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THE APPROXIMATE SOLUTIONS OF HIGH-ORDER LINEAR DIFFERENTIAL EQUATION SYSTEMS WITH VARIABLE COEFFICIENTS

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Abstract

In the present paper, a Taylor method is developed to find the approximate solution of high-order linear differential equation system with specified associated conditions in terms of Taylor polynomials at any point. In addition, examples that illustrate the pertinent features of the method are presented, and the results of the study are discussed.

1. Introduction

A Taylor method for solving Fredholm integral equations has been presented by Kanwall and Liu [1] and then this method has been extended by Sezer to Volterra integral equations [2] and to differential equations [3]. Similar approach has been used to solve linear Volterra-Fredholm integro-differential equations applied by Yalçinbaş and Sezer [6] and nonlinear Volterra-Fredholm integral equations by Yalçinbaş [4].

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The technique is based on, first, differentiating both sides of differential equation system n times and then substituting the Taylor series for the unknown function in the resulting equation. Here, the obtained linear algebraic system has been solved approximately by a suitable truncation scheme.

In this study, the basic ideas of the mentioned works are developed and applied to problems consisting of:

1. High order linear differential equation system

$$\left.\begin{aligned}
&{}^{k}r_{11}(t)\frac{d^{k}y_{1}}{dt^{k}} + {}^{k-1}r_{11}(t)\frac{d^{k-1}y_{1}}{dt^{k-1}} + \ldots + {}^{0}r_{11}(t)y_{1}(t)\\
&+ {}^{k}r_{12}(t)\frac{d^{k}y_{2}}{dt^{k}} + {}^{k-1}r_{12}(t)\frac{d^{k-1}y_{2}}{dt^{k-1}} + \ldots + {}^{0}r_{12}(t)y_{2}(t)\\
&+ \ldots + {}^{k}r_{1s}(t)\frac{d^{k}y_{s}}{dt^{k}} + {}^{k-1}r_{1s}(t)\frac{d^{k-1}y_{s}}{dt^{k-1}} + \ldots + {}^{0}r_{1s}(t)y_{s}(t) = f_{1}(t)\\
&{}^{k}r_{21}(t)\frac{d^{k}y_{1}}{dt^{k}} + {}^{k-1}r_{21}(t)\frac{d^{k-1}y_{1}}{dt^{k-1}} + \ldots + {}^{0}r_{21}(t)y_{1}(t)\\
&+ {}^{k}r_{22}(t)\frac{d^{k}y_{2}}{dt^{k}} + {}^{k-1}r_{22}(t)\frac{d^{k-1}y_{2}}{dt^{k-1}} + \ldots + {}^{0}r_{22}(t)y_{2}(t)\\
&+ \ldots + {}^{k}r_{2s}(t)\frac{d^{k}y_{s}}{dt^{k}} + {}^{k-1}r_{2s}(t)\frac{d^{k-1}y_{s}}{dt^{k-1}} + \ldots + {}^{0}r_{2s}(t)y_{s}(t) = f_{2}(t)\\
&\qquad\vdots\\
&{}^{k}r_{s1}(t)\frac{d^{k}y_{1}}{dt^{k}} + {}^{k-1}r_{s1}(t)\frac{d^{k-1}y_{1}}{dt^{k-1}} + \ldots + {}^{0}r_{s1}(t)y_{1}(t)\\
&+ {}^{k}r_{s2}(t)\frac{d^{k}y_{2}}{dt^{k}} + {}^{k-1}r_{s2}(t)\frac{d^{k-1}y_{2}}{dt^{k-1}} + \ldots + {}^{0}r_{s2}(t)y_{2}(t)\\
&+ \ldots + {}^{k}r_{ss}(t)\frac{d^{k}y_{s}}{dt^{k}} + {}^{k-1}r_{ss}(t)\frac{d^{k-1}y_{s}}{dt^{k-1}} + \ldots + {}^{0}r_{ss}(t)y_{s}(t) = f_{s}(t)
\end{aligned}\right\} \tag{1}$$

or briefly

$$\sum_{n=0}^{k}\sum_{m=1}^{s} {}^{n}r_{jm}(t)y_{m}^{(n)}(t) = f_{j}(t),\ (j = 1, 2, \ldots, s). \tag{1}$$

2. The conditions (in most general)

$$\left.\begin{aligned} \sum_{j=0}^{k-1}[\alpha_{ij}^{1}y_1^{(j)}(a)+\beta_{ij}^{1}y_1^{(j)}(b)+\gamma_{ij}^{1}y_1^{(j)}(c)] &= \lambda_{1i} \\ \sum_{j=0}^{k-1}[\alpha_{ij}^{2}y_2^{(j)}(a)+\beta_{ij}^{2}y_2^{(j)}(b)+\gamma_{ij}^{2}y_2^{(j)}(c)] &= \lambda_{2i} \\ &\vdots \\ \sum_{j=0}^{k-1}[\alpha_{ij}^{s}y_s^{(j)}(a)+\beta_{ij}^{s}y_s^{(j)}(b)+\gamma_{ij}^{s}y_s^{(j)}(c)] &= \lambda_{si} \end{aligned}\right\}, \tag{2}$$

where ${}^{n}r_{jm}(t), f_j(t), (m, j = 1, 2, ..., s; n = 0, 1, 2, ..., k)$ are functions having nth derivatives on an interval $a \le t, c \le b$, provided that the real coefficients α_{ij}^{m}, β_{ij}^{m}, γ_{ij}^{m} and λ_{mi} $(i, j = 0, 1, 2, ..., k-1, m = 1, 2, ..., s)$ appropriate constants, and the solution is expressed in the form

$$y_m(t) = \sum_{k=0}^{N}\frac{1}{k!}y_m^{(k)}(c)(t-c)^k; \quad \begin{pmatrix} m = 1, 2, ..., s \\ a \le t, c \le b, N \ge k \end{pmatrix} \tag{3}$$

which is a Taylor polynomial of degree N at $t = c$. Here $y_m^{(k)}(c)$, $(m = 1, 2, ..., s;\ k = 0, 1, 2, ..., N)$ are the coefficients to be determined.

