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ON A THIRD SINGULAR DIFFERENTIAL OPERATOR AND TRANSMUTATION

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Abstract

In this work we are concerned with the operator, called generalized Airy operator, studied recently in great detail by Cholewinsky and Reneke [Electron. J. Differential Equations 2003(87) (2003), 1-64]. We complete the analysis presented by the previous authors and give special attention for some integral equations associated with this operator in establishing the correspondent transmutation theory in a suitable space. As application we introduce the generalized translation and the convolution product related to this operator.

1. Introduction

The generalized Airy operator is third singular differential operator given by

$$\begin{aligned} Ł_{\nu} &= \frac{d}{dx}\left(x^{-3\nu}\frac{d}{dx}x^{3\nu}\frac{d}{dx}\right) \\ &= \frac{d^3}{dx^3} + \frac{3\nu}{x}\frac{d^2}{dx^2} - \frac{3\nu}{x^2}\frac{d}{dx}, \end{aligned} \tag{1.1}$$

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where ν is a nonnegative real number. When $\nu = 0$, this operator becomes the third derivative operator for which some analysis were studied by Widder [15] and for some special value of ν the operator L_ν appeared as a radial part of the generalized Airy equation of a nonlinear diffusion type partial differential equation in $\mathbb{R}^n$.

Recently, in a nice and longer paper, Cholewinski and Reneke [2] study and extend, for the operator Ł_ν, the well known theory related to some singular differential operator of second order for which the literature is extensive. The authors establish many notions related to this operator such that the eigenfunctions, the generalized translation, the heat equation, the heat polynomials,... .

This work is devoted to present the analysis related to the operator L_ν in same manner as in ([4], [12], [7]). We begin by recalling the notion of the 3-even trigonometric functions (see [3] for more information) and show that they are linked with the eigenfunction of Ł_ν via an integral representation of Mehler type. This is useful for establishing the transmutation operator between Ł_ν and $\frac{d^3}{dx^3}$ and play a central role for the study of the so called Riemann-Liouville and Weyl transformations in suitable spaces. As application we study the generalized translation, product convolution and Fourier transform. Some other properties of the operator Ł_ν are given such that the integral representation of Sonine type and some recursive relations which will be used for the study of the perturbed operator in $\text{Ł}_\nu + \chi(x)$, where $\chi(\cdot)$ is an analytic function in a coming paper.

2. The 3-trigonometric Functions and the Eigenfunctions of Ł_ν

Putting $\mu = e^{i\pi/3}$, $w_k = e^{2i\pi(k-1)/3}$; $k = 1, 2, 3$, and $l = 1, 2$, we begin by recalling the following definition:

Definition 2.1. A function $f(z)$ is called 3-*even* if

$$f(w_k z) = f(z) \tag{2.1}$$

and 3-*odd of order* l if

$$f(z) = w_k^l f(w_k z). \tag{2.2}$$

We define the 3-trigonometric cosine function which is 3-even by

$$\cos_3(z) = \sum_{m\geq 0} (-1)^m \frac{z^{3m}}{(3m)!} = \sum_{m\geq 0} (-1)^m b_m(z) \tag{2.3}$$

and the 3-trigonometric sine functions of order l which are 3-odd by

$$\sin_3(z) = \sum_{m\geq 0} (-1)^m \frac{z^{3m+l}}{(3m+l)!}. \tag{2.4}$$

These functions are entire. (For more information the reader can consult [3].) Using the fact that

$$\sum_{k=1}^{3} (w_k)^m = \begin{cases} 3 & \text{for } m \text{ divisible by } 3, \\ 0 & \text{for } m \text{ not divisible by } 3 \end{cases} \tag{2.5}$$

and the fact that

$$\cos_3(z) = \frac{1}{3}\sum_{k=1}^{3} e^{\mu w_k z} = \frac{1}{3}\left(e^{-z} + 2e^{z/2} \cos \frac{\sqrt{3}}{2} z\right),$$

we deduce

$$\left| \frac{d^n}{dz^n} \cos_3(z) \right| \leq \frac{1}{3} \sum_{k=1}^{3} e^{|z|} \leq e^{|z|}, \quad n = 0, 1, 2, \ldots. \tag{2.6}$$

Finally note that, for a complex number λ, the function $\cos_3(\lambda z)$ is the unique solution of the equation

$$\frac{d^3}{dx^3} u = -\lambda^3 u \tag{2.7}$$

under the initial conditions

$$u(0) = 1, \quad u'(0) = 0, \quad u''(0) = 0. \tag{2.8}$$

The Fröbenius method leads that, for λ complex, the fundamental solutions of the equations

$$\text{Ł}_{\nu} u = \frac{d}{dx}\left(x^{-3\nu}\left(\frac{d}{dx} x^{3\nu} \frac{d}{dx}\right)\right) u = -\lambda^3 u \tag{2.9}$$

are

$$\mathcal{G}_{\nu}^{(1)}(\lambda x) = \sum_{n=0}^{\infty} \frac{(-\lambda x)^{3n}}{3^{3n} n!\,(1/3)_n (\nu + 2/3)_n}$$

$$= {}_0F_2\left[1/3,\ \nu + 2/3 \,|\, \left(\frac{-\lambda x}{3}\right)^3\right],$$

$$\mathcal{G}_{\nu}^{(2)}(\lambda x) = x^2 \sum_{n=0}^{\infty} \frac{(-\lambda x)^{3n}}{3^{3n} n!\,(5/3)_n (\nu + 4/3)_n}$$

$$= x^2\, {}_0F_2\left[5/3,\ \nu + 4/3 \,|\, \left(\frac{-\lambda x}{3}\right)^3\right],$$

$$\mathcal{G}_{\nu}^{(3)}(\lambda x) = x^{1-3\nu} \sum_{n=0}^{\infty} \frac{(-\lambda x)^{3n}}{3^{3n} n!\,(4/3-\nu)_n (2/3-\nu)_n}$$

$$= x^{1-3\nu}\, {}_0F_2\left[4/3-\nu,\ 2/3-\nu \,|\, \left(\frac{-\lambda x}{3}\right)^3\right].$$

The reader notices that for the third solutions we need $4/3 - \nu,\ 2/3 - \nu \neq 0, -1, -2, \ldots$ and it is easy to show that the radius of convergence of these series is $R = \infty$.

In the remainder only the first solution is considered and noted $\mathcal{G}_{\nu} = \mathcal{G}_{\nu}^{(1)}$. We have $\mathcal{G}_0(x) = \cos_3(x)$ and for λ complex, the function $x \to \mathcal{G}_{\nu}(\lambda x)$ is 3-even and it is the unique solution of the problem

$$\begin{cases} L_{\nu} u(x) = -\lambda^3 u(x) \\ u(0) = 1, \quad u'(0) = 0, \quad u''(0) = 0. \end{cases} \tag{2.10}$$

3. Integral Representations of Mehler and Sonine Type of $\mathcal{G}_{\nu}(\lambda x)$

We rewrite the entire function $\mathcal{G}_{\nu}(\lambda x)$ with the help of the following functions $b_{n,\nu}(x)$ which is crucial in establishing the generalized translation related to the operator Ł_{ν}:

$$\mathcal{G}_{\nu}(\lambda x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}\lambda^{3n}}{3^{3n}(1)_n(1/3)_n(\nu + 2/3)_n}$$

$$= \sum_{n=0}^{\infty} (-1)^n b_{n,\nu}(x)\lambda^{3n}, \tag{3.1}$$

where

$$b_{n,\nu}(x) = \frac{x^{3n}}{3^{3n}(1)_n(1/3)_n(\nu + 2/3)_n}. \tag{3.2}$$

Here we have used the Pochammer symbol $(a)_0 = 1,\ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, $n = 1, 2, \ldots$.

