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THE EXISTENCE FOR PERIODIC SOLUTIONS OF SECOND-ORDER SYSTEM

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Abstract

In this paper, by using the saddle point theorem in Critical Point Theory, the existence theorems are obtained for periodic solutions of a class of nonautonomous second-order systems with a potential which is a sub-quadratic function.

1. Introduction

Consider the second-order system

$$\begin{cases} \ddot{u}(t) + A\dot{u}(t) + \nabla F(t, u(t)) = h(t), \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1}$$

where $T > 0$ and $F : [0, T] \times R^n \to R$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for every $x \in R^n$ and continuously differentiable in x for a.e. $t \in [0, T]$ and there exist $a \in C(R^+, R^+)$, $b \in L^1([0, T]; R^+)$ such that

$$|F(t, x)| \le a(|x|)b(t), \, |\nabla F(t, x)| \le a(|x|)b(t)$$

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for all $x \in R^n$ and a.e. $t \in [0, T]$.

For (1), the corresponding functional on H_T^1 given by

$$J(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \frac{1}{2}\int_0^T (Au(t), \dot{u}(t))dt$$
$$- \int_0^T F(t, u(t))dt + \int_0^T (h(t), u(t))dt,$$

where

$$H_T^1 = \{u : [0, T] \to R^n / u \text{ absolutely continues,}$$
$$u(0) = u(T) \text{ and } \dot{u} \in L^2(0, T; R^n)\}$$

is a Hilbert space with the norm defined by

$$\| u \| = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt\right)^{1/2}$$

for $u \in H_T^1$. Under assumption (A) and some other suitable conditions, many results are obtained about the existence of periodic solutions by minimax methods in [1]-[9]. However those results for system (1) is few. In this paper, we will give some main results about system (1) by using the minimax methods.

2. Theorem and Proof

Theorem. *Assume that* $F(t, x)$ *satisfying assumption* (A), A *is an inverse symmetric matrix, moreover* $\| A \| < 1,$ *and satisfying the following condition*:

$$F(t, x) \to +\infty, \quad (|x| \to +\infty), \tag{2}$$

uniformly for a.e. $t \in [0, T]$.

Further, there exist $0 < \gamma < 2,\ M > 0$ *such that*

$$(\nabla F(t, x), x) \le \gamma F(t, x), \quad |x| \ge M, \ a.e.\ t \in [0, T]. \tag{3}$$

Then the system (1) *has at least one solution.*

Proof. For convenience to prove, we let $T = 2\pi$. For $u \in H^1_{2\pi}$, let $\overline{u} = \frac{1}{2\pi}\int_0^{2\pi} u(t)dt$ and $\tilde{u}(t) = u(t) - \overline{u}$. Then one has

$$\| \tilde{u} \|_\infty^2 \leq \frac{\pi}{6}\int_0^{2\pi} | \dot{u}(t) |^2 dt \quad \text{(Sobolev's inequality)}$$

and

$$\int_0^{2\pi} | \tilde{u}(t) |^2 dt \leq \int_0^{2\pi} | \dot{u}(t) |^2 dt \quad \text{(Wirtinger's inequality).}$$

Lemma 1 [8]. *Suppose that F satisfies* (A) *and* (2), *then there exists a real function $g \in L^1(0, T)$ and $G \in C(R^n, R)$ which is sub-additive, that is,*

$$G(x + y) \leq G(x) + G(y), \quad x, y \in R^n$$

for all $x, y \in R^n$ and coercive, that is,

$$G(x) \leq | x | + 4, \quad x \in R^n$$

for all $x \in R^n$, such that

$$F(t, x) \geq G(x) + g(t),$$

for all $x \in R^n$ and a.e. $t \in [0, T]$.

Lemma 2 [9]. *Suppose that F satisfies* (A) *and* (3), *then there exists*

$$a_0 = \max_{| x | \leq M} a(| x |),$$

such that

$$F(t, x) \leq a_0 b(t)((| x |/M)^\gamma + 1)$$

for all $x \in R^n$ and a.e. $t \in [0, T]$.

Proof of Theorem. By the saddle point theorem (see Theorem 4.6 in [5]), we need to prove

(A1) $J(u) \to +\infty$, as $\| u \| \to \infty$ in $\tilde{H}^1_{2\pi}$, and

(A2) $J(u) \to -\infty$, as $\| u \| \to \infty$ in R^n.

For all $u(t) \in H^1_{2\pi}$, $u(t)$ is expressed as

$$\overline{u} = \frac{1}{2\pi}\int_0^{2\pi} u(t)dt; \quad \int_0^{2\pi} \widetilde{u}(t)dt = 0.$$

Hence $H^1_{2\pi} = \overline{H}^1_{2\pi} \oplus \widetilde{H}^1_{2\pi}$ and $\dim \overline{H}^1_{2\pi} = n < +\infty$.

(i) For all $u \in \overline{H}^1_{2\pi}$, $\| u \| \to \infty$, by Lemmas 1 and 2, we have

$$J(u) = -\int_0^{2\pi} F(t, u)dt \le -\int_0^{2\pi} G(u)dt - \int_0^{2\pi} g(t)dt$$

$$\le -2\pi G(u) \to -\infty$$

which implies (A1).

