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NONPERIODIC CONVEX INTEGRAL FUNCTIONALS AND EPICONVERGENCE

MESSAOUDI KHELIFA and BENHADID AYACHE

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Abstract

The limit analysis, by epiconvergence techniques, of the sequence of nonperiodic convex integral functionals, with time dependent case, is studied by a new approach. This approach allows to treat some cases more general than the ones considered in the previous literature. Our main result generalizes the one obtained by Mascarenhas [Trans. Amer. Math. Soc. 281(1) (1984), 179-195], in the time independent case. An example of such sequence is presented.

1. Introduction

In order to treat the periodic homogenization, the notion of two scale convergence is developed in [1] and [2]. As remarked by [11], the two scale limit represents in fact the barycenter of a Young measure. More recently, [6] introduced the scale convergence, which generalizes the multiscale convergence introduced by [1] and [2]. This new concept, seems to be a powerful tool to study by epiconvergence, the nonperiodic case. In this paper, we deal with the sequence of the nonperiodic integral functionals, of the form

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$$F_n(v) = \int_0^T \int_\Omega f(t, x, \alpha_n(x), v(t, x), v'(t, x))dxdt,$$

where, $v \in H^1([0, T]; L^2(\Omega))$ and $f : (t, x, \lambda, \xi, \zeta) \to f(t, x, \lambda, \xi, \zeta)$ is a function from $[0, T] \times \Omega \times \prod \times \mathbb{R}^2$ into $\mathbb{R}^+$, which satisfies the following conditions:

- f is (ξ, ζ)-convex;

- there exists $C > 0$, such that for every $(\xi, \zeta) \in \mathbb{R}^2$

$$\frac{1}{C}(|\xi|^2 + |\zeta|^2) \leq f(t, x, \lambda, \xi, \zeta) \leq C(1 + |\xi|^2 + |\zeta|^2),$$

$$\text{a.e. } (t, x, \lambda) \in [0, T] \times \Omega \times \prod; \tag{$*$}$$

- $f, \frac{\partial f}{\partial \xi}$ and $\frac{\partial f}{\partial \zeta}$ are (t, x)-measurable, (λ, ξ, ζ)-continuous.

By using, the theory of epiconvergence, the fundamental theorem for Young measures and the scale convergence, we will prove that the following integral functional

$$F^{\text{hom}}(v) = \inf\left\{\int_0^T \int_\Omega \int_\prod f(t, x, \lambda, w(t, x, \lambda), w'(t, x, \lambda))d\mu dt;\right.$$

$$\left. w \in \mathcal{H}, v(t, x) = \int_\prod w(t, x, \lambda) d\mu_x\right\}$$

is the epilimit of the sequence (F_n), with $\mathcal{H} = H^1([0, T]; L^2_\mu(\Omega \times \prod))$. The paper is organized as follows. Section 2 contains some useful results concerning Young measures, while Section 3 is devoted to introduce the so called scale convergence and related results. In Section 4, we prove our main result and in Section 5, we give an example.

2. Young Measures and the Measurable Functions

Let O be an open bounded subset of $\mathbb{R}^n$ and let S be a metrizable space. We denote by: dx the Lebesgue measure on $\mathbb{R}^n$; $\mathcal{F}(O)$ the family of all Lebesgue measurable subsets of O and $\mathcal{B}(S)$ the Borel σ-field of S; $\mathcal{M}^+(O \times S)$ the set of the positive Radon measures.

Definition 2.1. The *Young measure* on $O \times S$, is an any $\mu \in \mathcal{M}^+(O \times S)$, whose projection on O is dx, i.e., $\mu(A \times S) = dx(A)$, for all $A \in \mathcal{F}(O)$.

We denote by $\mathcal{Y}(O \times S)$, the set of all Young measures on $O \times S$, and we say that, the sequence μ_n narrow converges to μ in $\mathcal{Y}(O \times S)$ and we write $\mu_n \overset{nar}{\to} \mu$, if for each Ψ in $Cth^b(O \times S)$ (the set of the Carathéodory bounded integrands), we have $\langle \Psi, \mu_n \rangle \to \langle \Psi, \mu \rangle$.

Theorem 2.1 (disintegration [10])**.** *Let* $\mu \in \mathcal{Y}(O \times S)$. *Then for a.e.* x *in* O, *there exists a probability measures* μ_x *from* S, *such that for all* $\Psi : O \times S \to \mathbb{R}^+$, μ-*measurable*

$$\int_{O \times S} \Psi(x, \lambda) d\mu = \int_O \int_S \Psi(x, \lambda) d\mu_x dx.$$

Thus we write $\mu = \mu_x \otimes dx$.