The $b_{n,\nu}(x)$ are $\mathcal{C}^{\infty}$ functions and satisfy for $n = 1, 2, \ldots$

$$b_{0,\nu}(x) = 1,\quad b_{n,\nu}(0) = 0,\quad \text{Ł}_{\nu} b_{n,\nu}(x) = b_{n-1,\nu}(x) \tag{3.3}$$

$$b_{n,\nu}(x) = b_{n,\nu}(1)x^{3n},\quad \frac{b_{n,\nu+p}}{b_{n,\nu}} = \frac{(\nu + 2/3)_n}{(\nu + p + 2/3)_n} \tag{3.4}$$

$$0 \le b_{n,\nu}(x) \le \frac{x^{3n}}{(3n)!} = b_{n,0}(x) = b_n(x). \tag{3.5}$$

We conclude this section by giving the two important integral representation.

Proposition 3.1. *For* $\nu > 0$ *and* $p = 1, 2, \ldots$ *the function* $\mathcal{G}_{\nu}(\lambda x)$ *has the integral representations*

(1) *of Mehler type*

$$\mathcal{G}_{\nu}(z) = \frac{3\Gamma(\nu + 2/3)}{\Gamma(2/3)\Gamma(\nu)} \int_0^1 t(1 - t^3)^{\nu-1} \cos_3(zt)dt \tag{3.6}$$

$$\mathcal{G}_{\nu}(x) = \frac{3\Gamma(\nu + 2/3)}{\Gamma(2/3)\Gamma(\nu)} x^{1-3\nu} \int_0^x y(x^3 - y^3)^{\nu-1} \cos_3(y)dy \tag{3.7}$$

(2) *of Sonine type*

$$\mathcal{G}_{\nu+p}(x) = \frac{3\Gamma(\nu + p + 2/3)}{\Gamma(\nu + 2/3)\Gamma(p)} \int_0^1 t^{3\nu+1}(1 - t^3)^{p-1} \mathcal{G}_\nu(xt)dt. \tag{3.8}$$

Proof. We give just the proof of (2) because that of (1) is similar. Taking account of the expansion (2.3) the second member of (3.6) becomes

$$\frac{3\Gamma(\nu + p + 2/3)}{\Gamma(\nu + 2/3)\Gamma(p)} \int_0^1 t^{3\nu} t(1 - t^3)^{p-1} \sum_{n=0}^{\infty} (-1)^n b_{n,\nu}(1) x^{3n} t^{3n} dt$$

$$= \sum_{n=0}^{\infty} (-1)^n b_{n,\nu}(1) x^{3n} \int_0^1 (t^3)^{n+\nu+1/3} (1 - t^3)^{p-1} 3dt \, \frac{\Gamma(\nu + p + 2/3)}{\Gamma(\nu + 2/3)\Gamma(p)}$$

$$= \sum_{n=0}^{\infty} (-1)^n b_{n,\nu}(1) x^{3n} \frac{\Gamma(n + \nu + 2/3)}{\Gamma(\nu + 2/3)} \frac{\Gamma(p + \nu + 2/3)}{\Gamma(n + \nu + 2/3 + p)}$$

$$= \sum_{n=0}^{\infty} (-1)^n b_{n,\nu+p}(1) x^{3n} = \mathcal{G}_{\nu+p}(x).$$

This follows from (3.4). The calculus is valid since the radius of convergence of the series is ∞ and we integrate on [0,1].

As a consequence of (3.6) we obtain

$$\left| \frac{d^n}{dz^n} \mathcal{G}_\nu(z) \right| \le \frac{3\Gamma(\nu + 2/3)}{\Gamma(2/3)\Gamma(\nu)} e^{|z|} \tag{3.9}$$

and

$$b_{n,\nu}(x) = \frac{3\Gamma(\nu + 2/3)}{\Gamma(2/3)\Gamma(\nu)} x^{1-3\nu} \int_0^x y(x^3 - y^3)^{\nu-1} \frac{y^{3n}}{(3n)!} dy. \tag{3.10}$$

4. $Ł_\nu$-Riemann-Liouville Integral Transformation

Notation. We denote by $\mathcal{E}_*(\mathbb{R})$ the space of the $\mathcal{C}^\infty$ and 3-even functions defined on $\mathbb{R}$, equipped with the topology of uniform convergence for the functions on compact supports and its derivatives.

Our objective in this section is to construct the integral transformation called the Riemann-Liouville integral transformation which transmutes Ł_ν and $\dfrac{d^3}{dx^3}$.

Definition 4.1. We define on $\mathcal{E}_*(\mathbb{R})$, the *Riemann-Liouville integral transformation associated with the operator* Ł_ν by

$$\mathcal{X}(f)(x) = \begin{cases} \dfrac{3\Gamma(\nu + 2/3)}{\Gamma(2/3)\Gamma(\nu)} x^{1-3\nu} \int_0^x y(x^3 - y^3)^{\nu-1} f(y) dy & \text{if } x > 0, \\ f(0) & \text{if } x = 0. \end{cases} \tag{4.1}$$

Theorem 4.1. *The transformation* $\mathcal{X}$ *is an isomorphism of* $\mathcal{E}_*(\mathbb{R})$ *transmuting the operator* Ł_ν *and* $\dfrac{d^3}{dx^3}$ *in the following sense*:

$$\text{Ł}_\nu \mathcal{X}(f) = \mathcal{X} D^3(f), \quad f \in \mathcal{E}_*(\mathbb{R}),$$

$$\mathcal{X}(f)(0) = f(0), \quad f \in \mathcal{E}_*(\mathbb{R}). \tag{4.2}$$

Proof. Note that $W_\nu(t) = t(1 - t^3)^{\nu-1}$. For $f \in \mathcal{E}_*(\mathbb{R})$, we can rewrite

$$\mathcal{X}(f)(x) = c_\nu \int_0^1 W_\nu(t)\, f(xt) dt = c_\nu \int_0^1 t(1 - t^3)^{\nu-1} f(xt) dt,$$

where we have kept

$$c_\nu = \frac{3\Gamma(\nu + 2/3)}{\Gamma(2/3)\Gamma(\nu)}. \tag{4.3}$$

Put

$$I(x) = \frac{1}{c_\nu}(\text{Ł}_\nu \mathcal{X}(f)(x) - \mathcal{X} D^3(f)(x)).$$

We have

$$I(x) = -\int_0^1 (1 - t^3) W_\nu(t) D^3 f(xt) dt + \frac{3\nu}{x} \int_0^1 W_\nu(t) D_x^2 f(xt) dt$$

$$- \frac{3\nu}{x^2} \int_0^1 W_\nu(t) D_x f(xt) dt.$$

Using the fact that

$$D_t[(1-t^3)W_\nu(t)] = (1-t^3)^\nu - 3\nu t^2 W_\nu(t)$$

on two integrations by parts of the first quantity of the second member we obtain

$$tD_t f(xt) = xD_x f(xt); \quad t^2 D_t^2 f(xt) = x^2 D_x^2 f(xt)$$

and

$$Df(0) = 0; \quad D^2 f(0) = 0.$$

This shows that for $x \in \mathbb{R}^*$, $I(x) = 0$.

