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$\mathbb{R}$-LINEAR HOMOMORPHISMS BETWEEN BAIRE CLASSES

HAMID REZA SHATERY and MOSTAFA MIRBOD

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Abstract

The aim of this note is to prove that any multiplicative $\mathbb{R}$-linear operator between complex valued Baire-α classes, defined on two perfectly normal topological spaces, is characterized by a $\mathbb{C}$-linear between them.

1. Introduction

We know that

> *any multiplicative $\mathbb{R}$-linear operator between the rings of complex valued continuous functions, defined on two connected topological spaces, is $\mathbb{C}$-linear.*

The above fact is due to Krein and Krein [5]. A generalization of this theorem was given by Kaplansky [4]. The other useful related papers are [8, 9]. We are going to prove this theorem for Baire classes.

Throughout this paper, X is a perfectly normal topological space ([1], [3], [6]). A topological space X is perfectly normal, if it is Hausdorff and every closed subset is the zero set of some real continuous function.

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For a finite ordinal number α, we denote the Borel sets of multiplicative (additive) class α by $\mathcal{P}_\alpha$ $(\mathcal{S}_\alpha)$, beginning with $\mathcal{P}_0 = \mathcal{F}$ $(\mathcal{S}_0 = \mathcal{G})$, as the followings [6]:

$$\mathcal{P}_\alpha : \mathcal{F}, \mathcal{G}_\delta, \mathcal{F}_{\sigma\delta}, \ldots$$

$$\mathcal{S}_\alpha : \mathcal{G}, \mathcal{F}_\sigma, \mathcal{G}_{\delta\sigma}, \ldots.$$

As X is perfectly normal, so the $\mathcal{P}_\alpha$'s $(\mathcal{S}_\alpha$'s) form a chain and $\mathcal{F} \subseteq \mathcal{G}_\delta$, similar to the metric case (see [6]).

For each $A \in \mathcal{P}_\alpha$, there exists a sequence $(G_n)_{n=1}^{\infty} \subseteq \mathcal{S}_{\alpha-1}$ such that

$$A = \bigcap_{n=1}^{\infty} G_n.$$

For additive sets, "$\mathcal{S}$", "$\mathcal{P}$", and "$\bigcap$" are replaced respectively by "$\mathcal{P}$", "$\mathcal{S}$", and "$\bigcup$". See [6, Section 30] for details.

The ambiguous set of class α is denoted by $\mathcal{H}_\alpha$ [6] and defined as follows:

$$\mathcal{H}_\alpha = \mathcal{S}_\alpha \cap \mathcal{P}_\alpha.$$

Lemma 1.1. *Here we mention some facts about perfectly normal spaces.*

(a) *Every set in* $\mathcal{S}_\alpha$ $(\alpha \geq 1)$ *is the union of some countable disjoint sets in* $\mathcal{H}_\alpha$.

The proof is similar to that of metric spaces. See [6, Section 30, V, Theorem 1].

(b) *For each sequence* $(G_n)_{n=1}^{\infty} \subseteq \mathcal{S}_\alpha$ $(\alpha \geq 1)$, *there exists a mutually disjoint sequence* $(H_n)_{n=1}^{\infty}$ *in* $\mathcal{S}_\alpha$ *such that* $\bigcup_{n=1}^{\infty} H_n = \bigcup_{n=1}^{\infty} G_n$ *and* $H_n \subseteq G_n$ *for each n. In addition, if* $X = \bigcup_{n=1}^{\infty} H_n$, *then* H_n*'s belong to* $\mathcal{H}_\alpha$.

The proof is similar to that of metric spaces. See [6, Section 30, VII, Theorem 1].

(c) *For every sequence* $(F_n)_{n=1}^{\infty}$ *in* $\mathcal{P}_\alpha$ $(\alpha \geq 1)$ *such that* $\bigcap_{n=1}^{\infty} F_n = \varnothing$, *there exists a sequence* $(E_n)_{n=1}^{\infty}$ *in* $\mathcal{H}_\alpha$ *such that* $\bigcap_{n=1}^{\infty} E_n = \varnothing$ *and* $F_n \subseteq E_n$ *for each n. Therefore, if A and B are two disjoint* $\mathcal{P}_\alpha$ *sets, then there exists E in* $\mathcal{H}_\alpha$ *such that* $A \subseteq E$ *and* $B \cap E = \varnothing$. *That is, if* $A \in \mathcal{P}_\alpha$, $C \in \mathcal{S}_\alpha$ *and* $A \subseteq C$, *then there exists* $E \in \mathcal{H}_\alpha$ *such that* $A \subseteq E \subseteq C$.

See [6, Section 30, VII, Theorem 2].