Let $\alpha : O \to S$, be a measurable function and $G : x \to (x, \alpha(x))$ from O into $O \times S$, the graph map of α. Denoting by $\mu_\alpha = dx \circ G^{-1}$, the image measure of dx on O by G. Then $\mu_\alpha \in \mathcal{Y}(O \times S)$, and for every $A \in \mathcal{F}(O)$ and every $B \in \mathcal{B}(S)$ $\mu_\alpha(A \times B) := dx(A \cap \alpha^{-1}(B))$. So, for each μ_α-measurable function $\Psi : O \times S \to \mathbb{R}^+$, we have

$$\int_{O \times S} \Psi(x, \lambda) d\mu_\alpha = \int_O \Psi(x, \alpha(x)) dx.$$

By using Theorem 2.1, we obtain $\mu_\alpha = \delta_\alpha \otimes dx$, here δ_α denotes the Dirac measure of α. μ_α is said the *Young measure associated to* α. For a measurable sequence $\alpha_n : O \to S$, we say that

(i) μ in $\mathcal{Y}(O \times S)$ is generated by α_n, if the sequence of the Young measures associated to α_n, narrow converges to μ or equivalently for all

ϕ in $Cth^b(O \times S)$

$$\int_O \phi(x, \alpha_n(x))dx \to \int_{O\times S} \phi(x, \lambda)d\mu.$$

(ii) The sequence μ_n in $\mathcal{Y}(O \times S)$ is tight if, for every $\eta > 0$, there exists a compact space $K_\eta \subset S$, such that $\sup_n \mu_n\{O \times (S \setminus K_\eta)\} < \eta$, or $\sup_n dx\{x \in O;\, \alpha_n(x) \in (S \setminus K_\eta)\} < \eta$, if μ_n is associated to α_n.

Theorem 2.2 (Prokhorov's theorem, see [10]). *Every tight sequence μ_n in $\mathcal{Y}(O \times S)$, admits a subsequence μ_{n_k} which narrow converges in $\mathcal{Y}(O \times S)$.*

Note that, if $S = \mathbb{R}^d$, and α_n is a bounded sequence in $L^1(O; \mathbb{R}^d)$, then the sequence of their associated Young measure is tight. If now, S is a compact space, then the sequence μ_n of the Young measure associated to α_n is tight.

Proposition 2.1. *If the sequence (μ_n) is relatively compact in $\mathcal{Y}(O \times S_1)$ and if the sequence (ν_n) is relatively compact in $\mathcal{Y}(O \times S_2)$, then the sequence (μ_n, ν_n) is relatively compact in $\mathcal{Y}(O \times S_1 \times S_2)$.*

Theorem 2.3 (Fundamental theorem [10]). *Let $\alpha_n : O \to S$ be a sequence of measurable functions, such that the sequence of their associated Young measures narrow converges to μ.*

(a) *If $\psi : O \times S \to \mathbb{R}$ is a normal integrand such that, the sequence of the negative parts $\{\psi(x, \alpha_n(x))^-\}$ is uniformly integrable in O, then*

$$\int_{O\times S} \psi(x, \lambda)d\mu \le \liminf_n \int_O \psi(x, \alpha_n(x))dx.$$

(b) *If $\psi : O \times S \to \mathbb{R}$ is a Carathéodory integrand such that, the sequence $\{\psi(x, \alpha_n(x))\}$ is uniformly integrable in O, then*

$$\int_{O\times S} \psi(x, \lambda)d\mu = \lim_n \int_O \psi(x, \alpha_n(x))dx.$$

3. The Scale Convergence

In [6], the concept of the scale convergence, or α_n-convergence is given in $(L^2(\Omega), L^2_\mu(\Omega \times \prod))$, where $\prod$ is a metrizable compact space. In this section, we will extend this notion to $(H^1([0, T]; L^2(\Omega)), H^1([0, T]; L^2_\mu(\Omega \times \prod)))$ and we will demonstrate some related results, which used in the section below. In the sequel, $C(\prod)$ denotes the space of continuous functions from $\prod$ into $\mathbb{R}$.

Definition 3.1. The sequence (v_n) in $H^1([0, T]; L^2(\Omega))$, α_n-converges to $v \in H^1([0, T]; L^2_\mu(\Omega \times \prod))$ if for all $\phi \in L^2([0, T] \times \Omega; C(\prod))$;

$$\int_0^T \int_\Omega v_n(t, x)\phi(t, x, \alpha_n(x))dxdt \to \int_0^T \int_{\Omega\times\prod} v(t, x, \lambda)\phi(t, x, \lambda)d\mu dt;$$

and

$$\int_0^T \int_\Omega v_n'(t, x)\phi(t, x, \alpha_n(x))dxdt \to \int_0^T \int_{\Omega\times\prod} v'(t, x, \lambda)\phi(t, x, \lambda)d\mu dt.$$

We will say that v is the α_n*-limit* of the sequence v_n.

