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SOME APPLICATIONS OF TRANSVERSALITY IN EMBEDOLOGY

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Abstract

Transversality is a powerful technique for proving that some properties on spaces of maps, solutions of partial differential equations or vector fields are generic. In particular the properties of the evaluation map constructed to each problem is the key tool.

In the setting of dynamical systems given by the pair (M, f), where the state space M is a smooth manifold and $f : M \to M$ is a continuous map, frequently we have only a partial information of M throughout a scalar map $\alpha : M \to \mathbb{R}$ but we do not know how is M. As a consequence,

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a problem which arises is how to reconstruct M using α and getting the essential information on it. The technique of using embeddings maps is satisfactory since them keep that information. The embedology started with the works of Whitney in the 1930's. Since then some results have been obtained but using different techniques.

In this paper we present part of those results under the unified approach of the transversality theory and underlay the importance of the evaluation map associated to each problem of embedology.

1. Introduction

Searching for generic or typical properties of vector fields, maps, partial differential equations, etc. has been a subject of interest since the 1950's. It is well known that a general technique to establish such properties is called the *transversality theory* that has its beginning in the Thom's results of [16]. The Kupka and Smale's theorems ([7], [10] and [12]) on generosity of some properties for flows on compact manifolds is another good example.

Adequate topologies in the space of vector bundle sections are necessary to state the frame where a generic property can be precisely defined. Also the notion of *evaluation map* originally introduced by Abraham and Robbin in [1] is a basic tool in the transversality technique.

Embeddings in cartesian spaces have been widely considered after the seminal results by Merger and Nobeling [5]. Since this, Whitney in [19] got some results in the problem of embedding analytic manifolds in cartesian spaces. He proved that if an analytic manifold has dimension m, then it can be embedded *globally* into $\mathbb{R}^{2m+1}$ and that the set composed of embeddings is dense and open in the space of C^1-maps. These ideas were considered by Takens [15] and applied to the problem of the reconstruction of discrete dynamical systems, opening the door for other results (see [2], [11], [13], etc.).

Let $M \subset \mathbb{R}^n$ be a C^r-manifold (the state space) and $f : M \to M$ be a continuous map. Then the pair (M, f) is called *discrete dynamical system*, where the main problem is to understand the asymptotic behavior of the orbits of all points of M, that is, to know the asymptotic of the sequences $(f^n(x))_{n=0}^{\infty}$, where $f^n = f^{n-1} \circ f$ $(n \in \mathbb{N})$ and f^0 denotes the identity on *M*. When modelling real problems, in most cases we do not know *M* and have only partial information on it given by a scalar measure on *M*, that is, given by a map $\alpha : M \to \mathbb{R}$. This information has then the form of a time series obtained from α. From this information, we must be able to reconstruct the state space *M*.

The reconstruction of a space equivalent in some sense to *M* throughout α is in the basis of many nonlinear methods used in literature (see for example [6]).

A scalar measure is a projection of the unobserved internal variables of the system into an interval on the real line. Apart from the reduction in dimension, the projection may be nonlinear and mix different internal variables, producing great distortion on the output.

With these facts in mind, we need for the reconstructed space to have the property that the evolution of a trajectory in it be clearly stated and uniquely determined. But uniqueness of the dynamics in the reconstructed space is not the only property we want to hold. We want that others properties to be maintained such as dimensions, Lyapunov exponents, entropies, etc., but those features are invariant only under smooth nonsingular transformations. That means that in order to guarantee that the invariants coincide in the original state space and in the reconstructed (or at least in a part of it) we need to require that the structure of the tangent space be preserved by the reconstruction process. But it can be obtained through an *embedding* of a compact smooth manifold $M \subset \mathbb{R}^n$ which is a smooth map $f : M \to \mathbb{R}^n$ such that at every $x \in M$, $T_x M$ is injective (*f* is an immersive map) and maps homeomorphically *M* onto its image. The key point is to establish under what conditions the projection obtained from the scalar measure and the subsequent reconstruction form an embedding.

The problem with the embedding of scalar data in $\mathbb{R}^n$ has two aspects. First, if it is possible or not to obtain that the state of the system is uniquely characterized by *p* coordinates corresponding to *p* independent variables from the scalar time series. Once we know those variables, we also know that the dynamic lives in a p dimensional manifold. But this manifold could be curved. Then the second problem is to embed this manifold into a cartesian space where we will finally work. Thus to find maps having the former properties (called *embeddings maps*) is of great interest.

The aim of this paper is to present three results on embedology of discrete dynamical systems whose state spaces are compact manifolds, under the unified approach of the technique of transversality. In this case what is new is the unified approach we use.