2. Method of Solution

To obtain the solution of the given problem in the form of expression (3) we first differentiate equations (1) n times with respect to t to obtain

$$\left[{}^{k}r_{11}(t)\frac{d^k y_1}{dt^k}\right]^{(n)} + \left[{}^{k-1}r_{11}(t)\frac{d^{k-1}y_1}{dt^{k-1}}\right]^{(n)} + \dots + [{}^{0}r_{11}(t)y_1(t)]^{(n)}$$

$$+\left[{}^{k}r_{12}(t)\frac{d^k y_2}{dt^k}\right]^{(n)} + \left[{}^{k-1}r_{12}(t)\frac{d^{k-1}y_2}{dt^{k-1}}\right]^{(n)} + \dots + [{}^{0}r_{12}(t)y_2(t)]^{(n)}$$

$$+\dots+\left[{}^{k}r_{1s}(t)\frac{d^k y_s}{dt^k}\right]^{(n)} + \left[{}^{k-1}r_{1s}(t)\frac{d^{k-1}y_s}{dt^{k-1}}\right]^{(n)} + \dots + [{}^{0}r_{1s}(t)y_s(t)]^{(n)} = [f_1(t)]^{(n)}$$

$$\left[{}^{k}r_{21}(t)\frac{d^k y_1}{dt^k}\right]^{(n)} + \left[{}^{k-1}r_{21}(t)\frac{d^{k-1} y_1}{dt^{k-1}}\right]^{(n)} + \ldots + [{}^{0}r_{21}(t)y_1(t)]^{(n)}$$

$$+\left[{}^{k}r_{22}(t)\frac{d^k y_2}{dt^k}\right]^{(n)} + \left[{}^{k-1}r_{22}(t)\frac{d^{k-1} y_2}{dt^{k-1}}\right]^{(n)} + \ldots + [{}^{0}r_{22}(t)y_2(t)]^{(n)}$$

$$+\ldots+\left[{}^{k}r_{2s}(t)\frac{d^k y_s}{dt^k}\right]^{(n)} + \left[{}^{k-1}r_{2s}(t)\frac{d^{k-1} y_s}{dt^{k-1}}\right]^{(n)} + \ldots + [{}^{0}r_{2s}(t)y_s(t)]^{(n)} = [f_2(t)]^{(n)}$$

$$\vdots$$

$$\left[{}^{k}r_{s1}(t)\frac{d^k y_1}{dt^k}\right]^{(n)} + \left[{}^{k-1}r_{s1}(t)\frac{d^{k-1} y_1}{dt^{k-1}}\right]^{(n)} + \ldots + [{}^{0}r_{s1}(t)y_1(t)]^{(n)}$$

$$+\left[{}^{k}r_{s2}(t)\frac{d^k y_2}{dt^k}\right]^{(n)} + \left[{}^{k-1}r_{s2}(t)\frac{d^{k-1} y_2}{dt^{k-1}}\right]^{(n)} + \ldots + [{}^{0}r_{s2}(t)y_2(t)]^{(n)}$$

$$+\ldots+\left[{}^{k}r_{ss}(t)\frac{d^k y_s}{dt^k}\right]^{(n)} + \left[{}^{k-1}r_{ss}(t)\frac{d^{k-1} y_s}{dt^{k-1}}\right]^{(n)}$$

$$+\ldots+[{}^{0}r_{ss}(t)y_s(t)]^{(n)} = [f_s(t)]^{(n)}, \tag{4}$$

where $n = 0, 1, \ldots, N$. Using the Leibnitz's rule (dealing with differentiation of products of functions), simplifying and then substituting $t = c$ into the resulting equation, we have

$$\sum_{m=0}^{n}\binom{n}{m}\left\{\begin{aligned}&{}^{k}r_{11}^{(n-m)}(c)y_1^{(m+k)}(c) + {}^{k-1}r_{11}^{(n-m)}(c)y_1^{(m+k-1)}(c) + \ldots + {}^{1}r_{11}^{(n-m)}(c)y_1^{(m+1)}(c)\\ &+ {}^{0}r_{11}^{(n-m)}(c)y_1^{(m)}(c) + {}^{k}r_{12}^{(n-m)}(c)y_2^{(m+k)}(c) + {}^{k-1}r_{12}^{(n-m)}(c)y_2^{(m+k-1)}(c)\\ &+\ldots+ {}^{1}r_{12}^{(n-m)}(c)y_2^{(m+1)}(c) + {}^{0}r_{12}^{(n-m)}(c)y_2^{(m)}(c) + \ldots + {}^{k}r_{1s}^{(n-m)}(c)y_s^{(m+k)}(c)\\ &+ {}^{k-1}r_{1s}^{(n-m)}(c)y_s^{(m+k-1)}(c) + \ldots + {}^{1}r_{1s}^{(n-m)}(c)y_s^{(m+1)}(c) + {}^{0}r_{1s}^{(n-m)}(c)y_s^{(m)}(c)\end{aligned}\right\}$$

$$= f_1^{(n)}(c)$$

$$\sum_{m=0}^{n}\binom{n}{m}\left\{\begin{aligned}&{}^{k}r_{21}^{(n-m)}(c)y_1^{(m+k)}(c) + {}^{k-1}r_{21}^{(n-m)}(c)y_1^{(m+k-1)}(c) + \ldots + {}^{1}r_{21}^{(n-m)}(c)y_1^{(m+1)}(c)\\ &+ {}^{0}r_{21}^{(n-m)}(c)y_1^{(m)}(c) + {}^{k}r_{22}^{(n-m)}(c)y_2^{(m+k)}(c) + {}^{k-1}r_{22}^{(n-m)}(c)y_2^{(m+k-1)}(c)\\ &+\ldots+ {}^{1}r_{22}^{(n-m)}(c)y_2^{(m+1)}(c) + {}^{0}r_{22}^{(n-m)}(c)y_2^{(m)}(c) + \ldots + {}^{k}r_{2s}^{(n-m)}(c)y_s^{(m+k)}(c)\\ &+ {}^{k-1}r_{2s}^{(n-m)}(c)y_s^{(m+k-1)}(c) + \ldots + {}^{1}r_{2s}^{(n-m)}(c)y_s^{(m+1)}(c) + {}^{0}r_{2s}^{(n-m)}(c)y_s^{(m)}(c)\end{aligned}\right\}$$