Definition 3.1 is justified in view of the following compactness theorem:

Theorem 3.1. *From each bounded sequence* (v_n) *in* $H^1([0, T]; L^2(\Omega))$, *there exists a subsequence* (v_{n_k}), *which* α_{n_k}*-converges to* $w \in H^1([0, T]; L^2_\mu(\Omega \times \prod))$. *In particular, for a.e. x in* Ω

$$v_{n_k} \rightharpoonup \int_\prod w(t, x, \lambda)d\mu_x \text{ weakly in } H^1([0, T]; L^2(\Omega)).$$

Consequently, for every φ *in* $L^2(\Omega; C(\prod))$

$$\int_\Omega v_{n_k}(T, x)\varphi(x, \alpha_{n_k}(x))dx \to \int_\Omega \int_\prod w(T, x, \lambda)\varphi(x, \lambda)d\mu_x dx.$$

Proof. Let λ_n be the Young measure on $[0, T] \times \Omega \times \mathbb{R}^2$ associated to $\varphi_n : (t, x) \to (v_n(t, x), v_n'(t, x))$ from $[0, T] \times \Omega$ into $\mathbb{R}^2$. From Proposition 2.1, λ_n is relatively compact in $\mathcal{Y}([0, T] \times \Omega \times \mathbb{R}^2)$; the sequences of Young measures associated to $\alpha_n : \Omega \to \prod$ is relatively compact in $\mathcal{Y}(\Omega \times \prod)$. Hence the sequence of Young measures associated to (v_n, v_n', α_n) denoted (θ_n) is relatively compact in $\mathcal{Y}([0, T] \times \Omega \times \prod \times \mathbb{R}^2)$. Then, there exists a subsequence still denoted (θ_n) which narrow converges to some $\theta \in \mathcal{Y}([0, T] \times \Omega \times \prod \times \mathbb{R}^2)$.

Applying Theorem 2.3 (b) with $O = (0, T) \times \Omega$, $S = \mathbb{R}^2 \times \prod$ and $\psi(t, x, \xi, \zeta, \lambda) = \xi\phi(t, x, \lambda)$, we obtain

$$\int_0^T \int_\Omega v_n(t, x)\phi(t, x, \alpha_n(x))dxdt \to \int_{[0, T]\times\Omega\times\mathbb{R}^2\times\prod} \xi\phi(t, x, \lambda)d\theta.$$

By Theorem 2.1 there exists the family of the probability measures $(\theta_{(t,x,\lambda)})_{(t,x,\lambda)\in[0, T]\times\Omega\times\prod}$ such that

$$\int_{(0, T)\times\Omega\times\mathbb{R}^2\times\prod} \xi\phi(t, x, \lambda)d\theta = \int_0^T \int_\Omega \int_\prod \phi(t, x, \lambda) \int_{\mathbb{R}^2} \xi d\theta_{(t,x,\lambda)}(\xi, \zeta) d\mu dt;$$

therefore

$$\int_0^T \int_\Omega v_n(t, x)\phi(t, x, \alpha_n(x))dxdt \to \int_0^T \int_{\Omega\times\prod} w(t, x, \lambda)\phi(t, x, \lambda)d\mu dt,$$

where $w(t, x, \lambda) := \int_{\mathbb{R}^2} \xi d\theta_{(t,x,\lambda)}$.

It remains to prove that $w \in H^1((0, T); L^2_\mu(\Omega \times \prod))$. Applying Jensen's inequality to the probability measures $(\theta_{(t,x,\lambda)})_{(t,x,\lambda)\in[0,T]\times\Omega\times\prod}$, we have

$$\begin{aligned} \| w \|^2_{L^2((0, T);\, L^2_\mu(\Omega\times\prod))} &= \int_0^T \int_\Omega \int_\prod \left| \int_{\mathbb{R}^2} \xi d\theta_{(t,x,\lambda)}(\xi, \zeta) \right|^2 d\mu dt \\ &\le \int_0^T \int_\Omega \int_\prod \int_{\mathbb{R}^2} \xi^2 d\theta_{(t,x,\lambda)}(\xi, \zeta) d\mu dt. \end{aligned}$$

From Theorem 2.3 (a), with $O = (0, T) \times \Omega$, $S = \mathbb{R}^2 \times \prod$ and $\psi(t, x, \xi, \zeta, \lambda) = \xi^2$,

$$\int_{(0,\, T)\times\Omega\times\mathbb{R}^2\times\prod} \xi^2 d\theta \leq \liminf_{n\to+\infty} \int_0^T \int_\Omega |v_n(t, x)|^2 dxdt < +\infty,$$

therefore $w \in L^2((0, T); L^2_\mu(\Omega \times \prod))$.

Using the same argument for the sequence (v_n') and taking $\psi(t, x, \xi, \zeta, \lambda) = \zeta\phi(t, x, \lambda)$. We obtain

$$\int_0^T \int_\Omega v_n'(t, x)\phi(t, x, \alpha_n(x))dxdt \to \int_0^T \int_{\Omega\times\prod} \overline{w}(t, x, \lambda)\phi(t, x, \lambda)d\mu dt,$$

where $\overline{w}(t, x, \lambda) = \int_{\mathbb{R}^2} \zeta d\theta_{(t, x, \lambda)}(\xi, \zeta)$ and $\overline{w} \in L^2((0, T); L^2_\mu(\Omega \times \prod))$. In order to complete the proof we will show that $\overline{w}$ is the time derivative of w. Let $\phi \in \mathfrak{D}((0, T); C(\overline{\Omega} \times \prod))$, we have

$$\int_0^T \int_\Omega v_n'(t, x)\phi(t, x, \alpha_n(x))dxdt = -\int_0^T \int_\Omega v_n(t, x)\phi'(t, x, \alpha_n(x))dxdt,$$

when $n \to +\infty$

$$\int_0^T \int_{\Omega\times\prod} \overline{w}(t, x, \lambda)\phi(t, x, \lambda)d\mu dt = -\int_0^T \int_{\Omega\times\prod} w(t, x, \lambda)\phi'(t, x, \lambda)d\mu dt.$$

