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# STRATIFIED VISCOUS FLOW PAST AN ACCELERATED HORIZONTAL PLATE IN A ROTATING FLUID

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### Abstract

A semi-infinite mass of a stratified viscous fluid bounded by an infinite flat plate is initially rotating with uniform angular velocity $\Omega$ about an axis normal to the plate. An analysis is presented for the subsequent flow when the plate started impulsively from rest relative to the rotating fluid moves with uniform acceleration in its own plane. It is found that when $\Omega \neq 0$, the velocity profiles for varying times are non-similar in contrast to the velocity profiles which are similar in the absence of rotation $(\Omega = 0)$. An exact solution to the governing equations has been obtained by the Laplace transform technique. Velocity profiles are shown graphically and the skin friction components (both axial and transverse) are listed in a table for different values of angular velocity $\Omega$, time $T$ and stratification parameter $\lambda$.



## 1. Introduction

The first exact solution of the Navier-Stokes equation was given by Stokes [9] which is concerned with the flow past an impulsively started

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infinite horizontal plate and the flow past an oscillating plate. A similarity solution for the velocity distribution was obtained by Watson [10] while studying the flow of an incompressible viscous fluid near an infinite flat plate started impulsively from rest into motion in its own plane with velocity $At^{\alpha}$, $t$ being the time, $A$ and $\alpha$ being constants with $\alpha \geq 0$. An important characteristic of the above flows is that the convective acceleration terms vanish identically such that the viscous forces interact with the local acceleration. Deka et al. [3] studied the flow past an accelerated horizontal plate in a rotating fluid. Recently Deka et al. [4] studied the subsequent flow past an oscillating plate in a rotating fluid.

Fluid motion induced by the density and viscosity variations in the flow field characterizing as 'stratified flow' finding applications in a large number of technological fields. As an example the concept of colar pond and ocean thermal energy conversion may be mentioned. The intrusion of a heavy fluid into a lighter one occurs in the process of manufacturing glass. Channabasappa and Ranganna [2] studied the flow of viscous stratified fluid past a porous bed. Gupta and Sharma [5] studied the stratified viscous flow to investigate the effect of stratification on the mass flow rate in the channel formed by a moving impermeable wall and Singh [8] studied the stratified viscous fluid through a porous medium between two parallel plates with variable magnetic induction.



It would be interesting to see how the velocity distribution to the flow studied by Deka et al. [3] gets modified for stratified viscous flow of variable density and viscosity. This provides the motivation of the present study. Here a semi-infinite mass of stratified viscous fluid bounded by an infinite flat plate is initially rotating with angular velocity about an axis normal to the plate. We investigate the subsequent flow when the plate started impulsively from rest (relative to the rotating fluid) move with uniform acceleration in its own plane. The analysis reveals that the Coriolis force induces a flow parallel to the plate but transverse to the main flow direction. It may be noted that the flow situations studied in the present problem occur in many cases of engineering interest, e.g., tornadoes or flows past helicopter rotor hubs and flying saucers.

## 2. Mathematical Analysis

We consider here a semi-infinite mass of a stratified viscous fluid bounded by an infinite flat plate occupying the plane $z = 0$. Initially the fluid and the plate rotate in unison with a uniform angular velocity $\Omega$ about the $z$-axis. We introduce a co-ordinate system $(x, y, z)$ rotating with the fluid (Fig. 1). Relative to the fluid, the plate is impulsively started from rest and then moves with uniform acceleration in its own plane along the $x$-axis. The horizontal homogeneity of the problem shows that the flow quantities depend on $z$ and $t$ only, $t$ being the time variable. If $(u, v, w)$ are the components of the velocity vector $\mathbf{q}$, then the equation of continuity $\nabla \cdot \mathbf{q} = 0$ gives $w = 0$ everywhere in the flow such that the boundary condition $w = 0$ is satisfied at the plate. With respect to the rotating frame of reference, the equations of motion for the unsteady flow in the usual notation (see Prandtl [7]) are given by

$$\rho\left(\frac{\partial u}{\partial t} - 2\Omega v\right) = \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right), \tag{1}$$

$$\rho\left(\frac{\partial v}{\partial t} + 2\Omega u\right) = \frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right), \tag{2}$$

$$0 = -\frac{\partial p^*}{\partial z}, \tag{3}$$

where the second term in eqs. (1) and (2) represent the Coriolis force and $p^* = \frac{p}{\rho} - \frac{1}{2}\Omega^2(x^2 + y^2)$, $p$ being the fluid pressure. We assume that there is no imposed pressure gradient so that

$$\frac{\partial p^*}{\partial x} = \frac{\partial p^*}{\partial y} = 0. \tag{4}$$

Equations (3) and ( 4) then show that $p^*$ is a function of time only.

