Far East J. Math. Sci. (FJMS) **21**(1) (2006), 27-31

SPECTRUM PRESERVING MAPPINGS IN BANACH ALGEBRAS

ABDELAZIZ MAOUCHE

(Received February 19, 2006)

Submitted by K. K. Azad

Abstract

We give a new proof of a theorem stating that the socle of a Banach algebra is the largest algebraic (respectively, spectrum-finite) ideal. Then we prove that a surjective spectrum preserving map is linear on the socle of a Banach algebra.

1. Introduction

We assume throughout this paper that $\mathbf{A}$ and $\mathbf{B}$ are two complex semisimple Banach algebras. The socle of $\mathbf{A}$, denoted by $\mathbf{Soc}(\mathbf{A})$, is the sum of all minimal left ideals of $\mathbf{A}$. It is well known that the socle of $\mathbf{A}$ is a two-sided ideal of $\mathbf{A}$ and all its elements are algebraic. In fact we have the following characterization of the elements of the socle: $a \in \mathbf{Soc}(\mathbf{A})$ if and only if $\mathbf{Sp}(xa)$ is finite for all $x \in \mathbf{A}$. We recall from [3] that *rank one* elements of $\mathbf{A}$ are defined as the set $\mathcal{F}_1(\mathbf{A}) = \{a \in \mathbf{A} : \mathbf{Sp}(xa)$ contains at most one nonzero point for every $x \in \mathbf{A}\}$. This set $\mathcal{F}_1(\mathbf{A})$ is closed under multiplication by elements of $\mathbf{A}$ and $\mathcal{F}_1(\mathbf{A}) \subset \mathbf{Soc}(\mathbf{A})$. Examples of rank one elements are given by the minimal projections of $\mathbf{A}$. Furthermore,

---

2000 Mathematics Subject Classification: 46H70, 17A15.

Keywords and phrases: Banach algebra, spectrum preserving map, trace, socle, rank one element, minimal projection, spectrum-finite ideal.

every minimal left ideal of $\mathbf{A}$ has the form $\mathbf{A}_p$, where p is a minimal projection, so $\mathbf{Soc}(\mathbf{A})$ is equal to the set of all finite sums of rank one elements of A. Now, if $a \in \mathbf{Soc}(\mathbf{A})$, then we define the trace of a as in [3] by $\mathbf{Tr}a = \sum_{\lambda \in \mathbf{Sp}a} \lambda \cdot m(\lambda, a)$, where $m(\lambda, a)$ is the multiplicity of the spectral value λ. It is shown in [3] that $\mathbf{Tr}$ is linear and nondegenerate on $\mathbf{Soc}(\mathbf{A})$, that is, if $a \in \mathbf{Soc}(\mathbf{A})$ and $\mathbf{Tr}(ax) = 0$ for all $x \in \mathbf{Soc}(\mathbf{A})$, then $a = 0$. Moreover, this spectral trace coincides with the usual trace defined on matrices and extends the trace defined on the algebra $\mathcal{B}(\mathbf{X})$ of bounded linear operators on a Banach space $\mathbf{X}$ (see [2] and [3] for more results and details).

2. The Socle is the Largest Spectrum Finite-ideal

We give in this section a new analytic proof of a theorem first proved in [6] saying that the socle of a semisimple Banach algebra $\mathbf{A}$ is the largest spectrum-finite ideal of $\mathbf{A}$. It is easy to see that the socle is algebraic, so its elements have finite spectra. Our aim is to show that if $\mathbf{I}$ is a spectrum-finite ideal of $\mathbf{A}$, then $\mathbf{I} \subseteq \mathbf{Soc}(\mathbf{A})$.

Theorem 2.1. *Let* $\mathbf{A}$ *be a complex semisimple Banach algebra and* $\mathbf{I}$ *be an algebraic* (*or spectrum-finite*) *ideal of* $\mathbf{A}$. *Then* $\mathbf{I} \subseteq \mathbf{Soc}(\mathbf{A})$.