Definition 1.2. Let X be a topological space and $\beta_0(X) = C(X)$ be the set of all real valued continuous functions on X. Then for each ordinal α, we define Baire functions of class α as follows:

$$\beta_\alpha(X) = \{f : X \to \mathbb{R} : \text{there exists } (f_n)_{n=1}^{\infty} \subseteq \beta_{\alpha-1}(X)$$
$$\text{such that } \lim f_n(x) = f(x), \text{ for each } x \in X\}.$$

We also define Borel functions of class α as follows:

$$B_\alpha(X) = \{f : X \to \mathbb{R} : \text{for each closed set } F \text{ in } \mathbb{R},\ f^{-1}(F) \in \mathcal{P}_\alpha\}.$$

When X is a perfectly normal space, then by the same induction as in [7], $\beta_\alpha(X) \subseteq B_\alpha(X)$. It is obvious that $B_\alpha(X) \subseteq B_{\alpha+1}(X)$.

For a Banach space E, suppose that $C^\circ(X, E)$ is the set of all E-valued continuous functions with relatively compact ranges.

Definition 1.3. We define

$$\beta_0^\circ(X, E) = C^\circ(X, E),$$

$$\beta_\alpha^\circ(X, E) = \{f : X \to E : f \text{ is the pointwise limit of some sequence}$$
$$\text{in } \beta_{\alpha-1}(X, E) \text{ and range of } f \text{ is relatively compact}\},$$

$$B_\alpha^\circ(X, E) = \{f : X \to E : f^{-1}(F) \in \mathcal{P}_\alpha \text{ for each } F, \text{ closed in } E \text{ and}$$
$$\text{range of } f \text{ is relatively compact}\}.$$

1.1. Some results on Baire classes. In this part we obtain some results about Baire classes for using in the next section. Here first we give a Baire-α characterization of $\mathcal{H}_\alpha$ elements in X.

Lemma 1.4. *Let X be a perfectly normal space and E be a Banach space with $0 \neq e \in E$. Then we have $H \in \mathcal{H}_\alpha$ if and only if $e\chi_H \in \beta_\alpha(X, E)$.*

Proof. We prove by induction. Suppose that H is in $\mathcal{H}_\alpha$. As X is perfectly normal, then it is normal. Suppose that the statement holds for $(\alpha - 1)$. Then by Lemma 1.1 (b), there are a nondecreasing sequence $(F_n)_{n=1}^{\infty}$ of elements $\mathcal{P}_{\alpha-1}$ in X and a nonincreasing sequence $(G_n)_{n=1}^{\infty}$ of elements $\mathcal{S}_{\alpha-1}$ in X such that

$$\bigcup_{n=1}^{\infty} F_n = H = \bigcap_{n=1}^{\infty} G_n.$$

For each positive integer n, $F_n \subseteq G_n$. By use of induction, there is an $H_n \in \mathcal{H}_{\alpha-1}$ such that $f_n = e\chi_{H_n} \in \beta_{\alpha-1}(X, E)$, $f_n(F_n) = \{e\}$ and $f_n(G_n^c) = \{0\}$. Obviously, $e\chi_H$ is the pointwise limit of f_n. The proof of the other side is obvious and is omitted.

We define

$$\Sigma_{\alpha, E} = \left\{\sum_{i=1}^{n} e_i \chi_{H_i} : n \in \mathbb{N}, e_i \in E \text{ and } H_i \in \mathcal{H}_\alpha \text{ for each } i\right\}.$$

In the following theorem we give an approximation theorem for Baire functions by simple functions.

Theorem 1.5. *For a Fréchet space E, the uniform closure of $\Sigma_{\alpha, E}$ is $\beta_\alpha^\circ(X, E)$.*

Proof. Suppose that E is a Banach space. As the $range(f)$ is relatively compact, therefore there exists a countable set $Z \subseteq E$ such that the $range(f)$ is in the norm closure of Z. For each positive integer n, let $\mathcal{C}_n$ be the collection of open balls of radius $1/n$ in E with members of

Z as their centers. Hence the $range(f)$ is covered by finite members of $\mathcal{C}_n$, denoted by $\mathcal{B}_n$, labeled by the finite set I_n. Set $\mathcal{B}'_n = f^{-1}(\mathcal{B}_n)$. Thus $\mathcal{B}'_n$ is a cover of X and each of its elements belongs to $\mathcal{S}_\alpha$. Hence, its members are $\mathcal{H}_\alpha$ and by Lemma 1.1 (b), X has a finite refinement, consists of mutually disjoint elements of $\mathcal{H}_\alpha$ sets. Therefore, $\mathcal{A}_n = \{A_{i,n} : i \in I_n\}$ is a refinement of $\mathcal{B}'_n$ with $\mathcal{H}_\alpha$ sets. We can suppose that for any $n \geq 2$, $\mathcal{A}_n$ refines $\mathcal{A}_{n-1}$.