The initial and boundary conditions to be satisfied are:

$$\left.\begin{array}{lll} u = v = 0 & \text{at } t \le 0 & \text{for all } z \\ u = ct, v = 0 & \text{at } z = 0 & \text{for } t > 0 \\ u \to 0, v \to 0 & \text{as } z \to \infty & \text{for } t > 0 \end{array}\right\}, \text{ where } c(> 0) \text{ is a constant.} \tag{5}$$

Here we take

$$\rho = \rho_0 e^{-\lambda z}, \quad \mu = \mu_0 e^{-\lambda z}, \quad \Omega = \Omega_0 e^{-\lambda z}, \quad \nu_0 = \mu_0/\rho_0, \tag{6}$$

where $\rho_0$, $\mu_0$, $\Omega_0$, $\nu_0$, $\lambda$ are respectively density, viscosity, angular velocity, kinematic viscosity and stratification parameter at $z = 0$.

Introducing non-dimensional quantities

$$(U, V) = \frac{(u, v)}{(\nu_0 c)^{1/3}}, \quad Z = z\left(\frac{c}{\nu_0^2}\right)^{1/3}, \quad T = t\left(\frac{c^2}{\nu_0}\right)^{1/3}, \quad \Omega_1 = \Omega_0\left(\frac{\nu_0}{c^2}\right)^{1/3} \tag{7}$$

and using the eqs. (6) and (7), then the eqs. (1) and (2) become

$$\frac{\partial W}{\partial T} = \frac{\partial^2 W}{\partial Z^2} - 2iW\Omega_1 - \lambda\frac{\partial W}{\partial Z} \tag{8}$$

with initial and boundary conditions

$$\left.\begin{array}{lll} W = 0 & \text{at } T \le 0 & \text{for all } Z \\ W = T & \text{at } Z = 0 & \text{for } T > 0 \\ W \to 0 & \text{as } Z \to \infty & \text{for } T > 0 \end{array}\right\}, \tag{9}$$

where $W = U + iV$.

Taking the Laplace transform of eq. (8) along with the initial and boundary conditions (9), we get

$$\overline{W}(Z, s) = e^{\lambda Z/2}\frac{1}{s^2}\exp\left[-\left\{\frac{\lambda^2 + 8i\Omega_1}{4} + s\right\}^{1/2} Z\right], \tag{10}$$

where $\overline{W}(Z, s)$ is the Laplace transform of $W(Z, T)$.

Using Hetnarski's [6] algorithm for generating inverse Laplace transforms of the exponential form we obtain from (10)

$$W(Z, T) = e^{\lambda Z/2}\left[\frac{T}{2} - \frac{Z}{2\sqrt{(\lambda^2 + 8i\Omega_1)}}\right]$$

$$\cdot \exp\left[-\frac{Z}{2}\sqrt{(\lambda^2 + 8i\Omega_1)}\right] erfc\left[\frac{Z}{2\sqrt{T}} - \frac{1}{2}\sqrt{(\lambda^2 + 8i\Omega_1)T}\right]$$

$$+ e^{\lambda Z/2}\left[\frac{T}{2} + \frac{Z}{2\sqrt{(\lambda^2 + 8i\Omega_1)}}\right]\exp\left[\frac{Z}{2}\sqrt{(\lambda^2 + 8i\Omega_1)}\right]$$

$$\cdot\, erfc\left[\frac{Z}{2\sqrt{T}} + \frac{1}{2}\sqrt{(\lambda^2 + 8i\Omega_1)T}\right], \tag{11}$$

where $erfc(x)$ is the complementary error function defined by $erfc(x) = 1 - erf(x)$.

In order to gain an understanding of the flow pattern, we have carried out the numerical computations for the velocity components $(U, V)$ by separating $W$ given by (11) into real and imaginary parts. Since the argument of the error function involved in (11) is complex, we use the following well-known formula (Abramowitz and Stegun [1]) to separate it into real and imaginary parts:

$$erf(X + iY) = erf(X) + \frac{e^{-X^2}}{2\pi X}\{1 - \cos(2XY) + i\sin(2XY)\}$$

$$+ \frac{2}{\pi}e^{-X^2}\sum_{n=1}^{\infty}\frac{e^{-n^2/4}}{n^2 + 4X^2}\{f_n(X, Y) + ig_n(X, Y)\} + \varepsilon(X, Y), \tag{12}$$