$$= f_2^{(n)}(c)$$

$$\vdots$$

$$\sum_{m=0}^{n}\binom{n}{m}\left\{\begin{array}{l} {}^{k}r_{s1}^{(n-m)}(c)\,y_1^{(m+k)}(c) + {}^{k-1}r_{s1}^{(n-m)}(c)\,y_1^{(m+k-1)}(c) + \ldots + {}^{1}r_{s1}^{(n-m)}(c)\,y_1^{(m+1)}(c) \\ + {}^{0}r_{s1}^{(n-m)}(c)\,y_1^{(m)}(c) + {}^{k}r_{s2}^{(n-m)}(c)\,y_2^{(m+k)}(c) + {}^{k-1}r_{s2}^{(n-m)}(c)\,y_2^{(m+k-1)}(c) \\ + \ldots + {}^{1}r_{s2}^{(n-m)}(c)\,y_2^{(m+1)}(c) + {}^{0}r_{s2}^{(n-m)}(c)\,y_2^{(m)}(c) + \ldots + {}^{k}r_{ss}^{(n-m)}(c)\,y_s^{(m+k)}(c) \\ + {}^{k-1}r_{ss}^{(n-m)}(c)\,y_s^{(m+k-1)}(c) + \ldots + {}^{1}r_{ss}^{(n-m)}(c)\,y_s^{(m+1)}(c) + {}^{0}r_{ss}^{(n-m)}(c)\,y_s^{(m)}(c) \end{array}\right\}$$

$$= f_s^{(n)}(c). \tag{5}$$

This is a system of $s(N+1)$ linear equations for the $s(N+1)$ unknown coefficients $y_m^{(k)}(c)$, $(m = 1, 2, \ldots, s;\ k = 0, 1, \ldots, N)$. Here ${}^{n}r_{jm}^{(z)}(c)$, $f_j^{(z)}(c)$, $(n = 0, 1, \ldots, k;\ m, j = 1, 2, \ldots, s;\ z = 0, 1, \ldots, n;\ n = 0, 1, \ldots, N)$, respectively, denote the values of zth derivatives of the unknown functions ${}^{n}r_{jm}$, f_j at $t = c$, [3, 4, 6].

Note that, in general, system (5) cannot be directly used for solution of the given problem, but it is a fundamental relation.

To solve the more general problem consisting of equations (1) and conditions (2), we now write the matrix form of system (5) as

$$\left.\begin{array}{l} \mathbf{W}_{11}.\mathbf{Y}_1 + \mathbf{W}_{12}.\mathbf{Y}_2 + \ldots + \mathbf{W}_{1s}.\mathbf{Y}_s = \mathbf{F}_1 \\ \mathbf{W}_{21}.\mathbf{Y}_1 + \mathbf{W}_{22}.\mathbf{Y}_2 + \ldots + \mathbf{W}_{2s}.\mathbf{Y}_s = \mathbf{F}_2 \\ \qquad\vdots \\ \mathbf{W}_{s1}.\mathbf{Y}_1 + \mathbf{W}_{s2}.\mathbf{Y}_2 + \ldots + \mathbf{W}_{ss}.\mathbf{Y}_s = \mathbf{F}_s \end{array}\right\}, \tag{6}$$

where

$$\mathbf{Y}_1 = [y_1^{(0)}(c)\ \ y_1^{(1)}(c)\ \ \ldots\ \ y_1^{(N)}(c)]^T$$

$$\mathbf{Y}_2 = [y_2^{(0)}(c)\ \ y_2^{(1)}(c)\ \ \ldots\ \ y_2^{(N)}(c)]^T$$

$$\vdots$$

$$\mathbf{Y}_s = [y_s^{(0)}(c)\ \ y_s^{(1)}(c)\ \ \ldots\ \ y_s^{(N)}(c)]^T$$

$$\mathbf{F}_1 = [f_1^{(0)}(c)\ \ f_1^{(1)}(c)\ \ \ldots\ \ f_1^{(N)}(c)]^T$$

$$\mathbf{F}_2 = [f_2^{(0)}(c)\ \ f_2^{(1)}(c)\ \ \ldots\ \ f_2^{(N)}(c)]^T$$

$$\vdots$$

$$\mathbf{F}_s = [f_s^{(0)}(c)\ \ f_s^{(1)}(c)\ \ \ldots\ \ f_s^{(N)}(c)]^T$$

and for $n, m = 0, 1, \ldots, N$,

$$\left.\begin{array}{l} \mathbf{W}_{11} = [(w_{11})_{nm}],\ \mathbf{W}_{12} = [(w_{12})_{nm}],\ \ldots,\ \mathbf{W}_{1s} = [(w_{1s})_{nm}] \\ \mathbf{W}_{21} = [(w_{21})_{nm}],\ \mathbf{W}_{22} = [(w_{22})_{nm}],\ \ldots,\ \mathbf{W}_{2s} = [(w_{2s})_{nm}] \\ \mathbf{W}_{s1} = [(w_{s1})_{nm}],\ \mathbf{W}_{s2} = [(w_{s2})_{nm}],\ \ldots,\ \mathbf{W}_{ss} = [(w_{ss})_{nm}] \end{array}\right\}.$$

The elements of which are defined by

$$(w_{11})_{nm} = \binom{n}{m-k}{}^{k}r_{11}^{(n-m+k)}(c) + \binom{n}{m-(k-1)}{}^{k-1}r_{11}^{(n-m+(k-1))}(c)$$

$$+ \ldots + \binom{n}{m-1}{}^{1}r_{11}^{(n-m+1)}(c) + \binom{n}{m}{}^{0}r_{11}^{(n-m)}(c)$$

$$(w_{12})_{nm} = \binom{n}{m-k}{}^{k}r_{12}^{(n-m+k)}(c) + \binom{n}{m-(k-1)}{}^{k-1}r_{12}^{(n-m+(k-1))}(c)$$

$$+ \ldots + \binom{n}{m-1}{}^{1}r_{12}^{(n-m+1)}(c) + \binom{n}{m}{}^{0}r_{12}^{(n-m)}(c)$$

$$\vdots$$

$$(w_{1s})_{nm} = \binom{n}{m-k}{}^{k}r_{1s}^{(n-m+k)}(c) + \binom{n}{m-(k-1)}{}^{k-1}r_{1s}^{(n-m+(k-1))}(c)$$

