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B-SPLINE SOLUTION OF THE BOUNDARY LAYER FREE CONVECTION FLOW IN POROUS MEDIA

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Abstract

In this paper, we reconsider the problem of boundary layer free convection flow along a vertical flat plate with a power law surface temperature and embedded in a fluid-saturated porous medium. By using similarity analysis, the partial differential equations governing the flow are transformed into a boundary value problem (BVP) containing nonlinear ordinary differential equation. The resulting ordinary differential equation is solved numerically through the Quintic *B*-spline collocation method in combination with Quasilinearization. The accuracy of the method is discussed and the numerical solutions are found to be in excellent agreement with theoretical predictions of Banks [Theor. Appl. 2 (1983), 375].

1. Introduction

During the past two decades, tremendous progress has been made in the field of convective flow in saturated porous media as evidenced by the literature published in the recent book by Nield and Bejan [18]. Thus a major interest has existed to obtain solutions to the problem of boundary

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layer free convection from a vertical surface with variable surface temperature or variable surface heat flux embedded in a fluid saturated porous medium. The first work on this problem has been made by Cheng and Minkowycz [9] for a heated surface with the wall temperature proportional to x^m, where x is the distance measured along the surface and m is a constant. Numerical solutions were obtained in [9] for the resulting ordinary differential equation for $0 \le m \le 1$. The same equation was solved by Na and Pop [17] using the method of perturbation series in combination with Shanks transformation [21].

In this paper, we study a collocation method to compute the numerical solution of (scalar) third order two-point BVPs described by differential equation and boundary conditions of the form

$$L(u) = u'''(x) + p(x)u''(x) + q(x)u'(x) + r(x)u(x) = f(x),\ x \in [a, b] = I, \quad (1.1)$$

$$Bu(a) = v_a, \quad Bu(b) = v_b, \quad (1.2)$$

where $u(x)$ is an unknown function, v_a and v_b are given, $p(x), q(x), r(x)$ and $f(x)$ given functions and B is a differential operator (e.g., $Bu(a) = cu'(a)$ etc.).

In general, collocation proceeds as follows. We first choose an approximating space X of dimension n and a basis $\{\phi_1(x), \phi_2(x), \ldots, \phi_n(x)\}$ for X such that any $W_\Delta \in X$, where Δ is the uniform partition of I, can be written as $W_\Delta = \sum_{i=1}^{n} c_i\phi_i(x)$, where the c_i's are scalars referred to as coefficients or degrees of freedom. Note that the basis functions $\phi_1(x)$, $\phi_2(x), \ldots, \phi_n(x)$ are user chosen and thus known, while the degrees of freedom are unknown. Then a set T of data points called *collocation points* is selected, where $T = T_L \cup T_B$ with $T_B = \{a, b\}$. In the standard formulation of collocation methods we determine the degrees of freedom c_i, $i = 1, 2, \ldots, n$ and then the approximation W_Δ to the solution u of the BVPs by forcing W_Δ to satisfy conditions

$$LW_\Delta(x) = f(x); \quad x \in T_L$$

$$BW_{\Delta}(x) = g(x); \quad x \in T_L.$$

These conditions, called *collocation conditions*, give rise to a linear system with respect to the unknown degrees of freedom. The sets T_L and T_B are chosen so that the collocation conditions give rise to a uniquely solvable linear system. The choice of approximating space, basis functions and collocation points plays an important role in the calculations, especially those associated with the resulting linear system. The approximating space we choose is the space of quintic splines with respect to a partition Δ of I, that is, the space of piecewise quintic polynomials with C^4 continuity defined with respect to Δ, with quintic B-splines as basis functions. It is quite common in the literature, to pick as data points the grid points of the partition, when odd degree splines are used and the midpoints of the subintervals of Δ when even degree splines are used. See, for example, [10], [15] and [23].

In this paper, we present accurate numerical solutions of the governing equation given in [9] using the collocation method based on Quintic B-spline functions through the Quasilinearization technique [5]. B-splines are well known and are described in detail in text such as Farin [11] and Hoschek and Lasser [12]. The method of cubic B-spline is successfully used by Joshi and Doctor [14] for some specific flow and heat transfer problems. More information about the spline collocation methods is found in the references [7, 8, 15, 19, 22, 23]. Numerical results are presented in tabular form for various values of the parameter m. We believe that these results serve as a reference against which other approximate solutions for the present problem can be compared in the future. In addition, this method can be applied to solve variety of problems in the field of Applied Mathematics.