2. Notations, Definitions and Preliminary Results

It is supposed the reader knows some initial and elementary notions of General Topology and Differential Geometry such that those of C^k-*manifold*, C^1-*curve*, *coordinate charts* and *tangent vector* to a manifold at a point. To see the former notions and the subsequent properties and results, see for example [3] and [1].

Let us denote by M a C^k-compact manifold. Then the set of all tangent vectors at $x \in M$ is a vector space denoted by $T_x M$. The union of the tangent vectors at all points of M gives the *tangent bundle* or *tangent space* of M which is denoted by TM:

$$TM = \{(x, v) : x \in M, v \in T_x M\} = \bigcup_{x \in M} (\{x\} \times T_x M).$$

There is a natural map $\tau : T(M) \to M$ given by $\tau(v) = x$, if $v \in T_x M$ and $\tau^{-1}(x) = T_x M$. If M is a C^k-manifold of dimension $2m$, then τ is a C^{k-1}-map.

If we consider the tangent vectors of M with $\| v \| = 1$, then we obtain the set $\widetilde{T}M = \{(x, v) : x \in M, \| v \| = 1\} \subset TM$. This set is a compact manifold of dimension $2m - 1$.

In what follows M and N will denote smooth manifolds of dimensions m and n respectively, and $f : M \to N$ is a smooth map. The derivative of f at $x \in M$ is a linear map from $T_x M$ to $T_{f(x)}$ denoted by $T_x f$. The derivative of f is a map $Tf : TM \to TN$ mapping linearly $T_x M$ into $T_{f(x)} N$.

The notion of transversality is introduced throughout some definitions.

Definition 1. If $T_x M$ is injective, then f is *immersive* at $x \in M$. If f is immersive at every point of M, then f is an *immersion*. f is an *embedding* provided f is an immersion and maps homeomorphically m onto its image.

Remark 1. When M is compact, it is easy to see that f is an embedding if and only if f is injective and an immersion.

Definition 2. Let $f \in C^1$. Then a point $y \in N$ is a *regular value* of f if $f^{-1}(y) = \varnothing$ or if the map $T_x M$ is surjective at all $x \in f^{-1}(y)$.

Definition 3. Let $L \subset N$ be a submanifold. Then a C^1-map $f : M \to N$ is *transverse* to L at $x \in M$ provided $f(x) \notin L$ and if $f(x) \in L$, then the following hold:

(1) $(T_x f)^{-1}(T_{f(x)} L)$ has a closed complement in $T_x M$.

(2) $T_x f(T_x M)$ has a closed subspace V such that $T_{f(x)} N = V + T_{f(x)} L$.

Remark 2. When M or N are finite dimensional, f is transverse to the submanifold $L \subset N$ at $x \in M$ when $f(x) \notin L$ or $T_{f(x)} N = Tf(T_x M) + T_{f(x)} L$ when $f(x) \in L$.

Remark 3. Let f be a C^1-map. If f is transverse to L, then $f^{-1}(L)$ is a submanifold of M.

Let M denote now a C^r and m-dimensional compact manifold and $B = C^r(M, \mathbb{R}^n)$ be the set of C^r-maps from M to $\mathbb{R}^n$. Let $\{\varphi_j : U_j \subset M \to V_j \subset \mathbb{R}^m\}_{j=1}^N$ be a finite set of coordinate charts holding $\bigcup_{j=1}^N U_j = M$

(this is possible since the compactness of M) and let $C_j \subset U_j$ be compact subsets holding $\bigcup_{j=1}^{N} C_j = M$. Then the space B is complete when it is endowed with the topology given by the distance

$$d(f, g) = \sup\{|f(x) - g(x)|, \| T^{(i)}(f \circ \varphi_j^{-1})(\varphi_j(x)) - T^{(i)}(g \circ \varphi_j^{-1})(\varphi_j(x)) \|_{\mathbb{R}^n} : x \in C_j; 1 \leq j \leq N, 1 \leq i \leq r\}$$

for every $f, g \in C^r(M, \mathbb{R}^n)$ and where $T^{(i)}$ denotes the i-derivative of T.