Proof. Since $\mathbf{I}$ is an algebraic ideal, its elements have finite spectra. Let $a \in \mathbf{I}$ and $\mathbf{A}_n = \{x \in \mathbf{A} : \# \mathbf{Sp}(xa) \leq n\}$. By Newburgh's theorem, each $\mathbf{A}_n$ is closed and $\mathbf{A} = \bigcup_{n=1}^{\infty} \mathbf{A}_n$. By Baire's category theorem there exists an $n_0 \in \mathbf{N}$ such that $\mathbf{A}_{n_0}$ contains a nonempty open set Ω. Let $x \in \Omega$ and $y \in \mathbf{A}$ and consider the application $\mathbf{F} : \mathbf{C} \to \mathbf{A}a$, $\lambda \to [(1 - \lambda)x + \lambda y]a$. By the subharmonicity of $\log \delta_n$ and the fact that $\log \delta_n(\mathbf{F}(\lambda)) = -\infty$ on the segment [0, 1] which is of positive capacity, we deduce that $\delta_n(\mathbf{F}(\lambda)) = 0$, for all $\lambda \in \mathbf{C}$ (where δ_n denotes the n-th diameter (see [1])). In particular this is true for $\lambda = 1$. Then the spectrum of each element of $\mathbf{A}a$ has at most n points. Let $b \in \mathbf{A}$ such that

$\#\,\mathbf{Sp}(ba) = n$. Let $p_1, \ldots, p_n$ be the Riesz projections associated to the nonzero spectral values of $\mathbf{Sp}(ba)$. We intend to show that each projection p_i is minimal and $a = ap_1 + \cdots + ap_n$. Suppose that p_i is not minimal. Then there exists another nontrivial projection $p_i z p_i \in p_i \mathbf{A} p_i$. Thus the set $\{p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n, p_i z p_i, p_i - p_i z p_i\}$ is composed of orthogonal projections of $\mathbf{A}a$. But this is in contradiction with the fact that the spectrum of every element of $\mathbf{A}a$ consists of at most n elements. Let $p = p_1 + \cdots + p_n$ and suppose that $a \neq ap$. Then $\mathbf{A}(a - ap) \neq \{0\}$ and there exists a projection $q \in \mathbf{A}(a - ap)$, i.e., $q^2 = q = l(1-p)$ with $l \in \mathbf{A}a$. We see that $qp_i = 0$ for $i = 1, \ldots, n$ because $qp_i = l(1-p)p_i = l(p_i - p_i) = 0$. Now if $d = p_1 + 2p_2 + \cdots + np_n + (n+1)q$, then $d \in \mathbf{A}a$ and $\{1, 2, \ldots, n, n+1\} \subset \mathbf{Sp}(d)$, but this contradicts $\#\,\mathbf{Sp}(ba) = n$.

3. Spectrum Preserving Mappings

Let $\mathbf{A}$ and $\mathbf{B}$ be two semisimple Banach algebras and $\Phi : \mathbf{A} \mapsto \mathbf{B}$ be a surjective mapping such that $\mathbf{Sp}(\Phi(a)\Phi(x)) = \mathbf{Sp}(ax)$. We show that Φ is linear on the socle of $\mathbf{A}$. Notice that unlike most authors who are interested in spectrum-preserving problems, we do not assume linearity or multiplicativity of Φ in advance (see [7]). Our results are similar to those obtained in [5] in the case of the algebra of linear bounded operators $\mathbf{B}(\mathbf{H})$, where $\mathbf{H}$ is a Hilbert space.

Lemma 3.1. *Let* $a \in \mathbf{Soc}(\mathbf{A})$. *If* $\mathbf{Sp}(ax) = \{0\}$ *for all* $x \in \mathcal{F}_1(\mathbf{A})$, *then* $a = 0$ *and* Φ *is injective.*

Proof. If $\mathbf{Sp}(ax) = \{0\}$ for all $x \in \mathcal{F}_1(\mathbf{A})$, then $\mathbf{Tr}(au) = 0$ for all $u \in \mathcal{F}_1(\mathbf{A})$. Now if $y \in \mathbf{Soc}(\mathbf{A})$, then $y = \sum_{i=0}^{n} u_i$ with $u_i \in \mathcal{F}_1(\mathbf{A})$. So

$$\mathbf{Tr}(ay) = \mathbf{Tr}\left(a\sum_{i=0}^{n} u_i\right) = \sum_{i=0}^{n} \mathbf{Tr}(au_i) = \sum_{i=0}^{n} 0 = 0.$$

Since $\mathbf{Tr}$ is nondegenerate, we have $a = 0$. Suppose that $\Phi(a) = 0$. Then for each $x \in \mathcal{F}_1(\mathbf{A})$ we have $\mathbf{Sp}(ax) = \mathbf{Sp}(\Phi(a)\Phi(x)) = \{0\}$ and $a = 0$ by the first part of the proof.