For $\varphi_1 \in \mathfrak{D}(0, T)$ and $\varphi_2 \in C(\overline{\Omega} \times \prod)$, setting $\phi(t, x, \lambda) = \varphi_1(t)\varphi_2(x, \lambda)$, then

$$\int_0^T \int_{\Omega\times\prod} \overline{w}(t, x, \lambda)\varphi_1(t)\varphi_2(x, \lambda)d\mu dt = -\int_0^T \int_{\Omega\times\prod} w(t, x, \lambda)\varphi_1'(t)\varphi_2(x, \lambda)d\mu dt;$$

or

$$\int_{\Omega\times\prod} \int_0^T [\overline{w}(t, x, \lambda)\varphi_1(t) + w(t, x, \lambda)\varphi_1'(t)]\varphi_2(x, \lambda)dtd\mu = 0.$$

Therefore

$$\int_0^T \overline{w}(t, x, \lambda)\varphi_1(t) + w(t, x, \lambda)\varphi_1'(t)dt = 0 \ \ \mu \text{ a.e. in } \Omega \times \prod,$$

hence

$$\int_0^T \overline{w}\varphi_1(t)dt = -\int_0^T w\varphi_1'(t)dt,$$

and

$$\overline{w}(t, x, \lambda) = w'(t, x, \lambda) \text{ in } \mathfrak{D}'(0, T),$$

which implies that $\overline{w} = w'$ for all $(t, x, \lambda) \in [0, T] \times \Omega \times \prod$, so $w \in H^1((0, T); L^2_\mu(\Omega \times \prod))$.

Proposition 3.1. *Let w be in $H^1([0, T]; L^2(\Omega; C(\prod)))$ and $w_n(t, x) = w(t, x, \alpha_n(x))$. Then, for all Carathéodory integrand $\phi : [0, T] \times \Omega \times \prod \times \mathbb{R}^2 \to \mathbb{R}$ such that, there exist a positive constant C and $p \in L^1((0, T) \times \Omega)$ satisfying, for all $(\lambda, \xi, \zeta) \in \prod \times \mathbb{R}^2$*

$$|\phi(t, x, \lambda, \xi, \zeta)| \leq C(p(t, x) + |\xi|^2 + |\zeta|^2), \text{ a.e. } (t, x) \in [0, T] \times \Omega.$$

We have

$$\int_0^T \int_\Omega \phi(t, x, \alpha_n(x), w_n(t, x), w_n'(t, x))dxdt$$

$$\to \int_0^T \int_{\Omega \times \prod} \phi(t, x, \lambda, w(t, x, \lambda), w'(t, x, \lambda))d\mu dt.$$

In particular (w_n), α_n-converges to w.

Proof. It sufficient to remark that

$$|\phi(t, x, \alpha_n(x), w_n, w_n')| \leq C(p(t, x) + w_n^2 + (w_n')^2)$$

$$\leq C(p(t, x) + \sup_{\lambda \in \prod}[w^2 + (w')^2]),$$

and using Theorem 2.3 (b).

Proposition 3.2 (see for instance [11]). *$H^1([0, T]; L^2(\Omega; C(\prod)))$ is dense in $\mathcal{H} = H^1([0, T]; L^2_\mu(\Omega \times \prod))$.*

4. The Main Result

Let (X, τ) be a Banach space, and let $\{F_n, F, n \in \mathbb{N}\}$ be a family of functions mapping X into $\mathbb{R} \cup \{+\infty\}$. Let us recall the following notion of convergence, which is called *epiconvergence* or in its general setting Γ-*convergence*. For overview about epiconvergence, we refer the reader to [3] and [7].

Definition 4.1. We say that the sequence $(F_n)_{n\in\mathbb{N}}$ τ-epiconvergences to F at x in X iff the two following sentences hold:

(i) For every sequence $(x_n)_{n\in\mathbb{N}}$, τ-converging to x in X,

$$F(x) \leq \liminf_{n\to+\infty} F_n(x_n).$$

(ii) There exists a sequence $(x_n)_{n\in\mathbb{N}}$ of X, τ-converging to x such that, $F(x) \geq \limsup_{n\to+\infty} F_n(x_n)$.

If (i) and (ii) are satisfies for every x in X, then we say that $(F_n)_{n\in\mathbb{N}}$ τ-epiconvergences to F in X, and we write $F = \tau\text{-}epilimit\ F_n$.