Now we put

$$\frac{d}{dx^3} = \frac{d}{3x^2 dx}$$

and we introduce the space

$$\mathcal{M} = \{f \in \mathcal{E}_*(\mathbb{R}); f^{(3k)}(0) = 0, k \in \mathbb{N}\}.$$

We attempt to give explicitly $\mathcal{X}^{-1}$. To this end we proceed as in [12] in establishing the following lemma.

Lemma 4.2. *The operator* Q_k, $k \in \mathbb{N}$, *defined on* $\mathcal{E}_*(\mathbb{R})$ *by*

$$Q_k(f)(t) = \frac{3}{k!}\int_0^t y(t^3 - y^3)^k f(y)dy \tag{4.4}$$

is bijective on the space $\mathcal{E}_*(\mathbb{R})$ *of* $\mathcal{C}^\infty$ *functions on* $\mathbb{R}$, *having derivatives* $f^{(n)}(0) = 0$, $n = 0, 1, \ldots, 3k$. *Moreover the inverse operator is given by*

$$Q_k^{-1}(f)(t) = t\left(\frac{d}{dt^3}\right)^{k+1}(f)(t).$$

Proof. Put

$$F_0(t) = tf(t).$$

For $k = 1, 2, \ldots$

$$F_k(t) = t^2 \int_0^t F_{k-1}(u)du, \quad \frac{d}{t^2 dt} F_k(t) = F_{k-1}(t)$$

so

$$\left(\frac{d}{t^2 dt}\right)^k (F_k)(t) = tf(t), \quad k = 0, 1, 2, \ldots,$$

hence the operator Q_k becomes

$$Q_k(f)(t) = \frac{3}{k!} \int_0^t y(t^3 - y^3)^k f(y) dy$$

$$= \frac{3}{k!} \int_0^t (t^3 - y^3)^k \left(\frac{d}{y^2 dy}\right)^k (F_k)(y) dy$$

and k integrations by parts give

$$Q_k(f)(t) = (-1)^k \frac{3}{k!} \int_0^t \left(\frac{d}{y^2 dy}\right)^k [(t^3 - y^3)^k](F_k)(y) dy$$

$$= 3^{k+1} \int_0^t (F_k)(y) dy.$$

This gives that

$$f(t) = t\left(\frac{d}{dt^3}\right)^{k+1} Q_k(f)(t).$$

Theorem 4.3. *The operator* $\mathcal{X}^{-1}$ *is given as follows*:

(1) *If* $\nu = k + r,\ k \in \mathbb{N},\ 0 < r < 1,$ *then for* $t > 0,$

$$\mathcal{X}^{-1}(f)(t) = C_1 t\left(\frac{d}{dt^3}\right)^{k+1} \int_0^t \frac{y^{3\nu+1}}{(t^3 - y^3)^r} (f(y)) dy,$$

where

$$C_1 = \frac{3\Gamma(2/3)\Gamma(\nu - k)}{\Gamma(\nu)\Gamma(\nu + 2/3)\Gamma(1 - \nu)}.$$

(2) *If* $\nu = k + 1$, $k \in \mathbb{N}$, *then for* $t > 0$,

$$\mathcal{X}^{-1}(f)(t) = C_2 t \left(\frac{d}{dt^3}\right)^{k+1} (t^{3k+2}(f)(t)),$$

where

$$C_2 = \frac{\Gamma(2/3)}{\Gamma(\nu + 2/3)}.$$

Proof. We need to solve the integral equation with f as an unknown function

$$c_\nu t^{1-3\nu} \int_0^t y(t^3 - y^3)^{\nu-1} f(y) dy = g(t), \quad f, g \in \mathcal{E}_*(\mathbb{R}), \tag{4.5}$$

where c_ν is given by (4.3).

We proceed step by step

(1) For $0 < \nu < 1$

$$\int_0^t \frac{3y^2}{(t^3 - y^3)^\nu} \left(\int_0^y u(y^3 - u^3)^{\nu-1} f(u) du \right) dy = \frac{1}{c_\nu} \int_0^t \frac{3y^{3\nu+1}}{(t^3 - y^3)^\nu} g(y) dy.$$

By the use of the Fubini theorem, we obtain

$$\int_0^t \left(\int_u^t (t^3 - y^3)^{-\nu} (y^3 - u^3)^{\nu-1} 3y^2 dy \right) f(u) u \, du = \frac{1}{c_\nu} \int_0^t \frac{3y^{3\nu+1}}{(t^3 - y^3)^\nu} g(y) dy$$

since

$$\int_u^t (t^3 - y^3)^{-\nu} (y^3 - u^3)^{\nu-1} 3y^2 dy = \Gamma(1-\nu)\Gamma(\nu),$$

$$t\, f(t) = \frac{\Gamma(2/3)}{\Gamma(\nu + 2/3)\Gamma(1-\nu)} \frac{d}{dt} \left(\int_0^t \frac{y^{3\nu+1}}{(t^3 - y^3)^\nu} g(y) dy \right).$$

(2) If $\nu = k + r$, $k \in \mathbb{N}$, $0 < r < 1$, then we have for $t > 0$:

The integral equation (4.5) can be rewritten:

$$g(t) = c_\nu t^{1-3\nu} \int_0^t (t^3 - y^3)^{r+k-1} y\, f(y) dy, \quad f, g \in \mathcal{E}_*(\mathbb{R}),$$

integration by parts give

$$\frac{\Gamma(\nu+(2/3))}{\Gamma((2/3))\Gamma(\nu-k)}t^{1-3\nu}\int_0^t (t^3-y^3)^{r-1}\frac{d}{dy}(Q_k(f)(y))dy = g(t)$$

using (1) we have

$$Q_k(f)(t) = \frac{3\Gamma(\nu-k)\Gamma(2/3)}{\Gamma(\nu+(2/3))\Gamma(\nu)\Gamma(1-\nu)}\int_0^t \frac{y^{3\nu+1}}{(t^3-y^3)^r}g(y)dy.$$

The result is then the consequence of Lemma 4.2

(3) If $\nu = k+1$, since

$$\mathcal{X}(f)(t) = c_{k+1}t^{-3k-2}\int_0^t y(t^3-y^3)^k f(y)dy$$

$$= \frac{k!}{3}c_{k+1}t^{-3k-2}Q_k(f)(t).$$

Lemma 4.2 gives the result.