Theorem 3.2. *Let* $\Phi : \mathbf{A} \mapsto \mathbf{B}$ *be a surjective map preserving the spectrum in the sense* $\mathbf{Sp}(\Phi(a)\Phi(x)) = \mathbf{Sp}(ax)$ *for every* $a, x \in \mathbf{A}$. *Then* Φ *is injective,* $\Phi(\mathcal{F}_1(\mathbf{A})) = \mathcal{F}_1(\mathbf{B})$ *and* $\Phi(\mathbf{Soc}(\mathbf{A})) = \mathbf{Soc}(\mathbf{B})$.

Proof. (i) Let $u \in \mathcal{F}_1(\mathbf{A})$. Then $\#\,\mathbf{Sp}(ux)\setminus\{0\} = 1$ for all $x \in \mathbf{A}$. For each $y \in \mathbf{B}$, $\mathbf{Sp}(\Phi(u)y) = \mathbf{Sp}(\Phi(u)\Phi(x)) = \mathbf{Sp}(ux)$, for some x in $\mathbf{A}$. So $\Phi(u) \in \mathcal{F}_1(\mathbf{B})$. Similarly, we show that if $v \in \mathcal{F}_1(\mathbf{B})$, then there exists $u \in \mathcal{F}_1(\mathbf{A})$ such that $v = \Phi(u)$. Indeed, $\#\,\mathbf{Sp}(vy)\setminus\{0\} = 1$ for all $y \in \mathbf{B}$, $v = \Phi(u)$ and $y = \Phi(x)$ for u, x in $\mathbf{A}$ because Φ is surjective. Then $\mathbf{Sp}(vy) = \mathbf{Sp}(\Phi(u)\Phi(x)) = \mathbf{Sp}(ux)$ for all $x \in \mathbf{A}$. Thus $u \in \mathcal{F}_1(\mathbf{A})$ and $\Phi(\mathcal{F}_1(\mathbf{A})) = \mathcal{F}_1(\mathbf{B})$.

(ii) Let $a \in \mathbf{Soc}(\mathbf{A})$. So $a = axa$ for some $x \in \mathbf{A}$ because the socle is von Neumann regular. Then $\Phi(a) = \Phi(axa) \in \mathbf{Soc}(\mathbf{B})$ since for an arbitrary $y \in \mathbf{B}$, there exists $z \in \mathbf{A}$ such that $y = \Phi(z)$, so $\mathbf{Sp}(\Phi(a)y) = \mathbf{Sp}(\Phi(a)\Phi(z)) = \mathbf{Sp}(az)$ is finite; then $\Phi(a) \in \mathcal{F}_n(\mathbf{B}) \subset \mathbf{Soc}(\mathbf{B})$ for some $n \in \mathbf{N}$. Conversely, if $y \in \mathbf{Soc}(\mathbf{B})$, then $y = \Phi(a)$ for some $a \in \mathbf{Soc}(\mathbf{A})$ for the same reason. Then $\Phi(\mathbf{Soc}(\mathbf{A})) = \mathbf{Soc}(\mathbf{B})$.

Theorem 3.3. *Let* $\Phi : \mathbf{A} \mapsto \mathbf{B}$ *be a surjective map preserving the spectrum in the sense* $\mathbf{Sp}(\Phi(a)\Phi(x)) = \mathbf{Sp}(ax)$ *for every* $a, x \in \mathbf{A}$. *Then* $[\Phi(a+b) - \Phi(a) - \Phi(b)] \cdot y = 0$ *for all* a, b *in* $\mathbf{A}$, $y \in \mathbf{Soc}(\mathbf{B})$ *and* Φ *is linear on* $\mathbf{Soc}(\mathbf{A})$.