Proposition 4.1 (variational properties of epiconvergence)**.** *Assume that, (F_n) τ-epiconverges to F, and let H be a τ-continuous functional from X into $\mathbb{R}$. Then*

(i) *F is lsc and τ-epilimit $(F_n + H) = F + H$.*

(ii) *If now, $(x_n)_{n\in\mathbb{N}}$ is a sequence in (X, τ) such that, $F_n(x_n) \leq F_n(x) + \varepsilon_n$, where $\varepsilon_n > 0$, and if furthermore $(x_n)_{n\in\mathbb{N}}$ is τ-relatively compact. Then any cluster point $\bar{x}$ of $(x_n)_{n\in\mathbb{N}}$ is a minimiser of F and* $\lim_{n\to+\infty} \inf\{F_n(x);\, x \in X\} = \min\{F(x);\, x \in X\} = F(\bar{x})$.

We are now in a position to state the main result of this paper. Let $\{F_n;\, F^{\text{hom}}\}$ be the family of the integral functionals defined in the introduction. Then we have:

Theorem 4.1. *F_n epiconverges weakly to F^{hom} in $H^1([0, T]; L^2(\Omega))$.*

Proof. It remains to prove the assertions (i) and (ii), in Definition 4.1, of the epiconvergence.

(i) Let v, v_n be a sequence in $H^1([0, T]; L^2(\Omega))$, such that $v_n \rightharpoonup v$ weakly. From $(*)$ and Theorem 3.1, there exists a subsequence (v_{n_k}) of (v_n), still denoted (v_n), which α_{n_k}-converges to $w \in \mathcal{H}$, with $v(t, x) = \int_{\Pi} w(t, x, \lambda) d\mu_x$, a.e. $(t, x) \in [0, T] \times \Omega$ and $\liminf_{n \to +\infty} F_n(v_n) = \lim_{k \to +\infty} F_{n_k}(v_{n_k})$; hence from Proposition 3.2, there exists a sequence (w^k) in $H^1([0, T]; L^2(\Omega; C(\Pi)))$ such that $\| w^k - w \|_{\mathcal{H}} < \frac{1}{k}$. By $(*)$, and the fact that f is (ξ, ζ)-convex, we have, $\frac{\partial f}{\partial \xi}(t, x, \alpha_n, w_n^k, (w_n^k)')$ and $\frac{\partial f}{\partial \zeta}(t, x, \alpha_n, w_n^k, (w_n^k)')$ belong to $L^2((0, T) \times \Omega; C(\Pi))$, here $w_n^k(t, x) := w^k(t, x, \alpha_n(x))$, therefore

$$F_n(v_n) \geq F_n(w_n^k) + \int_0^T \int_{\Omega} \frac{\partial f}{\partial \xi}(t, x, \alpha_n, w_n^k, (w_n^k)') v_n dx dt$$

$$- \int_0^T \int_{\Omega} \frac{\partial f}{\partial \xi}(t, x, \alpha_n, w_n^k, (w_n^k)') w_n^k dx dt$$

$$+ \int_0^T \int_{\Omega} \frac{\partial f}{\partial \zeta}(t, x, \alpha_n, w_n^k, (w_n^k)') v_n' dx dt$$

$$- \int_0^T \int_{\Omega} \frac{\partial f}{\partial \xi}(t, x, \alpha_n, w_n^k, (w_n^k)') (w_n^k)' dx dt$$

by Definition 3.1, we obtain

$$\int_0^T \int_{\Omega} \frac{\partial f}{\partial \xi}(t, x, \alpha_n, w_n^k, (w_n^k)') v_n dx dt$$

$$\to \int_0^T \int_{\Omega \times \Pi} \frac{\partial f}{\partial \xi}(t, x, \lambda, w^k, (w^k)') w d\mu dt;$$

and

$$\int_0^T \int_\Omega \frac{\partial f}{\partial \zeta}(t, x, \alpha_n, w_n^k, (w_n^k)')v_n' dxdt$$

$$\to \int_0^T \int_{\Omega\times\Pi} \frac{\partial f}{\partial \xi}(t, x, \lambda, w^k, (w^k)')w' d\mu dt.$$

Taking respectively in Proposition 3.1, $\phi(t, x, \lambda, \xi, \zeta) = \xi \frac{\partial f}{\partial \xi}(t, x, \lambda, \xi, \zeta)$; $\phi(t, x, \lambda, \xi, \zeta) = \zeta \frac{\partial f}{\partial \zeta}(t, x, \lambda, \xi, \zeta)$ and $\phi = f$ we have respectively when $n \to +\infty$

$$\int_0^T \int_\Omega \frac{\partial f}{\partial \xi}(t, x, \alpha_n, w_n^k, (w_n^k)')w_n^k dxdt$$

$$\to \int_0^T \int_{\Omega\times\Pi} \frac{\partial f}{\partial \xi}(t, x, \lambda, w^k, (w^k)')w^k(t, x, \lambda) d\mu dt;$$

$$\int_0^T \int_\Omega \frac{\partial f}{\partial \xi}(t, x, \alpha_n, w_n^k, (w_n^k)')(w_n^k)' dxdt$$

$$\to \int_0^T \int_{\Omega\times\Pi} \frac{\partial f}{\partial \xi}(t, x, \lambda, w^k, (w^k)')(w^k)'(t, x, \lambda) d\mu dt$$

and

$$\lim_{n\to+\infty} F_n(w_n^k) = \int_0^T \int_{\Omega\times\Pi} f(t, x, \lambda, w^k(t, x, \lambda), (w^k)'(t, x, \lambda)) d\mu dt.$$