where

$$f_n(X, Y) = 2X - 2X\cosh(nY) + n\sinh(nY)\sin(2XY),$$

$$g_n(X, Y) = 2X\cosh(nY)\sin(2XY) + n\sinh(nY)\cos(2XY),$$

$$|\varepsilon(X, Y)| \approx 10^{-16}|erf(X + iY)|. \tag{13}$$

Figure 2 shows the variations of $U$ (the component of velocity in the direction of motion of the plate) and $V$ (the component of velocity transverse to the main flow direction) with $Z$ for different values of the stratification parameter $\lambda$ at time $T = 0.2$ assuming the rotation parameter $\Omega_1 = 0.4$. It can be seen that the axial velocity component $U$ at any given instant and at a given height from the plate increases with an increase in stratification, while the transverse velocity component $V$ decreases. Figure 3 shows that for fixed $T = 0.2$ and $\lambda = 3.0$, the axial velocity slightly decreases with increasing rotation parameter $\Omega_1$, while the transverse velocity decreases more with increasing rotation. Figure 4 shows that for fixed $\Omega_1 = 0.4$ and $\lambda = 3.0$, the axial velocity increases with increasing time, while the transverse velocity decreases. The negative values for $V$ indicates that this component is transverse to the main flow direction $x$ in the clockwise sense. The dimensionless skin-friction at the plate $Z = 0$ is calculated from eq. (11) as

$$-\left(\frac{dW}{dZ}\right)_{Z=0} = \left[\frac{1}{\sqrt{\lambda^2 + 8i\Omega_1}} + \frac{T}{2}\sqrt{\lambda^2 + 8i\Omega_1}\right] erf\sqrt{\left(\frac{(\lambda^2 + 8i\Omega_1)T}{4}\right)}$$

$$+ \sqrt{\frac{T}{\pi}} \exp\left[-\frac{T}{4}(\lambda^2 + 8i\Omega_1)\right] - \frac{\lambda T}{2}. \qquad (14)$$

Separating $-(dW/dZ)_{Z=0}$ into real and imaginary parts of the dimensionless skin-friction components $\tau_x (= -(dU/dZ)_{Z=0})$, $\tau_y (= -(dV/dZ)_{Z=0})$ can be computed from (14). The following table gives the variations of $\tau_x$ and $\tau_y$ with $\Omega_1$, $\lambda$ at different times.



| $T$ | $\Omega_1$ | $\lambda$ | $\tau_x$ | $\tau_y$ |
|---|---|---|---|---|
| 0.20 | 0.40 | 0.00 | 0.50520 | 0.02683 |
| 0.20 | 0.40 | 3.00 | 0.27753 | 0.02470 |
| 0.20 | 0.40 | 5.00 | 0.19287 | 0.02164 |
| 0.20 | 0.40 | 7.00 | 0.14217 | 0.01841 |
| 0.20 | 0.60 | 3.00 | 0.27797 | 0.03703 |
| 0.20 | 0.80 | 3.00 | 0.27859 | 0.04935 |
| 0.40 | 0.40 | 3.00 | 0.31240 | 0.06461 |
| 0.60 | 0.40 | 3.00 | 0.32682 | 0.11055 |
| 0.80 | 0.40 | 3.00 | 0.33506 | 0.15951 |

It can be seen that for fixed rotation parameter at certain instant both $\tau_x$ and $\tau_y$ decrease with an increase in the stratification parameter $\lambda$ but increases with an increase in rotation parameter $\Omega_1$ for fixed $\lambda$ and

$T$. Further for a fixed value of $\lambda$ and $\Omega_1$, both $\tau_x$ and $\tau_y$ increase with an increase in time. These results admit of a physical interpretation. For a fixed stratification parameter at certain instant, an increase in $\Omega_1$ causes a gradual thinning of the boundary layer which develops in the plate, while the opposite is the case with an increase in the stratification parameter $\lambda$ for fixing the value of $\Omega_1$ and time. This results in an increase of the shear stress at the plate with increasing value of $\Omega_1$ while a decrease of the shear stress with an increase in stratification parameter $\lambda$. On the other hand for a fixed value of $\Omega_1$ and $\lambda$ an increase in time results in an increase in the plate velocity and this in turn implies a gradual thinning of the boundary layer on the plate with increasing time $T$.

## 3. Conclusions

(i) The axial velocity at any given instant and at a given height from the plate increases with an increase in stratification while the transverse velocity component decreases. Also same trend is observed for increasing time. But the axial and transverse velocity decreases with increasing rotation parameter.

(ii) Due to the gradual thinning of the boundary layer with rotation, both the skin-friction components along and transverse to the direction of motion of the plate increase with increasing rotation and increasing time but decrease with increasing stratification.