2. Basic Equation

Referring to a Cartesian system of coordinates (x, y), the boundary layer equations which govern the free-convection boundary layer flow over vertical flat plate embedded in a porous medium are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$u = \frac{g\beta\kappa}{\upsilon}(T - T_\infty), \tag{2.2}$$

$$v = 0, \quad T = T_w, \tag{2.3}$$

where $y = 0$ is the plate, x is measured in the direction of the boundary force, and u and v are respectively the velocity components in the directions of x and y increasing. Also T, g, β, κ, υ and α are respectively the temperature, acceleration due to gravity, coefficient of thermal expansion, permeability of the saturated porous medium, kinematics' viscosity and thermal diffusivity, respectively; the suffix ∞ indicates conditions in the ambient fluid far from the plate. It is assumed here that Boussinesq approximation and Darcy's law are applicable, and that the Rayleigh number is large.

The boundary conditions for the problem are,

$$v = 0, \quad T = T_w, \quad \text{on} \quad y = 0, \tag{2.4}$$

$$u \to 0, \quad T \to T_\infty, \quad \text{as} \quad y \to \infty. \tag{2.5}$$

with $T_w = T_\infty + t(x)$. The similarity solutions were investigated by Cheng and Minkowycz [9] with $t(x) = ax^m$, where a is a constant. Equations (2.1-2.3) admit similarity solutions of the form

$$\psi = \alpha\left(\frac{2}{1+m}\right)^{1/2} Ra_x^{1/2} f(\eta), \; T = T_\infty + t(x)\theta(\eta) \tag{2.6}$$

and

$$\eta = \left(\frac{1+m}{2}\right)^{\frac{1}{2}} Ra_x^{1/2} \frac{y}{x}, \tag{2.7}$$

where ψ is the stream function defined in the usual way

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x} \tag{2.8}$$

and Ra_x is the modified Rayleigh number for a porous medium defined as

$$Ra_x = \frac{g\beta\kappa x t(x)}{\alpha v}. \tag{2.9}$$

Substituting (2.6) and (2.7) into equations (2.2) and (2.3), we obtain

$$f''' + ff'' - \beta f'^2 = 0, \tag{2.10}$$

where primes denote differentiation with respect to η and

$$\beta = \frac{2m}{1+m}. \tag{2.11}$$

The boundary conditions (2.4) and (2.5) become

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0. \tag{2.12}$$

It is worth mentioning that equation (2.10) subject to the boundary conditions (2.12) describes also the boundary layer on a stretching surface, which stretches with a speed proportional to x^m, where x is the distance measured along the stretching surface, and the exponent m can be regarded as materially dependent constant. Such stretching boundaries occur in the manufacture of polymer sheeting as well as in other real situations, see Banks and Zaturska [4]. Equation (2.10) has been also solved numerically by Banks [3] for values of β in the range $-1.9999 \le \beta \le 202$.

3. Solution by Quintic B-spline Collocation

In this section, we present quintic spline collocation method by solving the BVP given in equations (2.10), (2.12). We consider the uniform grid partition $\Delta = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of the interval $I = [a, b]$, and the set of data points $t_0, t_1, ..., t_n$ with $h = t_{i+1} - t_i$ be the mesh size of Δ.

Let $S_{5,\Delta}$ be the space of quintic splines with respect to Δ, that is, the space of quintic piecewise polynomials with respect to Δ and with continuity $C^4(I)$. Note that $S_{5,\Delta}$ has dimension $N+5$. In order to represent any quintic spline $S \in S_{5,\Delta}$, we choose a set of quintic spline basis functions, the quintic B-splines $B_j(t)$; $j = -2, -1, 0, 1, ..., N+2$. The quintic B-splines $B_j(t)$ with additional grid points $t_{-3} < t_{-2} < t_{-1}$

and $t_{n+3} > t_{n+2} > t_{n+1} > t_n$ are defined by

$$B_j(t) = 1/h^5[(t - t_{i-3})^5 - 6(t - t_{i-2})^5 + 15(t - t_{i-1})^5 - 20(t - t_i)^5$$