$$+ \ldots + \binom{n}{m-1}{}^{1}r_{1s}^{(n-m+1)}(c) + \binom{n}{m}{}^{0}r_{1s}^{(n-m)}(c)$$

$$(w_{21})_{nm} = \binom{n}{m-k}{}^{k}r_{21}^{(n-m+k)}(c) + \binom{n}{m-(k-1)}{}^{k-1}r_{21}^{(n-m+(k-1))}(c)$$

$$+ \ldots + \binom{n}{m-1}{}^{1}r_{21}^{(n-m+1)}(c) + \binom{n}{m}{}^{0}r_{21}^{(n-m)}(c)$$

$$(w_{22})_{nm} = \binom{n}{m-k} {}^{k}r_{22}^{(n-m+k)}(c) + \binom{n}{m-(k-1)} {}^{k-1}r_{22}^{(n-m+(k-1))}(c)$$

$$+ \dots + \binom{n}{m-1} {}^{1}r_{22}^{(n-m+1)}(c) + \binom{n}{m} {}^{0}r_{22}^{(n-m)}(c)$$

$$\vdots$$

$$(w_{2s})_{nm} = \binom{n}{m-k} {}^{k}r_{2s}^{(n-m+k)}(c) + \binom{n}{m-(k-1)} {}^{k-1}r_{2s}^{(n-m+(k-1))}(c)$$

$$+ \dots + \binom{n}{m-1} {}^{1}r_{2s}^{(n-m+1)}(c) + \binom{n}{m} {}^{0}r_{2s}^{(n-m)}(c)$$

$$\vdots$$

$$(w_{s1})_{nm} = \binom{n}{m-k} {}^{k}r_{s1}^{(n-m+k)}(c) + \binom{n}{m-(k-1)} {}^{k-1}r_{s1}^{(n-m+(k-1))}(c)$$

$$+ \dots + \binom{n}{m-1} {}^{1}r_{s1}^{(n-m+1)}(c) + \binom{n}{m} {}^{0}r_{s1}^{(n-m)}(c)$$

$$(w_{s2})_{nm} = \binom{n}{m-k} {}^{k}r_{s2}^{(n-m+k)}(c) + \binom{n}{m-(k-1)} {}^{k-1}r_{s2}^{(n-m+(k-1))}(c)$$

$$+ \dots + \binom{n}{m-1} {}^{1}r_{s2}^{(n-m+1)}(c) + \binom{n}{m} {}^{0}r_{s2}^{(n-m)}(c)$$

$$\vdots$$

$$(w_{ss})_{nm} = \binom{n}{m-k} {}^{k}r_{ss}^{(n-m+k)}(c) + \binom{n}{m-(k-1)} {}^{k-1}r_{ss}^{(n-m+(k-1))}(c)$$

$$+ \dots + \binom{n}{m-1} {}^{1}r_{ss}^{(n-m+1)}(c) + \binom{n}{m} {}^{0}r_{ss}^{(n-m)}(c) \quad [5]. \tag{7}$$

Note that in equations (7) that for $\ell < 0$,

$${}^{n}r_{jm}^{(\ell)} = 0 \tag{8}$$

and for $j < 0$ and $j > i$, $\binom{i}{j} = 0$, where i, j and k are integers. In this case, in equations (7), for $n < m - k$, $(n = 0, 1, \dots, N-(k-1);\ m = k+1, k+2, \dots, N)$

$$(w_{ij})_{nm} = 0.$$

Now write the system (6) in matrix form

$$\mathbf{WY} = \mathbf{F}, \tag{9}$$

where

$$\mathbf{Y} = \begin{bmatrix} y_1^{(0)}(c) \\ y_1^{(1)}(c) \\ \vdots \\ y_1^{(N)}(c) \\ y_2^{(0)}(c) \\ y_2^{(1)}(c) \\ \vdots \\ y_2^{(N)}(c) \\ \vdots \\ y_s^{(0)}(c) \\ y_s^{(1)}(c) \\ \vdots \\ y_s^{(N)}(c) \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \vdots \\ \mathbf{Y}_s \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} f_1^{(0)}(c) \\ f_1^{(1)}(c) \\ \vdots \\ f_1^{(N)}(c) \\ f_2^{(0)}(c) \\ f_2^{(1)}(c) \\ \vdots \\ f_2^{(N)}(c) \\ \vdots \\ f_s^{(0)}(c) \\ f_s^{(1)}(c) \\ \vdots \\ f_s^{(N)}(c) \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \vdots \\ \mathbf{F}_s \end{bmatrix} \tag{10}$$

and the matrix

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} & \cdots & \mathbf{W}_{1s} \\ \mathbf{W}_{21} & \mathbf{W}_{22} & \cdots & \mathbf{W}_{2s} \\ \vdots & \vdots & & \vdots \\ \mathbf{W}_{s1} & \mathbf{W}_{s2} & \cdots & \mathbf{W}_{ss} \end{bmatrix}$$

is formed by matrices $\mathbf{W}_{ij}$ $(i, j = 1, 2, \ldots, s)$ defined as

$$\mathbf{W}_{11} = \begin{bmatrix} (w_{11})_{00} & (w_{11})_{01} & \ldots & (w_{11})_{0N} \\ (w_{11})_{10} & (w_{11})_{11} & \ldots & (w_{11})_{1N} \\ \vdots & \vdots & & \vdots \\ (w_{11})_{N0} & (w_{11})_{N1} & \ldots & (w_{11})_{NN} \end{bmatrix}$$

$$\mathbf{W}_{12} = \begin{bmatrix} (w_{12})_{00} & (w_{12})_{01} & \ldots & (w_{12})_{0N} \\ (w_{12})_{10} & (w_{12})_{11} & \ldots & (w_{12})_{1N} \\ \vdots & \vdots & & \vdots \\ (w_{12})_{N0} & (w_{12})_{N1} & \ldots & (w_{12})_{NN} \end{bmatrix}$$

$$\vdots$$

$$\mathbf{W}_{1s} = \begin{bmatrix} (w_{1s})_{00} & (w_{1s})_{01} & \cdots & (w_{1s})_{0N} \\ (w_{1s})_{10} & (w_{1s})_{11} & \cdots & (w_{1s})_{1N} \\ \vdots & \vdots & & \vdots \\ (w_{1s})_{N0} & (w_{1s})_{N1} & \cdots & (w_{1s})_{NN} \end{bmatrix}$$