5. Weyl Integral Transformation associated with $Ł_\nu$

Notation. We denote by

- $\mathcal{D}_*(\mathbb{R})$ the space of $\mathcal{C}^\infty$ functions on $\mathbb{R}$, 3-even with compact support. It is known that

$$\mathcal{D}_*(\mathbb{R}) = \bigcup_{a>0}\mathcal{D}_a(\mathbb{R}),$$

where $\mathcal{D}_a(\mathbb{R})$ is the space of $\mathcal{C}^\infty$ functions on $\mathbb{R}$, 3-even with support in $[-a, a]$, $a>0$ equipped with the topology defined by the sequence of semi-norms.

$$q_n(f) = \sup_{0\le k\le n,\, x\in[-a,\, a]}|D^k f(x)|.$$

- $\mathcal{D}'_*(\mathbb{R})$ the space of distributions 3-even on $\mathbb{R}$.

- $\mathcal{E}'_*(\mathbb{R})$ the space of distributions 3-even on $\mathbb{R}$, with compact support.

Our objective is to construct for the operator $Ł_\nu$ an integral transform noted ${}^t\mathcal{X}(\cdot)$, called of Weyl type. We denote by T_f the distribution associated to the function f. As a direct consequence of the Theorem 4.1 we have

Theorem 5.1. *The operator ${}^t\mathcal{X}(f)$ defined on $\mathcal{E}'_*(\mathbb{R})$ by*

$$\langle {}^t\mathcal{X}\mathrm{T},\, g\rangle = \langle \mathrm{T},\, \mathcal{X}g\rangle; \quad g \in \mathcal{E}_*(\mathbb{R}) \tag{5.1}$$

is an isomorphism of $\mathcal{E}'_*(\mathbb{R})$.

The following theorem explicates the expression of ${}^t\mathcal{X}\mathrm{T}_f$.

Theorem 5.2. *For f in $\mathcal{D}_*(\mathbb{R})$ the distribution ${}^t\mathcal{X}\mathrm{T}_f$ is defined by the function ${}^t\mathcal{X}(f)$ given by*

$$ {}^t\mathcal{X}(f)(y) = \frac{3\Gamma(\nu + 2/3)}{\Gamma(2/3)\Gamma(\nu)} \int_y^\infty y(t^3 - y^3)^{\nu-1} t^{1-3\nu} f(t)dt. \tag{5.2}$$

Proof. For $g \in \mathcal{E}_*(\mathbb{R})$, we have

$$\begin{aligned}
\langle {}^t\mathcal{X}\mathrm{T}_f,\, g\rangle &= \left\langle \mathrm{T}_f,\, \mathcal{X}g\right\rangle \\
&= \int_0^\infty \mathcal{X}(g)(t) f(t)dt \\
&= \int_0^\infty f(t)\left(c_\nu t^{1-3\nu}\int_0^t y(t^3 - y^3)^{\nu-1} g(y)dy\right)dt \\
&= \int_0^\infty \left(c_\nu \int_y^\infty y(t^3 - y^3)^{\nu-1} t^{1-3\nu} f(t)dt\right) g(y)dy.
\end{aligned}$$

From the dominated convergence theorem we deduce that the map $y \mapsto {}^t\mathcal{X}(f)(y)$ is 3-even and continuous on $[0, \infty[$.

Remark 5.1. The integral transformation (5.2) is called the Weyl transformation related with the operator $Ł_\nu$.

Theorem 5.3. *For* $f \in \mathcal{D}_*(\mathbb{R})$, *the distribution* $({}^t\mathcal{X})^{-1}\mathrm{T}_f$ *defined by the function* $({}^t\mathcal{X})^{-1}(f)$ *given by*

(1) *If* $\nu = k + r$, $k \in \mathbb{N}$, $0 < r < 1$, *then*

$$({}^t\mathcal{X})^{-1}(f)(y) = (-1)^{k+1}C_1\int_y^\infty \frac{y^{3\nu+1}}{(z^3-y^3)^r}\left(\frac{d}{dz^3}\right)^{k+1}(f(z)z)dz.$$

(2) *If* $\nu = k + 1$, $k \in \mathbb{N}$, then

$$({}^t\mathcal{X})^{-1}(f)(y) = (-1)^{k+1}C_2 y^{3k+2}\left(\frac{d}{dy^3}\right)^{k+1}(f(y)y).$$

The constants C_1 *and* C_2 *are defined as in Theorem* 4.3. On the other hand, *the function* $({}^t\mathcal{X})^{-1}(f)(y)$ *is continuous on* $\mathbb{R}$, 3-*even with compact support*.

Proof. (1) Let $\nu = k + r$, $k \in \mathbb{N}$, $0 < r < 1$, and let $g \in \mathcal{E}_*(\mathbb{R})$. Then Theorem 4.3 gives

$$\begin{aligned}\langle({}^t\mathcal{X}^{-1})\mathrm{T}_f, g\rangle &= \langle{}^t(\mathcal{X}^{-1})\mathrm{T}_f, g\rangle \\ &= \langle\mathrm{T}_f, \mathcal{X}^{-1}g\rangle \\ &= C_1\int_0^\infty f(z)z\left(\frac{d}{dz^3}\right)^{k+1}\left(\int_0^z \frac{y^{3\nu+1}}{(z^3-y^3)^r}g(y)dy\right)dz.\end{aligned}$$

On integration by parts, we obtain

$$\langle({}^t\mathcal{X}^{-1})\mathrm{T}_f, g\rangle = (-1)^{k+1}C_1\int_0^\infty\left(\int_y^\infty \frac{y^{3\nu+1}}{(z^3-y^3)^r}\left(\frac{d}{dz^3}\right)^{k+1}(f(z)z)dz\right)g(y)dy.$$

So the distribution $({}^t\mathcal{X}^{-1})\mathrm{T}_f$ is well defined by the function $({}^t\mathcal{X}^{-1})f$, given by

$$({}^t\mathcal{X})^{-1}(f)(y) = (-1)^{k+1}C_1\int_y^\infty \frac{y^{3\nu+1}}{(z^3 - y^3)^r}\left(\frac{d}{dz^3}\right)^{k+1}(f(z)z)dz$$

$$= (-1)^{k+1}C_1\int_1^\infty \frac{(y)^{3\nu+1}}{((yv)^3 - y^3)^r}\left(\frac{d}{y^3dv^3}\right)^{k+1}(f(yv)yv)y\,dv$$

and the function $({}^t\mathcal{X})^{-1}(f)$ is 3-even with compact support on $\mathbb{R}$.

(2) If $\nu = k+1$, $k \in \mathbb{N}$, then Theorem 4.3 and same argument as (1) leads to the result.

