \cap \overline{Y_1}) \subset V_i$ for each $i = 1, \ldots, n+1$. By applying Lemma 7 at the restriction $g_1' |_{\overline{Y_1}}$ with $F = \overline{Y_1}$ and $K = B_{n+1}$, we obtain a function $g_2' \in C(X)$ prolonge $g_1' |_{\overline{Y_1}}$ such that $g_2'(B_{n+1}) \subset V_{n+1}$. Remark also that $g_2'(B_i \cap B_{n+1}) \subset V_i$ for each $i = 1, \ldots, n$.

We put $Y_2 = Y_1 \cup B_{n+1}$. It is easy to verify that $B_i \cap Y_2 = B_i \cap \overline{Y_2}$ for each $i = 1, \ldots, n$ and that $g_2'(B_i \cap \overline{Y_2}) \subset V_i$, for each $i = 1, \ldots, n$. Using the hypothesis of recurrence at $B_i \cap Y_2 = B_i \cap \overline{Y_2}$ for each $i = 1, \ldots, n$, there exists $g_3' \in C(X)$ prolonge $g_2' |_{\overline{Y_2}}$ such that $g_3'(B_i) \subset V_i$, for each $i = 1, \ldots, n$ and we have $g_3'(B_{n+1}) = g_2'(B_{n+1}) \subset V_{n+1}$. We take $g' = g_3'$.

Using the above result we obtain a necessary condition for $C_\gamma(X)$ to be weakly-α-favorable.

Proposition 9. *Let X be a normal space, γ be an admissible family of compact subsets of X contains the empty set and stable by finite union* *such that any point of X admits a member of* γ *as a neighborhood. Let* $Y = \overline{\bigcup\{K : K \in \gamma_0\}}$, *where* γ_0 *is a countably subfamily of* γ. *If* $C_\gamma(X)$ *is weakly-*α*-favorable and* γ *approximates* $\gamma \cap Y$, *then Y is* σ*-compact.*

Proof. The inclusion map $i : Y \to X$ induces a function $i^* : C_\gamma(X) \to C_\beta(Y)$, where $\beta = \gamma \cap Y$. By Theorem 1, i^* is continuous. By Theorem 2, i^* is an open function onto its image. To see that $i^*(C_\gamma(X))$ is dense in $C_\beta(Y)$, let $\bigcap_{i=1}^{n}[B_i, V_i]$ be a basic open subset of $C_\beta(Y)$, where $B_i \in \beta$, for each $i = 1, ..., n$ and $V_1, ..., V_n$ are bounded open intervals in $\mathbb{R}$ and let $g \in \bigcap_{i=1}^{n}[B_i, V_i]$. For each $i = 1, ..., n$ there exists $A_i \in \gamma$ such that $B_i = A_i \cap Y$. Since Y is closed in X normal, using Lemma 8, g has an extension $g_1 \in C_\gamma(X)$ with $g_1(A_i) \subset V_i$, for each $i = 1, ..., n$. It is easy to see that

$$g \circ i \in i^*(C_\gamma(X)) \cap \left(\bigcap_{i=1}^{n}[B_i, V_i]\right).$$

Then $C_\beta(Y)$ is a weakly-α-favorable space as continuous image of $C_\gamma(X)$. The player *II* has a winning strategy in the game $\Gamma_\beta^2(Y)$ by Theorem 3. Therefore Y is σ-compact by Proposition 4.

Acknowledgement

The authors acknowledge the support of M.E.R.S. (Project No. B 2301/04/04).

References

[1] A. Bouchair and S. Kelaiaia, Applications de jeux topologiques à l'étude de $C(X)$ muni d'une topologie set-open, Révue Sciences & Technologie, Univ. Mentouri Constantine A-No. 20 (2003), 17-20.

[2] A. Bouchair and S. Kelaiaia, α-favorability of $C(X)$ with a set-open topology, Far East J. Math. Sci. (FJMS) 18(3) (2005), 305-312.

[3] G. Choquet, Lectures in Analysis, Benjamin, New York, Amsterdam, 1969.

[4] R. Engelking, General Topology, Polish Scientific Publishing, 1977.

[5] S. Kelaiaia, Propriétés de certaines topologies set-open sur $C(X)$, Thèse de Doctorat de l'Université de Rouen, Soutenu le 30 Juin 1995.

[6] R. A. McCoy and I. Ntantu, Countability properties of function spaces with set-open topology, Topology Proc. 10 (1985), 329-345.

[7] R. A. McCoy and I. Ntantu, Completeness properties of function spaces, Topology Appl. 2 (1986), 191-206.

[8] J. C. Oxtoby, The Banach-Mazur game and Banach category theorem, Contributions to the theory of games, Vol. 3, pp. 159-163, Ann. Math. Stud. 39, Princeton University Press, Princeton, 1957.

[9] H. E. White, Topological spaces that are α-favorable for player with perfect information, Proc. Amer. Math. Soc. 50 (1975), 477-482.

Department of Mathematics
University of Jijel
P. O. Box 98, Jijel, Algeria

Department of Mathematics
University of Annaba
P. O. Box 12, Annaba, Algeria