Therefore

$$\begin{aligned}
&\liminf_{n\to+\infty} F_n(v_n) \\
&\geq \int_0^T \int_{\Omega\times\Pi} f(t, x, \lambda, w^k(t, x, \lambda), (w^k)'(t, x, \lambda)) d\mu dt \\
&\quad + \int_0^T \int_{\Omega\times\Pi} \frac{\partial f}{\partial \xi}(t, x, \lambda, w^k(t, x, \lambda), (w^k)'(t, x, \lambda))(w - w^k) d\mu dt \\
&\quad + \int_0^T \int_{\Omega\times\Pi} \frac{\partial f}{\partial \zeta}(t, x, \lambda, w^k(t, x, \lambda), (w^k)'(t, x, \lambda))(w' - (w^k)') d\mu dt.
\end{aligned}$$

Since there exists $C > 0$ such that

$$\left| \int_0^T \int_{\Omega\times\Pi} \frac{\partial f}{\partial \xi}(t, x, \lambda, w^k, (w^k)')(w - w^k) \right| \leq C\frac{1}{k};$$

$$\left| \int_0^T \int_{\Omega\times\Pi} \frac{\partial f}{\partial \zeta}(t, x, \lambda, w^k, (w^k)')(w' - (w^k)') \right| \leq C\frac{1}{k},$$

$$\lim_{k\to+\infty} \int_0^T \int_{\Omega\times\Pi} f(t, x, \lambda, w^k, (w^k)')d\mu dt = \int_0^T \int_{\Omega\times\Pi} f(t, x, \lambda, w, w')d\mu dt.$$

Finally,

$$\liminf_{n\to+\infty} F_n(v_n) \geq \int_0^T \int_{\Omega\times\Pi} f(t, x, \lambda, w(t, x, \lambda), w'(t, x, \lambda))d\mu dt \geq F^{\text{hom}}(v).$$

(ii) Let v be an element of $H^1([0, T]; L^2(\Omega))$. We prove that, there exists a sequence $(\overline{v}_n)$ in $H^1([0, T]; L^2(\Omega))$ such that

$$\begin{cases} \overline{v}_n \rightharpoonup v \text{ in } H^1([0, T]; L^2(\Omega)) \text{ weakly;} \\ \lim\limits_n F_n(v_n) = F^{\text{hom}}(v). \end{cases}$$

Let (w^k) be a minimizing sequence of the following minimizing problem

$$\inf\left\{ \int_0^T \int_\Omega \int_\Pi f(t, x, \lambda, w(t, x, \lambda), w'(t, x, \lambda))d\mu dt; \right.$$

$$\left. w \in \mathcal{H}, v(t, x) = \int_\Pi w(t, x, \lambda)d\mu_x \right\}.$$

By Proposition 3.2, there exists $(\overline{w}^k)$ in $H^1(0, T; L^2(\Omega; C(\Pi)))$ such that $\| \overline{w}^k - w^k \|_{\mathcal{H}} < \frac{1}{k}$. Then, for all $\varphi \in H^1([0, T]; L^2(\Omega))$

$$\left| \int_0^T \int_{\Omega\times\Pi} [\overline{w}^k(t, x, \lambda) - w^k(t, x, \lambda)]\varphi(t, x)d\mu dt \right|$$

$$\leq \frac{1}{k} \| \varphi \|_{H^1([0, T]; L^2(\Omega))},$$

therefore

$$\lim_{k\to+\infty}\int_0^T\int_{\Omega\times\Pi}\overline{w}^k\varphi d\mu dt = \lim_{k\to+\infty}\int_0^T\int_{\Omega\times\Pi} w^k\varphi d\mu dt$$

$$= \lim_{k\to+\infty}\int_0^T\int_\Omega\int_\Pi w^k\varphi d\mu_x(\lambda)dxdt = \int_0^T\int_\Omega \varphi(t, x)v(t, x)dxdt.$$

Since, w^k is bounded in $\mathcal{H}$ (see $(*)$), we have

$$\left|\int_0^T\int_{\Omega\times\Pi} f(t, x, \lambda, \overline{w}^k, (\overline{w}^k)') - f(t, x, \lambda, w^k, (w^k)')d\mu dt\right|$$

$$\leq \int_0^T\int_{\Omega\times\Pi} C(1+|\overline{w}^k|+|w^k|)|\overline{w}^k - w^k|d\mu dt \leq \frac{C}{k};$$

and

$$\left|\int_0^T\int_{\Omega\times\Pi} f(t, x, \lambda, w^k, (\overline{w}^k)') - f(t, x, \lambda, w^k, (w^k)')d\mu dt\right|$$

$$\leq \int_0^T\int_{\Omega\times\Pi} C(1+|(\overline{w}^k)'|+|(w^k)'|)|(\overline{w}^k)' - (w^k)'|d\mu dt \leq \frac{C}{k}.$$