Lemma 5.4. *For* $f \in \mathcal{D}_*(\mathbb{R})$, *the operators* ${}^t\mathcal{X}f$ *and* $({}^t\mathcal{X}^{-1})f$ *satisfy the relations*

$${}^t\mathcal{X}\,\mathrm{T}_{\text{Ł}_{-\nu}f} = D^3\,{}^t\mathcal{X}\,\mathrm{T}_f,$$

$$\text{Ł}_{-\nu}(({}^t\mathcal{X})^{-1})\mathrm{T}_f = ({}^t\mathcal{X})^{-1}D^3\mathrm{T}_f.$$

Proof. From Theorem 4.1, we obtain

$$\text{Ł}_\nu\mathcal{X}(g) = \mathcal{X}D^3(g), \quad g \in \mathcal{E}_*(\mathbb{R}).$$

Then for $f \in \mathcal{D}_*(\mathbb{R})$, we have

$$\langle \mathrm{T}_f, \text{Ł}_\nu\mathcal{X}g\rangle = -\langle \mathrm{T}_{\text{Ł}_{-\nu}f}, \mathcal{X}g\rangle = -\langle {}^t\mathcal{X}\,\mathrm{T}_{\text{Ł}_{-\nu}f}, g\rangle,$$

and

$$\langle \mathrm{T}_f, \mathcal{X}D^3g\rangle = \langle {}^t\mathcal{X}\,\mathrm{T}_f, D^3g\rangle = -\langle D^{3t}\mathcal{X}\,\mathrm{T}_f, g\rangle,$$

hence

$${}^t\mathcal{X}\,\mathrm{T}_{\text{Ł}_{-\nu}f} = D^{3t}\mathcal{X}\,\mathrm{T}_f.$$

A similar calculation shows that

$$\langle \text{Ł}_{-\nu}({}^t\mathcal{X})^{-1}\mathrm{T}_f, g\rangle = -\langle ({}^t\mathcal{X})^{-1}\mathrm{T}_f, \text{Ł}_\nu g\rangle = -\langle \mathrm{T}_f, \mathcal{X}^{-1}\text{Ł}_\nu g\rangle,$$

and

$$\langle ({}^t\mathcal{X})^{-1}D^3\mathrm{T}_f, (g)\rangle = \langle D^3\mathrm{T}_f, \mathcal{X}^{-1}g\rangle = -\langle \mathrm{T}_f, D^3\mathcal{X}^{-1}g\rangle,$$

so

$$\text{Ł}_{-\nu}(({}^t\mathcal{X})^{-1}\mathrm{T}_f) = ({}^t\mathcal{X})^{-1}D^3\mathrm{T}_f.$$

Theorem 5.5. *The Weyl integral transformation associated to* Ł_ν:

$$f \to {}^t\mathcal{X}(f),$$

is an isomorphism of $\mathcal{D}_*(\mathbb{R})$ *into itself.*

Proof. From Theorems 5.2, 5.3 and Lemma 5.4 we deduce that for f in $\mathcal{D}_*(\mathbb{R})$, the functions $({}^t\mathcal{X})(f)$ and $({}^t\mathcal{X})^{-1}(f)$ belong in $\mathcal{D}_*(\mathbb{R})$. Theorem 5.1 gives that the transformation $f \to {}^t\mathcal{X}(f)$, is bijective. It remains to show that it is continuous: For this, let f be a function of $\mathcal{D}_*(\mathbb{R})$ with compact support in $[-a, a]$. From Lemma 5.4, we have

$$D^{3n}\,{}^t\mathcal{X}(f)(y) = {}^t\mathcal{X}(\text{Ł}_{-\nu}^n f)(y)$$

so

$$|D^{3n}\,{}^t\mathcal{X}(f)(y)| \le \sup_{t\in[-a,\, a]} |(\text{Ł}_{-\nu}^n f(t))| \left|\int_y^a y(t^3 - y^3)^{\nu-1} t^{1-3\nu} f(t)dt\right.$$

$$\le M_1 \sup_{t\in[-a,\, a]} |(\text{Ł}_{-\nu}^n f(t))|.$$

Since

$$\sup_{t\in[-a, a]} |(\text{Ł}_{-\nu}^n f(t))| \le M_1 \sup_{0\le k\le 3n,\, t\in[-a, a]} |(D^k f(t))|$$

we obtain

$$\sup_{y\in[-a, a]} |D^{3n}\ {}^t\mathcal{X}(f)(y)| \le M_2 q_{3n}(f),$$

where M_2 is a constant and $q_{3n}(f)$ is the semi-norm defined previously.

This proves the continuity of the transformation $f \to {}^t\mathcal{X}(f)$ on $\mathcal{D}_*(\mathbb{R})$. We conclude that the inverse transformation $({}^t\mathcal{X})^{-1}(f)$ is continuous.

6. Fourier Transform of Distribution T on $\mathcal{E}'_*(\mathbb{R})$

In this section we denote by

- H_* the space of 3-even entire functions which decrease rapidly of exponential type. We have

$$\mathrm{H}_* = \bigcup_{a>0} H_a,$$

with

$$H_a = \begin{cases} \text{entire functions, 3-even such that for all integer } m, \\ \rho_m(\psi) = \sup_{\lambda\in\mathbb{C}} |(1+|\lambda|^3)^m \psi(\lambda)e^{-a|Im\lambda|}| < +\infty. \end{cases}$$

We equip H_a with the topology defined by the sequence of the semi-norms ρ_m.

- $\mathcal{H}_*$ the space of entire functions 3-even with slowly growth of exponential type. This means that there exists a positive real number a and an integer m such that

$$\sup_{\lambda\in\mathbb{C}} |(1+|\lambda|^3)^{-m} \psi(\lambda)e^{-a|Im\lambda|}| < +\infty.$$

Definition 6.1. (1) The Fourier transform related with $Ł_\nu$ of a distribution T of $\mathcal{E}'_*(\mathbb{R})$ is the function $\mathcal{F}_\nu(\mathrm{T})$ defined by

$$\mathcal{F}_\nu(\mathrm{T})(\lambda) = \langle \mathrm{T}, \mathcal{G}_\nu(\lambda t)\rangle. \tag{6.1}$$

(2) The Fourier transform related with $Ł_\nu$ of a function f of $\mathcal{D}_*(\mathbb{R})$ is the function $\mathcal{F}_\nu(f)$ defined by

$$\mathcal{F}_\nu(f)(\lambda) = \int_0^\infty f(t)\mathcal{G}_\nu(\lambda t)dt. \tag{6.2}$$

Noting that

$$\mathcal{F}_0(f)(\lambda) = \int_0^\infty f(t)\cos_3(\lambda t)dt \tag{6.3}$$

we obtain the following

Theorem 6.1. *The Fourier transform* $\mathcal{F}_0$ *is a linear map bijective and bicontinuous*

(1) *of* $\mathcal{D}_*(\mathbb{R})$ *on* H_*.

(2) *of* $\mathcal{E}'_*(\mathbb{R})$ *on* $\mathcal{H}_*$.