Now, for each $n \geq 1$ and for each $i \in I_n$, choose $y_{i,n} \in f(A_{i,n})$. Let $x \in X$ and for each $n \geq 1$ let $i(x, n) \in I_n$ be such that $x \in A_{i(x,n),n}$. If $m \geq n$, then $A_{i(x,m),m} \subseteq A_{i(x,n),n}$. Consequently, since $f(A_{i(x,n),n})$ has diameter at most $2/n$, $\{y_{i(x,n),n} : n \geq 1\}$ is a Cauchy sequence.

We define

$$g(x) = \lim_{n\to\infty} y_{i(x,n),n}.$$

Notice that

$$\| g(x) - y_{i(x,n),n} \| \leq 2/n.$$

If $x' \in A_{i(x,n),n}$, then $i(x, n) = i(x', n)$ and so

$$\| g(x) - g(x') \| \leq \| g(x') - y_{i(x,n),n} \| + \| g(x) - y_{i(x,n),n} \| \leq 4/n.$$

If $x \in A$, then for each $n \geq 1$, $x \in A_{i(x,n),n}$ and therefore

$$\| g(x) - f(x) \| \leq 4/n.$$

Hence $g = f$. Now, for each $n \in \mathbb{N}$, we define

$$f_n = \sum_{i \in I_n} y_{i,n} \chi_{A_{i,n}}.$$

It is obvious that f_n's are in $\beta^\circ_\alpha(X, E)$, and $f = g$ is the uniform limit of f_n's.

Now, suppose that E is a Fréchet space. It is enough to work with a countable collection of semi-norms that introduce its topology.

In the sequel we use the notation of [2, Chapter I] to obtain the dual of $\beta_{\alpha}^{\circ}(X)$. We denote by $VM(\mathcal{H}_{\alpha}, E)$, the space of all bounded finitely additive vector measures, $F : \mathcal{H}_{\alpha} \to E$ provided by semi-variation norm. Therefore, every F in $VM(\mathcal{H}_{\alpha}, E)$ is related to a T in $\mathcal{L}(\beta_{\alpha}^{\circ}(X), E)$ with the following correspondence:

$$T_F(f) = \int_X f dF$$

for every f in $\beta_{\alpha}^{\circ}(X, E)$.

In the particular case when $E = \mathbb{R}$, we have the following representation of the dual of $\beta_{\alpha}^{\circ}(X)$.

Corollary 1.6.

$$Dual(\beta_{\alpha}^{\circ}(X)) = (\beta_{\alpha}^{\circ}(X))^{*} = VM(\mathcal{H}_{\alpha}, \mathbb{R}).$$

Proof. The proof is the same as that of Theorem 13 page 6 of [2].

2. Main Result

It is obvious that any ring isomorphism between rings of continuous functions is an algebra isometric isomorphism between them [3]. First we prove this result for Baire-α classes.

Theorem 2.1. *Let* $\phi : \beta_{\alpha}^{\circ}(Y) \to \beta_{\alpha}^{\circ}(X)$ *be any ring homomorphism. Then* ϕ *is linear*, $\| \phi \| = 1$ (*unless* $\phi = 0$).

Proof. We prove this theorem similar to that of ring of continuous functions. Denote $\phi(\hat{1}_Y) = e$. Then $e^2 = \phi(\hat{1}_Y \hat{1}_Y) = e$ and therefore e is an idempotent so by Lemma 1.4, $e = \chi_H$ for a certain ambiguous subset H of X $(H \in \mathcal{H}_{\alpha})$. If $g \in \beta_{\alpha}^{\circ}(Y, \mathbb{R}^{+})$, then $g = h^2$ for a certain h in $\beta_{\alpha}^{\circ}(Y, \mathbb{R})$, and hence $\phi(g) = \phi(h^2) \geq 0$. If $\| g \| \leq 1$, then $-\hat{1}_Y \leq g \leq \hat{1}_Y$ and

$|\phi(g)| \le e$; therefore $\|\phi(g)\| \le 1$. Thus ϕ is continuous and hence it is linear.