If N is a C^r and n-dimensional manifold, then the topology of $C^r(M, N)$ is slightly more complicated. Better than define a distance between two functions, it is more convenient to describe a base of neighborhoods of the topology. Let $f \in C^r(M, N)$ and let $\varphi : U \subset M \to U' \subset \mathbb{R}^m$ and $\psi : V \subset N \to V' \subset \mathbb{R}^n$ be charts on M and N, respectively. Let $K \subset U$ be a compact set such that $f(K) \subset V$ and let $\varepsilon > 0$. Then a neighborhood of f is given by

$$N^r(f; \varphi, U), (\psi, V), K, \varepsilon)$$

$$= \left\{\begin{array}{l} g : M \to N : g(K) \subset V \text{ and} \\ \| T^k(\psi \circ f \circ \varphi^{-1})(\varphi(x)) - T^k(\psi \circ g \circ \varphi^{-1})(\varphi(x)) \|_{\mathbb{R}^n} < \varepsilon \\ \text{for all } x \in \varphi(K), k = 0, \ldots, r \end{array}\right\}.$$

Remember that a set S is residual in a topological space X if there is a countable number of dense open sets $\{U_j\}_{j \in \mathbb{N}}$ in X such that $S \subseteq \bigcap_{j \in \mathbb{N}} U_j$. If X is a complete metric space, then any residual subset S of it is dense.

With help of the notion of Lebesgue measure (see for example [17]), we introduce a measure on any manifold and we speak of zero measure sets and full measure sets in a similar way than we speak of zero Lebesgue measure or full Lebesgue measure sets.

Definition 4. Let M be an m-dimensional manifold. Then a subset $S \subset M$ has *zero measure* (respectively *full measure*) if for every chart

$\varphi : U \subset M \to V \subset \mathbb{R}^m$, the set $\varphi(U \cap S) \subset \mathbb{R}^m$ has zero Lebesgue measure (respectively full Lebesgue measure).

The following result is a key tool in this theory.

Theorem 1 (Morse-Sard)**.** *Let M and N be manifolds of dimensions m and n respectively, and* $f : M \to N$ *be a* C^r*-map. If* $r > \max\{0, m - n\}$, *then the set of regular values of f has full measure in N. Moreover, it is residual and therefore dense.*

If M is a compact manifold, then the set $C^r(M, \mathbb{R}^n)$ is a manifold [1] and associated we have

$$TC^r(M, \mathbb{R}^n) = \bigcup_{f \in C^r(M, \mathbb{R}^m)} \{f\} \times T_f C^r(M, \mathbb{R}^n),$$

where $T_f C^r(M, \mathbb{R}^n) = \{\eta \in C^r(M, T\mathbb{R}^n) : \tau_{\mathbb{R}^n} \circ \eta = f\}$ with $\tau_{\mathbb{R}^n} : T\mathbb{R}^n \to \mathbb{R}^n$ given by $\tau_{\mathbb{R}^n}(v) = f(x)$ and $v \in T_{f(x)}\mathbb{R}^n$. Since $C^r(\widetilde{T}M, T\mathbb{R}^n)$ is a manifold (recall that $\widetilde{T}M$ is a compact manifold), we introduce the set $TC^r(\widetilde{T}M, T\mathbb{R}^m)$ with $T_{T_f} C^r(\widetilde{T}M, T\mathbb{R}^n) = \{w \in C^r(\widetilde{T}M, TT\mathbb{R}^n) : \tau_{T\mathbb{R}^n} \circ w = Tf\}$.

The map $\omega : TTM \to TTM$, given by $\omega(x, v, u, w) = (x, u, v, w)$ is an isomorphism and is called the *canonical involution*. This map satisfies $\tau_{TM} \circ \omega = T\tau_M$ and $\omega \circ \omega =$ Identity.

Let M and N be two differential manifolds and consider a family of maps from M to N parametrized by a topological space K, that is, there is a map $F : K \to C^r(M, N)$ given by $F(k) = f_k$, where f_k denotes a member of the family. Associated to F we introduce the map

$$ev_F : K \times M \to N$$

given by $ev_F(k, x) = f_k(x)$ which it is called the *evaluation map of F*. This map plays an important role to obtain the transversality results. The same can be said of the following result, where its proof is made in [1] using an infinite dimensional version of the former Morse-Sard's theorem.

Theorem 2 (Parametric Transversality Theorem)**.** *Let K, M, N be C^r-manifolds, $L \subset N$ be a C^r-submanifold of codimension p and $F : K \to C^r(M, N)$ be a continuous map such that its evaluation map ev_F is C^r. Assume that*:

(i) *K and M are second countable* (*their topology has a countable base of neighborhoods*).

(ii) *ev_F is transverse to L.*

(iii) $r > \max\{0, m - p\}$, *where* $m = \dim M$.

Then $A = \{k \in K : f_k$ is transverse to $L\}$ is residual in K and therefore dense. Moreover if L is closed in N and M is compact, then A is also open.

When K is a finite dimensional manifold, we use Morse-Sard's theorem to obtain the next result. The proof we have made is inspired in Stark [14] and Hirsch [3].