Thus

$$\lim_{k\to+\infty}\int_0^T\int_{\Omega\times\Pi} f(t, x, \lambda, \overline{w}^k, (\overline{w}^k)')d\mu dt$$

$$= \lim_{k\to+\infty}\int_0^T\int_{\Omega\times\Pi} f(t, x, \lambda, w^k, (w^k)')d\mu dt = F^{\hom}(v).$$

Setting $\overline{w}_n^k(t, x) := \overline{w}^k(t, x, \alpha_n(x))$ and let (φ_j) be a countable dense family in $H^1([0, T]; L^2(\Omega))$, from Proposition 3.1, we have, when $n \to +\infty$

$$\int_0^T\int_\Omega f(t, x, \alpha_n(x), \overline{w}_n^k(t, x), (\overline{w}_n^k)'(t, x))dxdt$$

$$\to \int_0^T\int_{\Omega\times\Pi} f(t, x, \lambda, \overline{w}^k, (\overline{w}^k)')d\mu dt$$

and

$$\int_0^T \int_\Omega \overline{w}_n^k(t, x)\varphi_j(t, x)dxdt \to \int_0^T \int_{\Omega\times\Pi} \overline{w}^k(t, x, \lambda)\varphi_j(t, x)d\mu dt.$$

Therefore, for all $(j, k) \in \mathbb{N}^2$ and for all $\delta > 0$, there exists $n_0 = n_0(j, k, \delta) \in \mathbb{N}$ such that

$$\Bigg| \int_0^T \int_\Omega f(t, x, \alpha_n(x), \overline{w}_n^k(t, x), (\overline{w}_n^k)'(t, x))dxdt - \int_0^T \int_{\Omega\times\Pi} f(t, x, \lambda, \overline{w}_k, (\overline{w}^k)')d\mu dt \Bigg| < \delta$$

and

$$\left| \int_0^T \int_\Omega \overline{w}_n^k(t, x)\varphi_j(t, x)dxdt - \int_0^T \int_{\Omega\times\Pi} \overline{w}^k(t, x, \lambda)\varphi_j(t, x)d\mu dt \right| < \delta.$$

Taking $\delta = \frac{1}{k}$, $j = k$ and defining an increasing sequence (n_k) such that $n_k > n_0\left(k, k, \frac{1}{k}\right)$, we have for all $i \le k$

$$\lim_{k\to+\infty} \int_0^T \int_\Omega f(t, x, \alpha_{n_k}, \overline{w}_{n_k}^k(t, x), (\overline{w}_{n_k}^k)'(t, x))dxdt = F^{\text{hom}}(v),$$

and, for all $i \in \mathbb{N}$

$$\lim_{k\to+\infty} \int_0^T \int_\Omega \overline{w}_{n_k}^k \varphi_i(t, x)dxdt = \int_0^T \int_\Omega v(t, x)\varphi_i(t, x)dxdt.$$

Since $(\overline{w}^k)$ is bounded in $\mathcal{H}$, $(\overline{w}_{n_k}^k)$ is bounded in the separable space $H^1([0, T]; L^2(\Omega))$, we obtain the last convergence for all $\varphi \in H^1([0, T]; L^2(\Omega))$. Finally setting: $\overline{v}_n := \overline{w}_{n_k}^k$ if $n = n_k$, $\overline{v}_n := v$ if $n \ne n_k$, the sequence $\overline{v}_n$ satisfies

$$\begin{cases} \overline{v}_n \rightharpoonup v \text{ in } H^1([0, T]; L^2(\Omega)) \text{ weakly} \\ \lim\limits_{n\to+\infty} F_n(\overline{v}_n) = F^{\text{hom}}(v). \end{cases}$$

5. Example

We shall us the same notations as in the previous sections, and we consider the two following problems:

$$\begin{cases} \dot{u}_n(t, x) + a_n(t, x) g(u_n(t, x)) = h(t, x) \text{ in } (0, T) \times \Omega; \\ u_n(0, x) = 0 \text{ on } \Omega, \end{cases} \tag{E_n}$$

and

$$\min_{v \in K} J_n(v) \tag{P_n}$$

$K = \{v \in H^1(0, T; L^2(\Omega)) / v(0, x) = 0 \; dx \text{ a.e. in } \Omega\}$. $J_n(v)$ is the energy functional associated to (E_n), which takes the form, for every $v \in K$,

$$J_n(v) = \int_0^T \int_\Omega \left[a_n \Psi(v) + a_n \Psi^* \left(\frac{h - \dot{v}}{\alpha_n} \right) \right] dxdt + \int_0^T \int_\Omega (v\dot{v} - hv) dxdt;$$

where $\dot{u}_n$ is the time derivative of $u_n(t, x)$; $a_n \in L^\infty((0, T) \times \Omega)$, $\alpha \leq a_n(t, x) \leq \beta$, $\forall n \in \mathbb{N}^*$ $(0 < \alpha \leq \beta)$; $g(y) = \frac{d\Psi}{dy}$; with $\Psi \in C^1(\mathbb{R})$, which is strictly convex, $\theta y^2 - \gamma \leq \Psi(y) \leq \rho y^2 + \delta$ $(\theta, \rho > 0$ and $\gamma, \delta \geq 0)$, and $h \in L^2((0, T) \times \Omega)$.