$$+ 15(t - t_{i+1})^5 - 6(t - t_{i+2})^5 + (t - t_{i+3})^5] \tag{3.1}$$

with data points as $t_{-2} < t_{-1} < t_0 < \cdots < t_n < t_{n+1} < t_{n+2}$. Then the set $\{B_j(t)\}$ forms a basis for $S_{5,\Delta}$. Any quintic spline $S(t) \in S_{5,\Delta}$ can be written as

$$S(t) = \sum_{j=-2}^{N+2} c_j B_j(t), \tag{3.2}$$

where the scalars c_j, $j = -2, -1, 0, 1, \ldots, N + 2$, are degrees of freedom to be determined. The values of quintic B-spline $B_j(t)$ and its derivatives at different grid points are computed from equation (3.1) by using the function $(t - t_k)_+$ defined by,

$$(t - t_k)_+ = t - t_k, \quad \text{if} \quad t > t_k$$

$$= 0, \qquad \text{if} \quad t \le t_k.$$

These values are shown in Table 3.1.

Table 3.1. Values of quintic B-splines at grid points

	t_{j-3}	t_{j-2}	t_{j-1}	t_j	t_{j+3}	t_{j+2}
$B_j(t)$	0	1	26	66	1	0
$B'_j(t)$	0	$5/h$	$50/h$	0	$-5/h$	0
$B''_j(t)$	0	$20/h^2$	$40/h^2$	$-120/h^2$	$20/h^2$	0
$B'''_j(t)$	0	$60/h^3$	$-120/h^3$	0	$60/h^3$	0
$B''''_j(t)$	0	$120/h^4$	$-480/h^4$	$720/h^4$	$120/h^4$	0

Now we solve the problem given by equations (2.10), (2.12) by the collocation method based on spline function (3.1). For that we required the linear form of the problem. We use the Quasilinearization technique as discussed in [5, 16] to convert the nonlinear equation into linear equation. Due to Quasilinearization technique the equations (2.10) and (2.12) are transformed to

$$L(f) \equiv f'''_{i+1} + f_i f''_{i+1} - \beta f'_{i+1} + f'_i f_{i+1} = f_i f''_i - \beta (f'_i)^2 \tag{3.3}$$

with boundary conditions

$$Bf(0) \equiv f_{i+1}(0) = 0, \qquad f'_{i+1}(0) = 1, \tag{3.4a}$$

$$Bf(\infty) \equiv f'_{i+1}(\infty) = 0. \tag{3.4b}$$

For the numerical study the outer boundary is set at $\eta_\infty = 4$ and therefore the domain of the problem is restricted to [0, 4] and the end condition (3.4b) is considered as

$$Bf(4) \equiv f'_{i+1}(4) = 0. \tag{3.4c}$$

Now the quintic spline interpolation to $f(\eta)$ of equations (3.3), (3.4a, c) gives the collocation conditions:

$$Lf(\eta) \equiv LS(t), \tag{3.5}$$

$$Bf(0) \equiv BS(t_0) \equiv \{s(t_0) = 0,\ s'(t_0) = 1\}, \tag{3.6}$$

$$Bf(4) \equiv BS(t_N). \tag{3.7}$$

Using Table 3.1 the above collocation conditions with $N = 8$, give the following collocation equations:

$$\left[-\frac{60}{h^3} + \frac{20}{h^2} f_i + \frac{5}{h}\beta + f''_i\right] c_{i-2}$$

$$+ \left[\frac{120}{h^3} + \frac{40}{h^2} f_i + \frac{50}{h}\beta + 26 f''_i\right] c_{i-1}$$

$$+ \left[-\frac{120}{h^2} f_i + 66 f''_i\right] c_i$$

$$+\left[-\frac{120}{h^3}+\frac{40}{h^2}f_i-\frac{50}{h}\beta+26f_i''\right]c_{i+1}$$

$$+\left[\frac{60}{h^3}+\frac{20}{h^2}f_i-\frac{5}{h}\beta+f_i''\right]c_{i+2}$$

$$= f_i f_i'' - \beta(f_i')^2, \quad i = 0, 1, \ldots, 8, \tag{3.8}$$

$$c_{-2}+26c_{-1}+66c_0+26c_1+c_2=0, \tag{3.8a}$$

$$-\frac{5}{h}c_{-2}-\frac{50}{h}c_{-1}+(0)c_0+\frac{50}{h}c_1+\frac{5}{h}c_2=1, \tag{3.8b}$$

$$-\frac{5}{h}c_6-\frac{50}{h}c_7+(0)c_8+\frac{50}{h}c_9+\frac{5}{h}c_{10}=0. \tag{3.8c}$$