$$\mathbf{W}_{21} = \begin{bmatrix} (w_{21})_{00} & (w_{21})_{01} & \cdots & (w_{21})_{0N} \\ (w_{21})_{10} & (w_{21})_{11} & \cdots & (w_{21})_{1N} \\ \vdots & \vdots & & \vdots \\ (w_{21})_{N0} & (w_{21})_{N1} & \cdots & (w_{21})_{NN} \end{bmatrix}$$

$$\mathbf{W}_{22} = \begin{bmatrix} (w_{22})_{00} & (w_{22})_{01} & \cdots & (w_{22})_{0N} \\ (w_{22})_{10} & (w_{22})_{11} & \cdots & (w_{22})_{1N} \\ \vdots & \vdots & & \vdots \\ (w_{22})_{N0} & (w_{22})_{N1} & \cdots & (w_{22})_{NN} \end{bmatrix}$$

$$\vdots$$

$$\mathbf{W}_{2s} = \begin{bmatrix} (w_{2s})_{00} & (w_{2s})_{01} & \cdots & (w_{2s})_{0N} \\ (w_{2s})_{10} & (w_{2s})_{11} & \cdots & (w_{2s})_{1N} \\ \vdots & \vdots & & \vdots \\ (w_{2s})_{N0} & (w_{2s})_{N1} & \cdots & (w_{2s})_{NN} \end{bmatrix}$$

$$\vdots$$

$$\mathbf{W}_{s1} = \begin{bmatrix} (w_{s1})_{00} & (w_{s1})_{01} & \cdots & (w_{s1})_{0N} \\ (w_{s1})_{10} & (w_{s1})_{11} & \cdots & (w_{s1})_{1N} \\ \vdots & \vdots & & \vdots \\ (w_{s1})_{N0} & (w_{s1})_{N1} & \cdots & (w_{s1})_{NN} \end{bmatrix}$$

$$\mathbf{W}_{s2} = \begin{bmatrix} (w_{s2})_{00} & (w_{s2})_{01} & \cdots & (w_{s2})_{0N} \\ (w_{s2})_{10} & (w_{s2})_{11} & \cdots & (w_{s2})_{1N} \\ \vdots & \vdots & & \vdots \\ (w_{s2})_{N0} & (w_{s2})_{N1} & \cdots & (w_{s2})_{NN} \end{bmatrix}$$

$$\vdots$$

$$\mathbf{W}_{ss} = \begin{bmatrix} (w_{ss})_{00} & (w_{ss})_{01} & \dots & (w_{ss})_{0N} \\ (w_{ss})_{10} & (w_{ss})_{11} & \dots & (w_{ss})_{1N} \\ \vdots & \vdots & & \vdots \\ (w_{ss})_{N0} & (w_{ss})_{N1} & \dots & (w_{ss})_{NN} \end{bmatrix}.$$

Next we can obtain the corresponding matrix forms for the conditions (2) as follows. The expression (3) and its derivative are equivalent to the matrix equations

$$\left.\begin{aligned} y_m^{(0)}(t) &= \begin{bmatrix} \frac{1}{0!} & \frac{(t-c)}{1!} & \frac{(t-c)^2}{2!} & \dots & \frac{(t-c)^N}{N!} \end{bmatrix}.\mathbf{Y}_m \\ y_m^{(1)}(t) &= \begin{bmatrix} 0 & \frac{1}{0!} & \frac{(t-c)}{2!} & \dots & \frac{(t-c)^{N-1}}{(N-1)!} \end{bmatrix}.\mathbf{Y}_m \\ &\vdots \\ y_m^{(k-1)}(t) &= \begin{bmatrix} 0 & \dots & 0 & \frac{1}{0!} & \frac{(t-c)}{1!} & \dots & \frac{(t-c)^{N-(k-1)}}{(N-(k-1))!} \end{bmatrix}.\mathbf{Y}_m \end{aligned}\right\},$$

where $\mathbf{Y}_m$ $(m = 1, 2, \dots, s)$ is defined in equations (6). By using these equations, the quantities $y_m^{(n)}(c)$, $y_m^{n}(a)$ and $y_m^{(n)}(b)$, $(n = 0, 1, 2, \dots, k-1)$, can be written as

$$\left.\begin{aligned} y_m^{(0)}(c) &= [1 \; 0 \; 0 \; \dots \; 0].\mathbf{Y}_m \\ y_m^{(0)}(a) &= \begin{bmatrix} 1 & \frac{h}{1!} & \frac{h^2}{2!} & \dots & \frac{h^N}{N!} \end{bmatrix}.\mathbf{Y}_m \\ y_m^{(0)}(b) &= \begin{bmatrix} 1 & \frac{k}{1!} & \frac{k^2}{2!} & \dots & \frac{k^N}{N!} \end{bmatrix}.\mathbf{Y}_m \\ y_m^{(1)}(c) &= [0 \; 1 \; 0 \; \dots \; 0].\mathbf{Y}_m \\ y_m^{(1)}(a) &= \begin{bmatrix} 0 & \frac{1}{0!} & \frac{h}{1!} & \dots & \frac{h^{N-1}}{(N-1)!} \end{bmatrix}.\mathbf{Y}_m \\ y_m^{(1)}(b) &= \begin{bmatrix} 0 & \frac{1}{0!} & \frac{k}{1!} & \dots & \frac{k^{N-1}}{(N-1)!} \end{bmatrix}.\mathbf{Y}_m \\ &\vdots \\ y_m^{(k-1)}(c) &= [0 \; \dots \; 0 \; 1 \; 0 \; \dots \; 0].\mathbf{Y}_m \\ y_m^{(k-1)}(a) &= \begin{bmatrix} 0 & \dots & 0 & \frac{1}{0!} & \frac{h}{1!} & \dots & \frac{h^{N-(k-1)}}{(N-(k-1))!} \end{bmatrix}.\mathbf{Y}_m \\ y_m^{(k-1)}(b) &= \begin{bmatrix} 0 & \dots & 0 & \frac{1}{0!} & \frac{k}{1!} & \dots & \frac{k^{N-(k-1)}}{(N-(k-1))!} \end{bmatrix}.\mathbf{Y}_m \end{aligned}\right\}, \qquad (11)$$

where $h = a - c$ and $k = b - c$.