Theorem 2.2. *Let* $\phi : \beta_\alpha^\circ(Y, \mathbb{C}) \to \beta_\alpha^\circ(X, \mathbb{C})$ *be a nonzero multiplicative* $\mathbb{R}$*-linear operator. Then* $\|\phi\| = 1$ *and there exist disjoint* $\mathcal{H}_\alpha$ *sets,* H_1 *and* H_2 *of* X *and a multiplicative* $\mathbb{C}$*-linear operator* $\psi : \beta_\alpha^\circ(Y, \mathbb{C}) \to \beta_\alpha^\circ(X, \mathbb{C})$ *such that for every g in* $\beta_\alpha^\circ(Y, \mathbb{C})$,

$$(\phi g)(x) = \begin{cases} (\psi g)(x) & x \in H_1 \\ 0 & x \in X - (H_1 \cup H_2) \\ \overline{(\psi g)(x)} & x \in H_2. \end{cases}$$

Proof. Denote $\phi(\hat{1}_Y) = e$. Then $e^2 = e$ and $e = \chi_H$ for an $\mathcal{H}_\alpha$-subset H of X. Denote $u = \phi(i\hat{1}_Y)$. Then $u^2 = \phi(i^2\hat{1}_Y) = -e = -\chi_H$. Therefore, for $x \in H$, $u^2(x) = -1$ and thus $u(x) = i$ or $u(x) = -i$. So there exist disjoint sets H_1 and H_2 such that

$$H_1 = u^{-1}(\{i\}), \quad H_2 = u^{-1}(\{-i\}).$$

For $x \notin H$, $u(x) = 0$, define

$$\psi(g + ih) = \phi(g) + i\phi(h), \quad \forall g, h \in \beta_\alpha^\circ(Y, \mathbb{R}).$$

Straightforward verification shows that ψ is a multiplicative linear operator from $\beta_\alpha^\circ(Y, \mathbb{C})$ into $\beta_\alpha^\circ(X, \mathbb{C})$; in particular, $\psi(if) = i\psi(f)$ for each $f \in \beta_\alpha^\circ(Y, \mathbb{C})$. It will be shown that $\|\psi\| = 1$. If $x \in X$, then the functional $\eta_x = \psi^*(\delta_x)$ is multiplicative and linear on $\beta_\alpha^\circ(Y, \mathbb{C})$. Claim: $\|\eta_x\| = 1$. Indeed, if $\eta_x(\hat{1}_Y)$ were equal to 0, η_x would be identically 0. Hence, $\eta_x(\hat{1}_Y) = \eta_x(\hat{1}_Y\hat{1}_Y) = \eta_x(\hat{1}_Y)^2$, $\eta_x(\hat{1}_Y)$ is equal to 1. Let $f \in \beta_\alpha^\circ(Y, \mathbb{C})$ such that $\|f\| = 1$. Suppose, if possible, that $|\eta_x(f) > 1|$. Denote $a = \eta_x(f)$ and

$$f' = \frac{1}{\hat{1}_Y - \frac{f}{a}}.$$

Therefore $\left\| \frac{f}{a} \right\| < 1$, f' is in $\beta_\alpha^\circ(Y, \mathbb{C})$ and bounded on Y. Consequently,

$$\eta_x(f)\eta_x\left(\hat{1}_Y - \frac{f}{a}\right) = 1.$$

Therefore $\eta_x\left(\hat{1}_Y - \frac{f}{a}\right) \neq 0$; i.e., $\eta_x(f) \neq a$. This contradiction shows that $\eta_x(f) = 1$. Thus, if $f \in \beta_\alpha^\circ(Y, \mathbb{C})$, then $|\psi f \cdot x| = |\eta_x(f)| = \| f \|$. Consequently $\| \psi \| \leq 1$. Since $\| e \| = \| \psi(\hat{1}_Y) \| = 1$, $\| \psi \| = 1$. Now, let $g \in \beta_\alpha^\circ(Y, \mathbb{R})$; it will be shown that $\psi(g)$ is real-valued. If $x \in X - H$, then $\psi g \cdot x = 0$. Let $x \in H$ and $\psi g \cdot x = a + ib$, where $a, b \in \mathbb{R}$. For every $t \in \mathbb{R}$,

$$\begin{aligned} a^2 + b^2 + 2bt + t^2 &= |\psi g \cdot x + ite(x)|^2 \\ &\leq \| \psi(g + it\hat{1}_Y) \|^2 \\ &\leq \| g + it\hat{1}_Y \|^2 \\ &= \| g \|^2 + t^2. \end{aligned}$$

Hence t may be an arbitrary number, b equals to 0; i.e., $\psi g \cdot x$ is real. Consequently, if f and g are in $\beta_\alpha^\circ(Y, \mathbb{R})$, then $\mathrm{re}\psi(g + ih) = \psi(g)$. Thus,

$$\phi(g + ih) = \phi(g) + i\phi(h) = \mathrm{re}\psi(g + ih) + i\mathrm{im}\psi(g + ih),$$

for every $g, h \in \beta_\alpha^\circ(Y, \mathbb{R})$. Therefore, ψ is the desired operator.

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Department of Mathematics
University of Isfahan
Isfahan, 81745-163, Iran
e-mail: shatery@math.ui.ac.ir

Department of Management
Azad University of Dehaghan
Dehaghan, 864-1611161, Iran
e-mail: mirbod-332@yahoo.com