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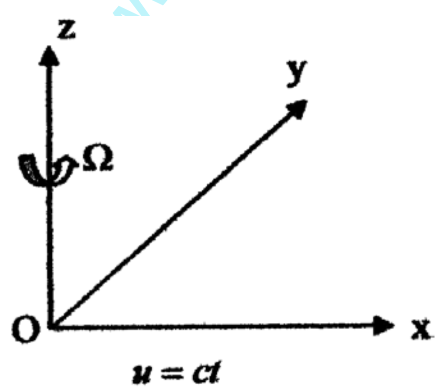


**Figure 1.** A sketch of the physical problem.

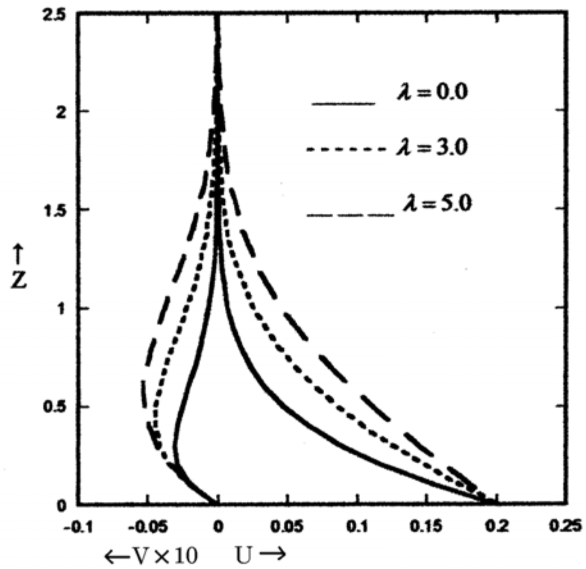


**Figure 2.** Axial and transverse velocity profiles $(U, V)$, $T = 0.2$, $\Omega_1 = 0.4$.

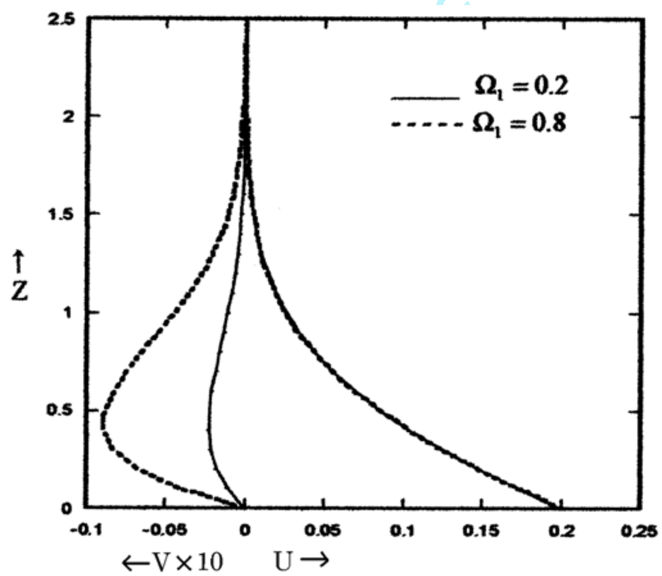


**Figure 3.** Axial and transverse velocity profiles $(U, V)$, $T = 0.2$, $\lambda = 3.0$.

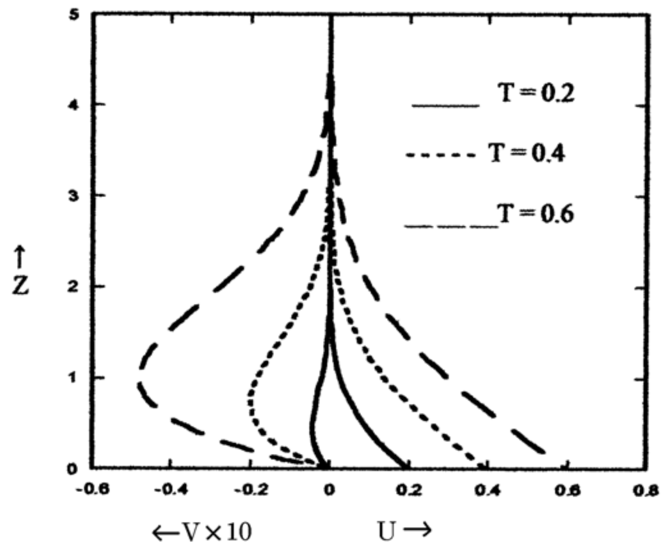


**Figure 4.** Axial and transverse velocity profiles $(U, V)$, $\Omega_1 = 0.4$, $\lambda = 3.0$.

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