From the above system we get twelve equations for thirteen coefficients c_j $(j = -2, -1, 0, 1, \ldots, 10)$. So that we require one more equation to complete the system. Numerical oscillation will occur if a fourth equation is not properly chosen. Due to the physical behavior of the flowing fluid, we may choose the additional condition as $f''(4) = 0$. With the help of equation (3.5) and Table 3.1 this condition is written as

$$\frac{20}{h^2}c_6+\frac{40}{h^2}c_7-\frac{120}{h^2}c_8+\frac{40}{h^2}c_9+\frac{20}{h^2}c_{10}=0. \tag{3.8d}$$

Now the system given by equations (3.8), (3.8a, b, c, d) is a complete system of thirteen equations for the solution of thirteen coefficients c_j $(j = -2, -1, 0, 1, \ldots, 10)$. Eliminating $c_{-2}, c_{-1}, c_9, c_{10}$ from this system we get a system of nine equations in nine unknown $c_0, c_1, \ldots, c_8$. The coefficient matrix of which is a diagonally dominant five-band, nonsingular matrix. Therefore the linear system has a unique solution. By simple manipulation, we can calculate c_{-2}, c_{-1}, c_9, c_{10}. Hence there exists a uniquely determined quintic spline interpolation S of f satisfying (3.5)-(3.7). In order to obtain spline approximations begin with a curve $f(\eta)=\frac{-1}{8}\eta^2+\eta$ satisfying the condition (3.4a, b) as an initial guess. This curve gives the values f_i, f_i', f_i'' $(i=0,1,\ldots,n)$ that are used in equation (3.8) for the initial iteration. For the second iteration the values f_i, f_i', f_i'' are calculated from equation (3.2) and the iteration process is continued until the two successive approximate solutions are agree well with each other.

4. Result and Discussion

The BVP (2.10), (2.12) was solved by the Quintic *B*-spline collocation method. For the numerical study, the outer boundary is set at $\eta_\infty = 4$ with length of subinterval of [0, 4], taken as $h = 0.5$. For $-1.8 \leq \beta \leq 1.5$, the values $f''(0)$, are calculated and presented in Table 4.1. For comparison, the results of Banks [3] are also included in the table. It is seen that these results are in excellent agreement. We have thus shown that by using spline collocation to problem (2.10), (2.12) that the applicability of the method can be extended to higher order nonlinear boundary value problems. It is worth noticing that spline collocation uses two data point per subinterval of the partition and the coefficient matrix is a five band matrix, which shows that spline collocation gives rise to smaller linear systems and therefore in that respect, is advantageous. The quintic *B*-spline collocation is therefore further demonstrated as a useful tool in the analysis of such problems.

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Table 4.1. Comparisons of values of $f''(0)$ and $f(\infty = 4)$

		Banks (4)		Present Method	
β	m	$f''(0)$	$f(\infty = 4)$	$f''(0)$	$f(\infty = 4)$
–1.8	– 0.47368	2.14633	2.20605	2.14739	2.20744
–1.6	– 0.44444	1.00913	1.83302	1.00956	1.83337
–1.4	– 0.41176	0.51774	1.63840	0.51790	1.63859
–1.1	– 0.35484	0.10053	1.45868	0.10058	1.45878
–1.0	– 0.33333	0.00000	1.41421	0.00004	1.41430
– 0.9	– 0.31034	– 0.08901	1.37475	– 0.08898	1.37482
– 0.5	– 0.20000	– 0.37039	1.25112	– 0.37308	1.25116
0.0	0.00000	– 0.62755	1.14277	– 0.62756	1.11280
0.5	0.333333	– 0.82995	1.06277	– 0.82996	1.06277
0.9	0.81818	– 0.96796	1.01151	– 0.96798	1.01149
1.0	1.0	– 1.00000	1.00000	– 1.00002	0.99999
1.1	1.22222	– 1.03119	0.08805	– 1.03121	0.98894
1.5	3.00000	– 1.14860	0.94827	– 1.14862	0.94871

■