Now it is easy to obtain

Proposition 6.2. *We have*

$$\begin{cases} \textit{for } \mathrm{T} \in \mathcal{E}'_*(\mathbb{R});\ \mathcal{F}_\nu(\mathrm{T}) = \mathcal{F}_0 \circ {}^t\mathcal{X}(\mathrm{T}), \\ \textit{for } \mathrm{T} \in \mathcal{E}'_*(\mathbb{R});\ \mathcal{F}_0(\mathrm{T}) = \mathcal{F}_\nu \circ ({}^t\mathcal{X})^{-1}(\mathrm{T}). \end{cases} \tag{6.4}$$

$$\begin{cases} \textit{for } f \in \mathcal{D}_*(\mathbb{R});\ \mathcal{F}_\nu(f) = \mathcal{F}_0 \circ {}^t\mathcal{X}(f), \\ \textit{for } f \in \mathcal{D}_*(\mathbb{R});\ \mathcal{F}_0(f) = \mathcal{F}_\nu \circ ({}^t\mathcal{X})^{-1}(f). \end{cases} \tag{6.5}$$

A consequence of Theorems 5.1, 5.5, 6.1 and Proposition 6.2 is

Theorem 6.3. *The Fourier transform* $\mathcal{F}$ *is a linear map bijective and bicontinuous*

(1) *of* $\mathcal{D}_*(\mathbb{R})$ *on* H_*.

(2) *of* $\mathcal{E}'_*(\mathbb{R})$ *on* $\mathcal{H}_*$.

7. Translation Operator T_x^ν associated with $Ł_\nu$

In this section we study the generalized translation operator associated with the operator $Ł_\nu$ in the Delsartes sense. We begin by giving the following definition related to $D^3 = \frac{d^3}{dx^3}$.

Definition 7.1. The translation operator τ_x, $x \in \mathbb{R}$ associated with the third order derivative operator D^3 is defined for f in $\mathcal{E}_*(\mathbb{R})$ and $y \in \mathbb{R}$ by

$$\tau_x(f)(y) = \sum_{n=0}^{\infty} b_{n,0}(y) D^{3n} f(x). \tag{7.1}$$

The function $b_{n,0}(y) = b_n(y)$ are given by (2.3).

The reader can show easily that it preserves the 3-parity and we have

$$\tau_x(y^{3n}) = \tau_y(x^{3n}) = \sum_{k=0}^{n} \begin{bmatrix} b_{n,0} \\ b_{k,0} \end{bmatrix} y^{3k} x^{3(n-k)}$$

and

$$\begin{bmatrix} b_{n,0} \\ b_{k,0} \end{bmatrix} = \frac{b_{k,0}(1) b_{n-k,0}(1)}{b_{n,0}(1)}. \tag{7.2}$$

This translation operator acts on a function as

$$\tau_x(f) = \frac{1}{3}(f(x + w_1 y) + f(x + w_2 y) + f(x + w_3 y)),$$

where w_k being the third roots of unity, and we have the product formula

$$\cos_3(\lambda x) \cdot \cos_3(\lambda y) = \tau_x \cos_3(\lambda y) = \tau_y \cos_3(\lambda x).$$

Proposition 7.1. *The operator* τ_x *satisfy*

(1) *For* $x \in \mathbb{R}$, τ_x *belong in* $\mathcal{L}(\mathcal{E}_*(\mathbb{R}), \mathcal{E}_*(\mathbb{R}))$

(2) *The map* $x \to \tau_x$ *is indefinitely differentiable*, 3-*even*.

(3) *We have*

$$\tau_0 = \text{unity operator},$$

$$D_x^3 \tau_x = \tau_x D_x^3, \quad \text{where} \quad D = \frac{d}{dx}.$$

(4) *For* $f \in \mathcal{D}_*(\mathbb{R})$ *we have*

$$\mathcal{F}_0({}^t\tau_x)(f)(\lambda) = \cos_3(\lambda x)\mathcal{F}_0(f)(\lambda).$$

Definition 7.2. The *convolution product* of two functions f and g of $\mathcal{D}_*(\mathbb{R})$ is defined by

$$f \star g(x) = \int_0^\infty {}^t\tau_x f(y) g(y) dy = \int_0^\infty f(y) \tau_x g(y) dy. \tag{7.3}$$

Proposition 7.2. *For f and g in* $\mathcal{D}_*(\mathbb{R})$,

$$\mathcal{F}_0(f \star g)(\lambda) = \mathcal{F}_0(f) \cdot \mathcal{F}_0(g). \tag{7.4}$$

These previous properties can be extended for the operator L_ν which suggest the following definition.

Definition 7.3. The operators $\mathrm{T}^\nu_{x,}$ $x \in \mathbb{R}$, defined on $\mathcal{E}_*(\mathbb{R})$ by

$$\mathrm{T}^\nu_x(f)(y) = \sum_{n=0}^{\infty} b_{n,\nu}(y) Ł^n_\nu(f)(x), \quad y \in \mathbb{R}, \tag{7.5}$$

are called *generalized translation operators associated with* $Ł_\nu$, where the functions $b_{n,\nu}(y)$ are given by (3.1).

As a direct consequence of this definition we deduce that the function $u(x, y) = \mathrm{T}^\nu_x(y^{3n})$ is the solution of

$$\begin{cases} Ł_{x,\nu} u(x, y) = Ł_{y,\nu} u(x, y) & x^{3n} \in \mathcal{E}_*(\mathbb{R}), \\ u(x, 0) = x^{3n}, \quad u(0, y) = y^{3n} & y^{3n} \in \mathcal{E}_*(\mathbb{R}) \end{cases} \tag{7.6}$$

and

$$\mathrm{T}^\nu_x(y^{3n}) = x^{3n} {}_3F_2\left[\begin{matrix} -n, -n+2/3, -n+1/3-\nu \\ 1/3, \nu+2/3 \end{matrix} \middle| -\left(\frac{y}{x}\right)^3\right]$$

$$= (x^3+y^3)^n {}_3F_2\left[\begin{matrix} -n/2, \frac{1-n}{2}, -n+\nu \\ 1/3, \nu+2/3 \end{matrix} \middle| \frac{4(xy)^3}{(x^3+y^3)^2}\right]. \tag{7.7}$$

When y is fixed and $x \to \infty$, we have (see [2, p. 4])

$$\mathrm{T}^\nu_x(y^{3n}) \sim (x^3+y^3)^n.$$

Now we prove the connection between the translate operator (7.5) and the transmutation operator (4.1).

Proposition 7.3. *For* $f \in \mathcal{D}_*(\mathbb{R})$, x *and* $y \in \mathbb{R}$,

$$\mathrm{T}^\nu_x f(y) = \mathcal{X}_x \mathcal{X}_y[\tau_x \mathcal{X}^{-1} f(y)].$$

Proof. This can be shown as follows

$$\mathrm{T}_x^{\nu} f(y) = \sum_{n=0}^{\infty} b_{n,\nu}(x) Ł_{\nu}^{n}(f)(y)$$

$$= \sum_{n=0}^{\infty} \mathcal{X}_x b_n(x) Ł_{\nu}^{n} \mathcal{X}_y (\mathcal{X}_y^{-1}(f))(y)$$

$$= \sum_{n=0}^{\infty} \mathcal{X}_x \mathcal{X}_y b_n(x) D^{3n} (\mathcal{X}_y^{-1}(f))(y).$$

We summarize the properties of T_x^{ν} in the proposition.