Note that, under consideration, the above minimization problem has a unique solution, and a close relationship between (E_n) and (P_n) is given by the following variational principal, that is: "u_n is the solution of (E_n) iff u_n is the solution of (P_n)" see for instance [4].

We shall show in the sense of Theorem 4.1 and Proposition 4.1 (ii), that the limit problem of (P_n) is

$$\min_{v \in K^h} J^{\text{hom}}(v), \tag{P^{hom}}$$

where

$K^h = \{w \in \mathcal{H}; w(0, x, \lambda) = 0, \mu \text{ - a.e. in } \Omega \times \prod\}; \mathcal{H} = H^1((0, T); L^2_\mu(\Omega \times \prod))$

and

$$J^{\hom}(v) = \min\left\{\int_0^T \int_\Omega \int_\Pi \left[\lambda\Psi(w) + \lambda\Psi^*\left(\frac{h - \dot{w}}{\lambda}\right) - hw\right] d\mu_x dxdt\right.$$

$$\left. + \frac{1}{2}\| w(T)\|^2_{L^2_\mu},\ w \in K^h,\ v = \int_\Pi w d\mu_x\right\}.$$

Ψ^* denotes the Fenchel conjugate of Ψ.

For the sake of the simplicity, we assume that, $\Psi(y) = \frac{1}{2}y^2$; then $\forall z \in \mathbb{R}$; $\Psi^*(z) = \frac{1}{2}z^2$; and

$$J_n(v) = \frac{1}{2}\int_0^T \int_\Omega a_n v^2 + \frac{\dot{v}^2}{a_n} + \frac{h^2}{a_n} dxdt - \int_0^T \int_\Omega \frac{h}{a_n}\dot{v} dxdt - \int_0^T \int_\Omega hv dxdt$$

$$+ \frac{1}{2}\int_0^T \int_\Omega v^2 dxdt.$$

Since, the epiconvergence is stable by the continuous perturbation, see Proposition 4.1 (i), it sufficient to study the sequence of the integral functionals

$$F_n(v) = \frac{1}{2}\int_0^T \int_\Omega a_n v^2 + \frac{\dot{v}^2}{a_n} + \frac{h^2}{a_n} dxdt - \int_0^T \int_\Omega \frac{h}{a_n}\dot{v} dxdt.$$

Let

$$f(t, x, \lambda, \xi, \zeta) = \frac{1}{2}\lambda\xi^2 + \frac{1}{2}\frac{\zeta^2}{\lambda} - \frac{\zeta}{\lambda}h + \frac{1}{2}\frac{h^2}{\lambda}.$$

Then

$$\frac{\partial f}{\partial \xi}(t, x, \lambda, \xi, \zeta) = \lambda\xi,\ \frac{\partial f}{\partial \zeta}(t, x, \lambda, \xi, \zeta) = \frac{\zeta}{\lambda} - \frac{h}{\lambda}.$$

It is obvious that f, $\frac{\partial f}{\partial \xi}$ and $\frac{\partial f}{\partial \zeta}$ are Carathéodory functions, f is (ξ, ζ)-convex, (λ, ξ, ζ)-continuous and satisfies $(*)$ with $C = \max\left\{\frac{1}{2}\beta, \frac{1}{\alpha}, \frac{1}{\alpha}\| h \|^2_{L^\infty((0,T)\times\Omega)}\right\}$. So, in this particular case the epilimit functional takes the form:

$$J^{\text{hom}}(v) = \min\left\{\int_0^T \int_\Omega \int_\pi \left[\frac{\lambda}{2} w^2 + \frac{\lambda}{2}\left(\frac{h - \dot{w}}{\lambda}\right)^2 - hw\right] d\mu_x dx dt \right.$$

$$\left. + \frac{1}{2} \| w(T) \|^2_{L^2_\mu}, \; w \in K^h, \; v = \int_\Pi w d\mu_x \right\}.$$

As the consequence of Proposition 4.1 (ii), we have

$$\lim_{n \to +\infty} \min\left\{J_n(v);\; w \in K^h,\; v = \int_\Pi w d\mu_x\right\}$$

$$= \min\left\{J^{\text{hom}}(v);\; w \in K^h,\; v = \int_\Pi w d\mu_x\right\} = J^{\text{hom}}(\overline{v});$$

where $\overline{v}$ is the cluster point of the sequence of the solution of the minimizing problem $\min\left\{F_n(v);\; w \in K^h,\; v = \int_\Pi w d\mu_x\right\}$.

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Département de Mathématiques
Faculté des Sciences
Université de Batna
05000 Batna, Algeria