Substituting quantities (11) into equations (2) and then simplifying, we obtain the matrix forms of the first, second and kth conditions defined in equations (2), respectively, as

$$^{1}\mathbf{U}_i.\mathbf{Y}_1 = [\lambda_{1i}],\ ^{2}\mathbf{U}_i.\mathbf{Y}_2 = [\lambda_{2i}], \ldots,\ ^{k}\mathbf{U}_i.\mathbf{Y}_s = [\lambda_{si}]\ (i = 0, 1, 2, \ldots, k-1)$$

or more clearly,

$$[{}^{1}u_{i0}\ \ {}^{1}u_{i1}\ \ {}^{1}u_{i2}\ \ldots\ {}^{1}u_{iN}].\mathbf{Y}_1 = \lambda_{1i} \quad (12)$$

$$[{}^{2}u_{i0}\ \ {}^{2}u_{i1}\ \ {}^{2}u_{i2}\ \ldots\ {}^{2}u_{iN}].\mathbf{Y}_2 = \lambda_{2i} \quad (13)$$

$$\vdots$$

$$[{}^{k}u_{i0}\ \ {}^{k}u_{i1}\ \ {}^{k}u_{i2}\ \ldots\ {}^{k}u_{iN}].\ \mathbf{Y}_s = \lambda_{si}, \quad (14)$$

where ${}^{1}u_{ij}, {}^{2}u_{ij}, {}^{3}u_{ij}, \ldots, {}^{k}u_{ij}$ are constants related to the coefficients $\alpha_{ij}^{m}, \beta_{ij}^{m}, \gamma_{ij}^{m}$ and λ_{mi} $(m = 1, 2, \ldots, s;\ i = 0, 1, \ldots, k-1;\ j = 0, 1, \ldots, N)$ in equations (2), h and k in equations (11). Of course we should be careful in the choice of coefficients of the conditions given by equations (2).

Now, by replacing the k rows matrices $\mathbf{W}_{11}$ and $\mathbf{F}_1$ in (10) by the last k rows of the matrices ${}^{1}\mathbf{U}_i$ and λ_{1i} $(i = 0, 1, \ldots, k-1)$ in (12), respectively, we have

$$\mathbf{W}_{11}^{*} = \begin{bmatrix} (w_{11})_{00} & (w_{11})_{01} & \cdots & (w_{11})_{0N} \\ (w_{11})_{10} & (w_{11})_{11} & \cdots & (w_{11})_{1N} \\ \vdots & \vdots & & \vdots \\ (w_{11})_{(N-k)0} & (w_{11})_{(N-k)1} & \cdots & (w_{11})_{(N-k)N} \\ {}^{1}u_{00} & {}^{1}u_{01} & \cdots & {}^{1}u_{0N} \\ {}^{1}u_{10} & {}^{1}u_{11} & \cdots & {}^{1}u_{1N} \\ \vdots & \vdots & & \vdots \\ {}^{1}u_{(k-1)0} & {}^{1}u_{(k-1)1} & \cdots & {}^{1}u_{(k-1)N} \end{bmatrix}, \mathbf{F}_1^{*} = \begin{bmatrix} f_1^{(0)}(c) \\ f_1^{(1)}(c) \\ \vdots \\ f_1^{(N-k)}(c) \\ \lambda_{10} \\ \lambda_{11} \\ \vdots \\ \lambda_{1(k-1)} \end{bmatrix}. \quad (15)$$

Similarly, by replacing the k rows matrices $\mathbf{W}_{22}$ and $\mathbf{F}_2$ in (10) by the last k rows of the matrices ${}^{2}\mathbf{U}_i$ and λ_{2i} $(i = 0, 1, \ldots, k-1)$, respectively, we obtain

$$\mathbf{W}_{22}^* = \begin{bmatrix} (w_{22})_{00} & (w_{22})_{01} & \cdots & (w_{22})_{0N} \\ (w_{22})_{10} & (w_{22})_{11} & \cdots & (w_{22})_{1N} \\ \vdots & \vdots & & \vdots \\ (w_{22})_{(N-k)0} & (w_{22})_{(N-k)1} & \cdots & (w_{22})_{(N-k)N} \\ {}^2u_{00} & {}^2u_{01} & \cdots & {}^2u_{0N} \\ {}^2u_{10} & {}^2u_{11} & \cdots & {}^2u_{1N} \\ \vdots & \vdots & & \vdots \\ {}^2u_{(k-1)0} & {}^2u_{(k-1)1} & \cdots & {}^2u_{(k-1)N} \end{bmatrix}, \mathbf{F}_2^* = \begin{bmatrix} f_2^{(0)}(c) \\ f_2^{(1)}(c) \\ \vdots \\ f_2^{(N-k)}(c) \\ \lambda_{20} \\ \vdots \\ \lambda_{2(k-1)} \end{bmatrix}. \quad (16)$$

Finally, by replacing the k rows matrices $\mathbf{W}_{ss}$ and $\mathbf{F}_s$ in (10) by the last k rows of the matrices ${}^k\mathbf{U}_i$ and λ_{si} $(i = 0, 1, \ldots, k-1)$ in (13), respectively, we get

$$\mathbf{W}_{ss}^* = \begin{bmatrix} (w_{ss})_{00} & (w_{ss})_{01} & \cdots & (w_{ss})_{0N} \\ (w_{ss})_{10} & (w_{ss})_{11} & \cdots & (w_{ss})_{1N} \\ \vdots & \vdots & & \vdots \\ (w_{ss})_{(N-k)0} & (w_{ss})_{(N-k)1} & \cdots & (w_{ss})_{(N-k)N} \\ {}^ku_{00} & {}^ku_{01} & \cdots & {}^ku_{0N} \\ {}^ku_{10} & {}^ku_{11} & \cdots & {}^ku_{1N} \\ \vdots & \vdots & & \vdots \\ {}^ku_{(k-1)0} & {}^ku_{(k-1)1} & \cdots & {}^ku_{(k-1)N} \end{bmatrix}, \mathbf{F}_s^* = \begin{bmatrix} f_s^{(0)}(c) \\ f_s^{(1)}(c) \\ \vdots \\ f_s^{(N-k)}(c) \\ \lambda_{s0} \\ \vdots \\ \lambda_{s(k-1)} \end{bmatrix}. \quad (17)$$