Proposition 7.4. *The operators* T_x^{ν} *satisfy*

(1) *For* $x \in \mathbb{R}$, T_x^{ν} *in* $\mathcal{L}(\mathcal{E}_*(\mathbb{R}), \mathcal{E}_*(\mathbb{R}))$

(2) *The map* $x \to \mathrm{T}_x^{\nu}$ *is infinitely differentiable*, 3*-even.*

(3) *For all functions f in* $\mathcal{E}_*(\mathbb{R})$,

$$\mathrm{T}_x^{\nu} f(y) = \mathrm{T}_y^{\nu} f(x);$$

$$\mathrm{T}_0^{\nu} f(y) = f(y)$$

(4) *For given f in* $\mathcal{E}_*(\mathbb{R})$, *put*

$$u(x, y) = \mathrm{T}_x^{\nu} f(y).$$

Then the function u is solution of the Cauchy problem:

$$(\mathbb{I}) \qquad \begin{cases} Ł_{x,\nu} u(x, y) = Ł_{y,\nu} u(x, y), \\ u(x, 0) = f(x); \ \dfrac{d}{dy} u(x, 0) = 0; \ \dfrac{d^2}{dy^2} u(x, 0) = 0. \end{cases}$$

(5) *The product formula holds*

$$\mathrm{T}_x^{\nu} \mathcal{G}_{\nu}(\lambda y) = \mathcal{G}_{\nu}(\lambda x) \mathcal{G}_{\nu}(\lambda y) = \mathrm{T}_y^{\nu} \mathcal{G}_{\nu}(\lambda x). \tag{7.8}$$

Proof. These results are consequences of the properties of the operators $\mathcal{X}$, $\mathcal{X}^{-1}$ and Proposition 7.1.

Theorem 7.5. *For f in* $\mathcal{D}_*(\mathbb{R})$,

$$ {}^t\mathrm{T}_x^{\nu} f(y) = \mathcal{X}_x ({}^t\mathcal{X})_y^{-1} [\tau_{-x} {}^t\mathcal{X}(f)(y)], $$

where the operator ${}^t\mathrm{T}_x$ *is the transposed of the operator* T_x.

Proof. For f and g in $\mathcal{D}_*(\mathbb{R})$,

$$ \langle \mathrm{T}_x^{\nu} f(y),\ g(y)\rangle = \left\langle \mathcal{X}_x \mathcal{X}_y \sum_{n=0}^{\infty} b_n(x) D^{3n} (\mathcal{X}^{-1} f(y)),\ g(y) \right\rangle $$

$$ = \left\langle f(y),\ \mathcal{X}_x \sum_{n=0}^{\infty} (-1)^n b_n(x) {}^t\mathcal{X}_y^{-1} D^{3n}\, {}^t\mathcal{X}_y g(y) \right\rangle $$

$$ = \left\langle f(y),\ \mathcal{X}_x \sum_{n=0}^{\infty} b_n(-x) {}^t\mathcal{X}_y^{-1} D^{3n}\, {}^t\mathcal{X}_y g(y) \right\rangle $$

and thus the result follows.

As an important deduction of the previous theorem we have the following expression of the convolution product $\star$ related to the operator D^3.

Corollary 7.6. *For* $f \in \mathcal{D}_*(\mathbb{R})$ *and* $x > 0$,

$$ {}^t\mathrm{T}_x^{\nu} f(y) = ({}^t\mathcal{X}_y^{-1})[{}^t\mathcal{X}(f) \star k(x, \cdot)](y), \tag{7.9} $$

where $k(t, y) = c_\nu t^{1-3\nu} y(t^3 - y^3)^{\nu-1}$ *and* c_ν *is given by* (4.3).

Proof. Let f be in $\mathcal{D}_*(\mathbb{R})$ and $x > 0$. Then

$$ {}^t\mathrm{T}_x^{\nu} f(y) = \mathcal{X}_x ({}^t\mathcal{X})_y^{-1} [{}^t\tau_y {}^t\mathcal{X}(f)(x)] $$

$$ = \int_0^x k(x, u) ({}^t\mathcal{X})_y^{-1}\ {}^t\tau_y {}^t\mathcal{X}(f)(u) du $$

$$= ({}^t\mathcal{X})_y^{-1}\int_0^x k(x, u)\,{}^t\tau_y\,{}^t\mathcal{X}(f)(u)du$$

$$= ({}^t\mathcal{X}_y^{-1})[{}^t\mathcal{X}(f)\star k(x, \cdot)](y).$$

Corollary 7.7. *Let f be in $\mathcal{D}_*(\mathbb{R})$. Then the function $y \to {}^t\mathrm{T}_x^\nu f(y)$ belong in $\mathcal{D}_*(\mathbb{R})$ and*

$$\mathcal{F}_\nu({}^t\mathrm{T}_x^\nu f)(\lambda) = \mathcal{G}_\nu(\lambda x)\mathcal{F}_\nu(f)(\lambda). \tag{7.10}$$

Proof. Since ${}^t\mathrm{T}_0^\nu f(y) = f(y)$, the result follows by inspection for $x = 0$. Let us prove the result for $x > 0$. From the Corollary 7.6 it is easy to see that the function $y \to {}^t\mathrm{T}_x^\nu f(y)$ belong in $\mathcal{D}_*(\mathbb{R})$. From Theorem 6.1 and Corollary 7.6, we have

$$\mathcal{F}_\nu({}^t\mathrm{T}_x^\nu f(\lambda)) = \mathcal{F}_0[{}^t\mathcal{X}{}^t\mathrm{T}_x^\nu f](\lambda)$$

$$= \mathcal{F}_0[{}^t\mathcal{X}(f)\star k(x, \cdot)](y)$$

$$= \mathcal{F}_0(k(x, \cdot))(\lambda)\cdot\mathcal{F}_0({}^t\mathcal{X}(f))(\lambda).$$

It suffices to recall (3.7)

$$\mathcal{F}_0(k(x, \cdot))(\lambda) = \mathcal{G}_\nu(\lambda x)$$

to obtain the result.

Now we are able to define the convolution product related to the operator $Ł_\nu$.

Definition 7.4. The convolution product associated to $Ł_\nu$ of two functions f and g in $\mathcal{D}_*(\mathbb{R})$ is the function $f \star_\nu g$ defined by

$$f\star_\nu g(y) = \int_0^\infty f(x)\mathrm{T}_y^\nu g(x)dx$$

$$= \int_0^\infty {}^t\mathrm{T}_y^\nu f(x)g(x)dx. \tag{7.11}$$

In the following theorem we give the essential properties of this convolution product.

Theorem 7.8. *For f and g in $\mathcal{D}_*(\mathbb{R})$,*

(1) ${}^t\mathcal{X}[f \star_\nu g] = {}^t\mathcal{X}(f) \star {}^t\mathcal{X}(g)$

(2) $\mathcal{F}_\nu(f \star_\nu g)(\lambda) = \mathcal{F}_\nu f(\lambda)\mathcal{F}_\nu g(\lambda)$

(3) $\mathcal{F}_\nu(f) = \mathcal{F}_0 \circ^t \mathcal{X}(f)$

(4) $\mathcal{F}_0(f) = \mathcal{F}_\nu \circ^t \mathcal{X}^{-1}(f)$.