Proof. Let $y \in \mathbf{Soc}(\mathbf{B})$, so $y = y_1 + \cdots + y_n$, with $y_i \in \mathcal{F}_1(\mathbf{B})$. By the previous theorem there exists $u_i \in \mathbf{Soc}(\mathbf{A})$ such that $y_i = \Phi(u_i)$ and $y = \sum_{i=1}^{n} \Phi(u_i) \in \mathbf{Soc}(\mathbf{B})$. Consequently,

$$\mathbf{Tr}[(\Phi(a+b) - \Phi(a) - \Phi(b)) \cdot y] = \sum_{i=1}^{n} \mathbf{Tr}([\Phi(a+b) - \Phi(a) - \Phi(b)] \cdot \Phi(u_i)) = 0$$

because for $s \in \mathcal{F}_1(\mathbf{A})$ and by linearity of the trace we have:

$$\begin{aligned}\mathbf{Tr}[(\Phi(a) + \Phi(b)) \cdot \Phi(s)] &= \mathbf{Tr}(\Phi(a)\Phi(s) + \Phi(b)\Phi(s)) \\ &= \mathbf{Tr}(\Phi(a)\Phi(s)) + \mathbf{Tr}(\Phi(b)\Phi(s))\end{aligned}$$

$$= \mathbf{Tr}(as) + \mathbf{Tr}(bs) \quad (\text{because } as, bs \in \mathcal{F}_1(\mathbf{A}))$$
$$= \mathbf{Tr}((a+b)s).$$

Then $\mathbf{Tr}[(\Phi(a)+\Phi(b))\cdot(s)] = \mathbf{Tr}[(a+b)s] = \mathbf{Tr}[\Phi(a+b)\Phi(s)]$ and $\mathbf{Tr}[(\Phi(a) + \Phi(b) - \Phi(a+b))\cdot\Phi(s)] = 0$ for all $s \in \mathcal{F}_1(\mathbf{A})$. Hence we conclude that $\mathbf{Tr}[(\Phi(a)+\Phi(b)-\Phi(a+b))\cdot\Phi(s)] = 0$ for all $s \in \mathbf{Soc}(\mathbf{A})$. That is $\Phi(a) + \Phi(b) - \Phi(a+b) \in \mathbf{Soc}(\mathbf{B})^{\perp}$. It is well known that $\mathbf{Tr}$ is nondegenerate on the socle, i.e., if $\mathbf{Tr}(xu) = 0$ for every $u \in \mathbf{Soc}(\mathbf{A})$ and some $x \in \mathbf{Soc}(\mathbf{A})$, then $x = 0$ [3]. It follows that Φ is additive on $\mathbf{Soc}(\mathbf{A})$. Also, since $\mathbf{Sp}[\Phi(a)\Phi(x)] = \mathbf{Sp}(ax)$ for every $a, x \in \mathbf{A}$, we have $\Phi(\lambda a) = \lambda\Phi(a)$ for all $\lambda \in \mathbf{C}$ and $a \in \mathbf{Soc}(\mathbf{A})$. Indeed, $\mathbf{Tr}[(\Phi(\lambda a) - \lambda\Phi(a))\Phi(u)] = \mathbf{Tr}[\Phi(\lambda a)\Phi(u)] - \lambda\mathbf{Tr}[\Phi(a)\Phi(u)] = \mathbf{Tr}(\lambda au) - \lambda\mathbf{Tr}(au) = 0$, which implies that $\Phi(\lambda a) = \lambda\Phi(a)$ for every $a \in \mathbf{Soc}(\mathbf{A})$. Hence Φ is linear on $\mathbf{Soc}(\mathbf{A})$.

Remark 3.4. Similar results hold also for Jordan-Banach algebras.

References

[1] B. Aupetit, A Primer on Spectral Theory, Springer-Verlag, New York, 1991.

[2] B. Aupetit and A. Maouche, Trace and determinant in Jordan-Banach algebras, Publ. Mat. 46 (2002), 3-16.

[3] B. Aupetit and H. du Toit Mouton, Trace and determinant in Banach algebras, Studia Math. 121(2) (1996), 115-136.

[4] A. Maouche, Formes multiplicatives à valeurs dans le spectre, Colloq. Math. 71 (1996), 43-45.

[5] L. Molnár, Characterizations of the automorphisms of $B(H)$ and $C(X)$, Proc. Amer. Math. Soc. 130 (2001), 111-120.

[6] M. R. F. Smyth, Riesz theory in Banach algebras, Math. Z. 145(2) (1975), 145-155.

[7] J. Wu, Spectrum preserving multiplicative maps, Acta Math. Sinica (N.S.) 14 (1998), 719-722.

Department of Mathematics and Statistics
College of Science
Sultan Qaboos University
P. O. Box 36, AL Khod 123, Oman
e-mail: maouche@squ.edu.om