(ii) $u \in \widetilde{H}^1_{2\pi}$, by Lemmas 1 and 2, we have

$$J(u(t)) = \frac{1}{2}\int_0^{2\pi} | \dot{u}(t) |^2 dt + \frac{1}{2}\int_0^{2\pi} (Au(t), \dot{u}(t))dt$$

$$- \int_0^{2\pi} F(t, u(t))dt + \int_0^{2\pi} (h(t), u(t))dt$$

$$\ge \frac{1}{2}(1 - \| A \|)\| \dot{u} \|^2_{L^2} - \int_0^{2\pi} a_0 b(t)(| u |/M)^{\gamma} dt - a_0\int_0^{2\pi} b(t)dt$$

$$+ \int_0^{2\pi} (h(t), u(t))dt$$

$$= \frac{1}{2}(1 - \| A \|)\| \dot{u} \|^2_{L^2} - 1/M^{\gamma}\int_0^{2\pi} a_0 b(t)| u |^{\gamma} dt - a_0\int_0^{2\pi} b(t)dt$$

$$+ \int_0^{2\pi} (h(t), u(t))dt$$

$$\ge \frac{1}{4}(1 - \| A \|)\| u \|^2_{L^2} - c_1\| u \|^{\gamma}_{L^2} - c_2$$

for some constants $c_1, c_2 > 0$. The above inequality implies that there exists some real constant $R > 0$ by $0 < \gamma < 2, 1 - \| A \| > 0$ such that

$$\sup_{u \in \overline{s}_R} J(u) < \inf_{u \in \widetilde{H}^1_{2\pi}} J(u),$$

where $\overline{s}_R = \{u \,|\, u \in \overline{H}^1_{2\pi}, | u | = R\}$.

Now by saddle point theorem we only need to prove that $J(u)$ satisfy condition (C) in [1], that is, (u_k) has a convergent sequence in $H^1_{2\pi}$ whenever $J(u_k)$ is bounded and $\| J'(u_k) \| (1 + \| u_k \|) \to 0$, as $k \to +\infty$.

Let $\{u_k(t)\} \in H^1_{2\pi}$ satisfying that $J(u_k)$ is bounded, $\| J'(u_k) \|$ $(1 + \| u_k \|) \to 0$, as $k \to +\infty$. Then there exists some constant c such that

$$| J(u_k) | \leq c, \quad \| J'(u_k) \| (1 + \| u_k \|) \leq c \quad (k \geq k_0). \tag{4}$$

So we have by assumption (A) and (3)

$$\begin{aligned}
3c &\geq \| J'(u_k) \| (1 + \| u_k \|) - 2J(u_k) \geq (J'(u_k), u_k) - 2J(u_k) \\
&= -\int_0^{2\pi} (Au_k(t), \dot{u}_k(t))dt \\
&\quad - \int_0^{2\pi} (\nabla F(t, u_k(t)), u_k(t))dt - \int_0^{2\pi} (Au_k(t), \dot{u}_k(t))dt \\
&\quad + \int_0^{2\pi} F(t, u_k(t))dt - \int_0^{2\pi} (h(t), u_k(t))dt \\
&= \int_0^{2\pi} [2F(t, u_k(t)) - (\nabla F(t, u_k(t)), u_k(t))]dt \\
&\quad - \int_0^{2\pi} (Au_k(t), \dot{u}_k(t))dt - \int_0^{2\pi} (h(t), u_k(t))dt + \int_0^{2\pi} (\dot{u}_k(t), Au_k(t))dt \\
&= \int_0^{2\pi} [2F(t, u_k(t)) - (\nabla F(t, u_k(t)), u_k(t))]dt - \int_0^{2\pi} (h(t), u_k(t))dt \\
&\geq (2 - \gamma) \int_0^{2\pi} F(t, u_k(t))dt - c_3,
\end{aligned}$$

for some constant $c_3 > 0$, the above inequality implies

$$\int_0^{2\pi} F(t, u_k(t))dt \leq c_4, \tag{5}$$

for some constant $c_4 > 0$. By (4) and (5), we have

$$c \geq J(u_k) = \frac{1}{2} \int_0^{2\pi} | \dot{u}_k |^2 dt + \frac{1}{2} \int_0^{2\pi} (Au_k(t), \dot{u}_k(t))dt$$

$$-\int_0^{2\pi} F(t, u_k(t))dt + \int_0^{2\pi} (h(t), u_k(t))dt$$

$$= \frac{1}{2}\int_0^{2\pi} |\dot{u}_k|^2 dt - \int_0^{2\pi} (A\dot{u}_k(t), u_k(t))dt + c_4$$

$$\geq \frac{1}{2}(1 - \|A\|)\int_0^{2\pi} |\dot{u}_k(t)|^2 dt + c_4.$$

Then we have

$$\int_0^{2\pi} |\dot{u}_k(t)|^2 dt \leq c_5.$$

It follows from Wirtinger's inequality that

$$\|\tilde{u}_k\|_\infty \leq c_5, \tag{6}$$

for some constant $c_5 > 0$. By (2) and Lemma 1,

$$c_4 \geq \int_0^{2\pi} F(t, u_k(t))dt \geq \int_0^{2\pi} G(u_k)dt + \int_0^{2\pi} g(t)dt$$

$$= \int_0^{2\pi} G(\overline{u}_k + \tilde{u}_k)dt - \int_0^{2\pi} g(t)dt$$

$$\geq 2\pi G(\overline{u}_k) - \int_0^{2\pi} (|\tilde{u}_k| + 4)dt - \int_0^{2\pi} g(t)dt$$

$$\geq 2\pi G(\overline{u}_k) - 2\pi(c_4 + 4) - \int_0^{2\pi} g(t)dt$$

which implies that $\{\overline{u}_k\}$ is bounded thus $\{u_k\}$ is bounded in $H^1_{2\pi}$ by (6). Hence $J(u)$ satisfy condition (C). Theorem holds.

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