Proof. For f and g in $\mathcal{D}_*(\mathbb{R})$ the proofs are consequences of previous results.

1. $$f \star_\nu g = \langle f, \mathrm{T}_x^\nu g\rangle = \mathcal{X}_x\langle {}^t\mathcal{X}_y^{-1}\, {}^t\tau_y {}^t\mathcal{X}(f), g\rangle$$

$$= {}^t\mathcal{X}_y^{-1}\langle \mathcal{X}_x\ {}^t\tau_y {}^t\mathcal{X}(f)(x), g\rangle = {}^t\mathcal{X}_y^{-1}\langle {}^t\tau_y {}^t\mathcal{X}(f), {}^t\mathcal{X}_x g\rangle$$

$$= {}^t\mathcal{X}_y^{-1}\langle {}^t\tau_y {}^t\mathcal{X}(f), {}^t\mathcal{X}_x g\rangle = {}^t\mathcal{X}_y^{-1}\ {}^t\mathcal{X}(f) \star {}^t\mathcal{X}(g).$$

2. $$\mathcal{F}_\nu(f \star_\nu g)(\lambda) = \int_0^\infty \int_0^\infty f(x)\, {}^t\tau_y \mathcal{G}_\nu(\lambda x)dx g(y)dy$$

$$= \int_0^\infty f(x)\mathcal{G}_\nu(\lambda x)dx \int_0^\infty g(y)\mathcal{G}_\nu(\lambda y)dy$$

$$= \mathcal{F}_\nu f(\lambda)\mathcal{F}_\nu g(\lambda).$$

3. $$\mathcal{F}_\nu(f)(\lambda) = \int_0^\infty f(t)\mathcal{X}\cos_3(\lambda t)dt$$

$$= \int_0^\infty f(t)^t\mathcal{X}\cos_3(\lambda t)dt.$$

4. $$\mathcal{F}_0(f)(\lambda) = \int_0^\infty f(t)\cos_3(\lambda t)dt = \int_0^\infty f(t)\mathcal{X}^{-1}\mathcal{G}_\nu(\lambda t)dt$$

$$= \int_0^\infty {}^t\mathcal{X}^{-1}f(t)\mathcal{G}_\nu(\lambda t)dt.$$

8. Application: Heat Equation

In this section we introduce the space $L^1_\nu(\mathbb{R}^*_+, dx)$ of functions f satisfying

$$\int_0^\infty |f(x)|\mathcal{G}_\nu(\lambda x)dx < \infty, \quad \nu > 0; \lambda > 0.$$

for $\nu > 0; t > 0, x > 0.$ We consider the problem

$$(\mathbb{II}) \quad \begin{cases} Ł_\nu u(x, t) = \dfrac{d}{dt}u(x, t) \\ u(x, 0) = f(x); \quad f \in L^1_\nu(\mathbb{R}^*_+, dx) \\ D_x u(0, t)| \ = 0, \quad D_x^2 u(0, t)| = 0 \\ u(w_k x, t) = u(x, t), \quad k = 1, 2, 3, \end{cases}$$

where w_k's are the third roots of unity.

To look for a solution of this problem, we put

$$U(\lambda; t) = \mathcal{F}_\nu(u(.; t))(\lambda). \tag{8.1}$$

Then

$$Ł_\nu U(\lambda, t) = -\lambda^3 U(\lambda; t) \tag{8.2}$$

and

$$U(\lambda; 0) = \mathcal{F}_\nu(f)(\lambda). \tag{8.3}$$

Hence

$$U(\lambda; t) = \mathcal{F}_\nu(f)(\lambda)\exp(-\lambda^3 t). \tag{8.4}$$

We obtain that the solution is given by

Proposition 8.1. *We have*

$$u(x; t) = (F_\nu(\cdot, t) \star_\nu f)(x) \tag{8.5}$$

with

$$F_\nu(x, t) = \frac{1}{\Gamma(1/3)\Gamma(\nu + 2/3)} \frac{2}{t^{1/3}} \left(\frac{x}{3t^{1/3}}\right)^{(3\nu+1)/2} K_{-\nu-\frac{1}{3}}\left(2\left(\frac{x}{3t^{1/3}}\right)^{3/2}\right), \tag{8.6}$$

where K_ν is the Mac-Donald function ([8], [13]).

Note that the function $F_\nu(x, t)$ is 3-even in x.

Proof. We start by recalling the formula for $\beta > 0$ and $\beta/2 - \lambda > 0$ [13], we have

$$\int_0^\infty u^{\beta-\lambda-1} K_\lambda(2u)du = \frac{1}{4}\Gamma(\beta/2)\Gamma(\beta/2 - \lambda).$$

We deduce for $\beta = 2(n + 1/3)$ and $\lambda = -(1/3 + \nu)$ that

$$\int_0^\infty (u)^{(2n+\nu)} K_{-\nu-1/3}(2u)du = \frac{1}{4}\Gamma(n + 1/3)\Gamma(n + \nu + 2/3).$$

The last one with suitable change of variable lead to

$$\int_0^\infty F_\nu(x, t^3)\left(\frac{x}{3}\right)^{3n} dx = t^{3n}\,\frac{\Gamma(n + 1/3)\Gamma(n + \nu + 2/3)}{\Gamma(1/3)\Gamma(\nu + 2/3)}.$$

So

$$\int_0^\infty F_\nu(x, t^3)\mathcal{G}_\nu(\lambda x)dx$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{n!}\int_0^\infty F_\nu(x, t^3)\left(\frac{\lambda x}{3}\right)^{3n} dx\,\frac{\Gamma(1/3)\Gamma(\nu + 2/3)}{\Gamma(n + 1/3)\Gamma(n + \nu + 2/3)}$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{n!}\lambda^{3n} t^{3n} = \exp(-\lambda^3 t^3).$$

This means

$$\int_0^\infty F_\nu(x, t)\mathcal{G}_\nu(\lambda x)dx = \exp(-\lambda^3 t).$$

The function $F_\nu(x, t)$ plays the role of Gaussian kernel associated with this heat equation. Contrary to the case of the kernel associated with the second order operator, the previous function has oscillator behavior when x tends to infinity. We can discover this phenomena as follows.

It is clear that $K_\nu = K_{-\nu}$ and from the integral representation [2, p. 252] valid for $\operatorname{Re}(\nu) > -1/2$, $x > 0$

$$K_\nu(x) = \frac{\Gamma(\nu + 1/2)}{\Gamma(1/2)} \frac{2^\nu}{x^\nu} \int_0^\infty \cos(xu)(1 + u^2)^{-\nu-1/2} du. \tag{8.7}$$

We deduce for $\nu > 0$ the following integral representation

$$F_\nu(x, t) = \frac{\Gamma(\nu + 5/6)}{\Gamma(1/2)\Gamma(1/3)\Gamma(\nu + 2/3)} \int_0^\infty (1 + u^2)^{-\nu-5/6} \cos\left(2\left(\frac{x}{3t^{1/3}}\right)^{3/2} u\right) du. \tag{8.8}$$

The function $F_\nu(x, t)$ is 3-even for $x > 0$ and $\nu > 0$.

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