Taking into account (15), (16), (17) the matrix equations (10) can be written into the form

$$\mathbf{W}^*\mathbf{Y} = \mathbf{F}^* \quad (18)$$

or more clearly,

$$\begin{bmatrix} \mathbf{W}_{11}^* & \mathbf{W}_{12} & \cdots & \mathbf{W}_{1s} \\ \mathbf{W}_{21} & \mathbf{W}_{22}^* & \cdots & \mathbf{W}_{2s} \\ \vdots & \vdots & & \vdots \\ \mathbf{W}_{s1} & \mathbf{W}_{s2} & \cdots & \mathbf{W}_{ss}^* \end{bmatrix} \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \vdots \\ \mathbf{Y}_s \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1^* \\ \mathbf{F}_2^* \\ \vdots \\ \mathbf{F}_s^* \end{bmatrix}.$$

If $|\mathbf{W}^*| \neq 0,$ then we can write

$$\mathbf{Y} = (\mathbf{W}^*)^{-1}.\mathbf{F}^*. \tag{19}$$

Thus, the coefficients $y_m^{(k)}(c), (m = 1, 2, ..., s;\ k = 0, 1, ..., N)$ are uniquely determined by equation (19). Thereby the differential equation system (1) with the condition (2) has only unique solution. This solution is given by the Taylor polynomials

$$y_m(t) = \sum_{k=0}^{N} \frac{1}{k!} y_m^{(k)}(c).(t - c)^k,\ (a \leq t,\ c \leq b;\ m = 1, 2, ..., s;\ N \geq k). \tag{20}$$

3. Accuracy of Solution

We can easily check the accuracy of the solution obtained in the form (20) as follows. Since the truncated Taylor series (20) or the corresponding polynomial expansion is an approximate solution of equations (1), when the solutions $y_m(t)$ and its derivatives are substituted in equations (1), the resulting equations must be satisfied approximately, that is, for $t, t_i \in [a, b], i = 0, 1, 2, ...$

$$D_j(t_i) = \left| \sum_{n=0}^{k} \sum_{m=1}^{s} {}^n r_{jm}(t) y_m^{(n)}(t) - f_j(t) \right| \cong 0,\ (j = 1, 2, ..., s;\ i = 0, 1, 2, ...) \tag{21}$$

or

$$D_j(k_i) \leq 10^{-k_i}\ (k_i \text{ is any positive integer}).$$

If $\max|10^{-k_i}| = |10^{-k}|$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the differences $|D_j(t_i)|$ at each of the points becomes smaller than the prescribed 10^{-k}.

4. Examples

The method of this study is useful in finding the solution of differential equation system in terms of Taylor polynomials. We illustrate it by the following examples.

Example 4.1. Let us first consider the problem

$$ty_1 + 2y_2' + 2y_2 = te^t + 3e^t + 2$$

$$y_1' + y_1 + 3y_2 = 5e^t + 3$$

$$y_1(0) = 1, \;\; y_2(0) = 2 \tag{22}$$

and approximate the solution $y_m(t)$ by the Taylor polynomial

$$y_m(t) = \sum_{k=0}^{N} \frac{1}{k!} y_m^{(k)}(c).(t-c)^k,$$

where $N = 4,\ c = 0,\ a = 0$ and $b = 0.$

Then, by using these quantities ${}^{n}r_{jm}^{(z)}(t),\ f_j^{(z)}(t)$ in (1) and relation (7) for $N = 4,$ we obtain the matrix $\mathbf{W}^*$ and $\mathbf{F}^*$ in (18) as

$$\mathbf{W}^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{F}^* = \begin{bmatrix} 5 \\ 4 \\ 5 \\ 6 \\ 1 \\ 8 \\ 5 \\ 5 \\ 5 \\ 2 \end{bmatrix}.$$

From the solution of equation (19), the coefficients $y_m^{(k)}(0),\ (m = 1, 2, \ldots, s;\ \ k = 0, 1, \ldots, N)$ are uniquely determined as

$$\mathbf{Y} = \begin{bmatrix} 1.08280 \\ 0.90016 \\ 1.13390 \\ 1.04630 \\ -5.6818 \times 10^{-3} \\ 2.00570 \\ 0.98864 \\ 0.93994 \\ 1.31980 \\ -4.13960 \times 10^{-2} \end{bmatrix}.$$

By substituting the obtained coefficients in (20) the solution of (22) becomes

$$y_1 = 1.0828 + 0.90016t + 0.56695t^2 + 0.17438t^3 - 2.3674 \times 10^{-4} t^4$$

$$y_2 = 2.0057 + 0.98864t + 0.46997t^2 + 0.21997t^3 - 1.7248 \times 10^{-3} t^4.$$

The comparison of solutions (for $c = 0$, $N = 4$) with exact solution $y_1 = e^t$ and $y_2 = 1 + e^t$ is given in Table 1.

Table 1. Numeric results of Example 4.1

i	t_i	Exact solution $y_1 = e^t$	Exact solution $y_2 = 1 + e^t$	Present method $c = 0,\ N = 4$ y_1	Present method $c = 0,\ N = 4$ y_2
0	– 0.5	0.60653	1.60650	0.75264	1.60130
1	– 0.4	0.67032	1.67030	0.80228	1.67130
2	– 0.3	0.74082	1.74080	0.85907	1.74550
3	– 0.2	0.81873	1.81870	0.92405	1.82500
4	– 0.1	0.90484	1.90480	0.99828	1.91130
5	0	1.00000	2.00000	1.08280	2.00570
6	0.1	1.10520	2.10520	1.17870	2.10950
7	0.2	1.22140	2.22140	1.28690	2.22400
8	0.3	1.34990	2.34990	1.40860	2.35050
9	0.4	1.49180	2.49180	1.54470	2.49040
10	0.5	1.64870	2.64870	1.69640	2.64490

Example 4.2. Let us now consider the differential equation system

$$(t+1)y_1'' + 3y_2 - 3y_3 = 3t^6 - 3t^5 + 20t^4 + 20t^3 - 3t - 3$$

$$y_1 + ty_2 + y_3'' = t^7 + t^5 + 20t^3 + 4t^2 + 5t + 1$$

$$y_1 + 2y_2'' + y_3 = 2t^5 + 60t^4 + 8t + 4 \qquad (23)$$

with conditions

$$y_1(0) = 1, \;\; y_1'(0) = 3, \;\; y_2(0) = 2, \;\; y_2'(0) = 4, \;\; y_3(0) = 3, \;\; y_3'(0) = 5.$$

To find a Taylor polynomial solutions the problem above, we first take $c = 0$ and $N = 4$, and then proceed as before. Then we obtain the matrix $\mathbf{W}^*$ and $\mathbf{F}^*$ in (18) as

$$\mathbf{W}^* = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{F}^* = [-3 \;\; -3 \;\; 0 \;\; 1 \;\; 3 \;\; 1 \;\; 5 \;\; 8 \;\; 2 \;\; 4 \;\; 4 \;\; 8 \;\; 0 \;\; 3 \;\; 5]^T.$$

From the solution of equation (19), the coefficients $y_m^{(k)}(0)$, $(m = 1, 2, 3;$ $k = 0, 1, ..., 4)$ are uniquely determined as

$$\mathbf{Y} = [1 \;\; 3 \;\; 0 \;\; 0 \;\; 0 \;\; 2 \;\; 4 \;\; 0 \;\; 0 \;\; 0 \;\; 3 \;\; 5 \;\; 0 \;\; 0 \;\; 0]^T.$$

By substituting the obtained coefficients in (20) the solution of (23) becomes

$$y_1 = 3t + 1, \;\; y_2 = 4t + 2, \;\; y_3 = 5t + 3. \tag{24}$$

The values $y_m(t_i)$, $(m = 1, 2, 3; \; i = 0, 1, ..., 10)$ of solution (24) in an interval $-0.5 < x < 0.5$ and the values $D_j(t_i)$ are demonstrated in Table 2.

Table 2. Numeric results of Example 4.2

i	t_i	Present method $c = 0, N = 4$ y_1	Present method $c = 0, N = 4$ y_2	Present method $c = 0, N = 4$ y_3	$\lvert D_1(t_i)\rvert$	$\lvert D_2(t_i)\rvert$	$\lvert D_3(t_i)\rvert$
0	– 0.5	– 0.50000	0.00000	0.50000	1.1094	2.5391	3.6875
1	– 0.4	– 0.20000	0.40000	1.00000	0.72499	1.2919	1.5155
2	– 0.3	0.10000	0.80000	1.50000	0.36852	0.54265	0.48114
3	– 0.2	0.40000	1.20000	2.00000	0.12685	0.16033	0.9536
4	– 0.1	0.70000	1.60000	2.50000	0.017967	0.02001	0.00598
5	0	1.00000	2.00000	3.00000	0	0	0
6	0.1	1.30000	2.40000	3.50000	0.021973	0.02001	0.00602
7	0.2	1.60030	2.80010	4.00000	0.19123	0.15033	0.9664
8	0.3	1.90240	3.20070	4.50240	0.6969	0.54265	0.49086
9	0.4	2.21020	3.60410	5.01020	1.7736	1.2919	1.5565
10	0.5	2.53130	4.01560	5.53130	3.7031	2.5391	3.8125

Example 4.3. Let us consider the problem

$$\frac{d^4y_1}{dt^4} + t\frac{d^3y_1}{dt^3} + y_1 + \frac{d^3y_2}{dt^3} + (2t-1)\frac{d^2y_2}{dt^2} + 2y_2 = 3t^2 + 8t - 1$$

$$\frac{d^4y_2}{dt^4} + \frac{d^2y_2}{dt^2} + y_2 + t\frac{d^3y_1}{dt^3} + y_1 = 2t^2 + 2t + 4 \tag{25}$$

$$y_1(0) = 3,\ \ y_1'(0) = 0,\ \ y_1''(0) = 2,\ \ y_1'''(0) = 0,\ \ y_2(0) = -1,$$

$$y_2'(0) = 2,\ \ y_2''(0) = 2,\ \ y_2'''(0) = 0$$

and approximate the solution $y_m(t)$ by the Taylor polynomial

$$y_m(t) = \sum_{k=0}^{5}\frac{1}{k!}y_m^{(k)}(c).(t-c)^k, \tag{26}$$

where $N = 5,\ c = 0,\ a = 0$ and $b = 0.$

Following the procedures in the previous examples, we find the unknown coefficients $y_m^{(k)}(0)$ $(n = 0, 1, ..., 5)$

$$\mathbf{Y} = [3 \ \ 0 \ \ 2 \ \ 0 \ \ 0 \ \ 0 \ \ -1 \ \ 2 \ \ 2 \ \ 0 \ \ 0 \ \ 0]^T. \tag{27}$$

Substituting the elements of column matrices (27) into equation (26), we obtain the approximate solutions, in term of the Taylor polynomial of degree two about $t = c = 0$, as

$$y_1(t) = 3 + t^2, \ \ y_2(t) = -1 + 2t + t^2.$$

Of course these are exact solutions.

5. Conclusions

High-order linear differential equation systems with variable coefficients are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. For this purpose, the presented method can be proposed. A considerable advantage of the method is that the solution is expressed as a truncated Taylor series and thereby a Taylor polynomial at $t = c$. Furthermore, after calculation of the series coefficients, the solution $y(t)$ can be easily evaluated for arbitrary values of t at low computation effort.

If the functions ${}^{n}r_{jm}(t)$, $f_j(t)$, $(m, j = 1, 2, ..., s; \ n = 0, 1, 2, ..., k)$ are functions having nth derivatives on the interval $a \leq t \leq b$, then we can approach the solutions $y_m(t)$ by the Taylor polynomial

$$y_m(t) = \sum_{k=0}^{N} \frac{1}{k!} y_m^{(k)}(c).(t-c)^k$$

about $t = c$; otherwise, the method cannot be used.

On the other hand, it is observed that this method shows the best advantage when the known functions in equation can be expanded to Taylor series about $t = c$ which converge rapidly. The method can be developed and applied to another high-order linear and nonlinear differential equation systems with variable coefficients.

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