Theorem 3. *Let K, M, N be C^r-manifolds, $L \subset N$ be a C^r-submanifold of codimension p and suppose F is continuous and ev_F is C^r. Assume that*:

(i) *K and M are finite dimensional with* $\dim(M) = m$.

(ii) *ev_F is transverse to L.*

(iii) $r > \max\{0, m - p\}$.

Then $A = \{k \in K : f_k$ is tranverse to $L\}$ has full measure in K. Moreover if L is closed in N and M is compact, then A is also open.

Proof. Since ev_F is transverse to L, the set $ev_F^{-1}(L)$ is a submanifold of $K \times M$. Let $\pi : ev_F^{-1}(L) \to K$ be the first projection. Then we claim that the set of regular values of π belongs to the set $A = \{k \in K : f_k$ is transverse to $L\}$.

Let $k \in K$ be a regular value of π.

(i) If $\pi^{-1}(k) = \emptyset$, then $f_k(x) \notin L$ for all $x \in M$ and $F(k) = f_k$ is transverse to L.

(ii) If $\pi^{-1}(k) \neq \emptyset$, then $y = f_k(x) \in L$ for some $x \in M$ and we need to prove that f_k is transverse to L, that is, $T_x f_k(T_x M) + T_y L = T_y N$. As ev_F is transverse to L, for any $w \in T_y N$, there exists $v \in T_y L$, $\alpha \in T_k K$ and $u \in T_x M$ such that $T_{k,x} ev_F(\alpha, u) + v = w$, that is, $\eta_\alpha(x) + T_x f_k(u) + v = w$, where $\eta_\alpha : M \to TN$ such that $\tau_N \circ \eta_\alpha = f_k$.

Also we know that $T_{k,x}\pi$ is surjective and then for $\alpha \in T_k K$, exists $u' \in T_x M$ is such that $(\alpha, u') \in T_{k,x}(ev_F^{-1}(L))$ and $\eta_\alpha(x) + T_x f_k(u') \in T_y L$. Finally we have $T_x f_k(u - u') + (\eta_\alpha(x) + T_x f_k(u') + v) = w$ and then f_k is transverse to L, proving the claim. Hence A has full measure by Morse-Sard's theorem.

The technical Lemmas 1 and 2 are proved in [13].

Lemma 1. *Let $f \in C^r(M, \mathbb{R}^m)$. Given $x \in M$, $v \in T_{f(x)}\mathbb{R}^m$ and $U \subset M$ an open neighborhood of x, there exists a C^r function $\eta \in T_f C^r(M, \mathbb{R}^m)$ such that $\eta(x) = v$ whose support is contained in U.*

Lemma 2. *Let $f \in C^r(M, \mathbb{R}^m)$. Given $v \in T_x M$ with $v \neq 0$, $\overline{w} \in T_{Tf(v)}(T\mathbb{R}^m)$ and an open neighborhood $U \subset M$ of x, there exists $\eta \in T_f C^r(M, \mathbb{R}^m)$ such that $\omega(T_x \eta(v)) = \overline{w}$, whose support is contained in U.*

Lemmas 3, 4 and 5 are key points for proving the Whitney's theorem and its prevalent version using transversality theorems.

Lemma 3. *Let M be a compact smooth manifold of dimension m. Consider the maps σ and its evaluation*

$$\sigma : C^r(M, \mathbb{R}^{2m+1}) \to C^{r-1}(\widetilde{T}M, T\mathbb{R}^{2m+1})$$

$$ev_\sigma(f, v) = Tf(v).$$

Then ev_σ is transverse to every submanifold of $T\mathbb{R}^{2m+1}$, for $r \geq 1$.

Proof. Consider

$$T_{f,v} ev_\sigma : T_f C^r(M, \mathbb{R}^{2m+1}) \times T_v \widetilde{T}M \to T_{Tf(v)} T\mathbb{R}^{2m+1}$$

given by

$$T_{f,v}ev_{\sigma}(\eta, \overline{v}) = T_v Tf(\overline{v}) + \omega T_x \eta(v).$$

By Lemma 2, $T_{f,v}ev_{\sigma}$ is surjective for all $(f, v) \in C^r(M, \mathbb{R}^{2m+1}) \times \widetilde{T}M$ and then is transverse to every submanifold of $T\mathbb{R}^{2m+1}$.

Lemma 4. *Let M be a compact smooth manifold of dimension m contained in $\mathbb{R}^k$ and let $\{f_1, f_2, ..., f_p\}$ be a basis for the finite-dimensional space of polynomial functions from $\mathbb{R}^k$ to $\mathbb{R}^{2m+1}$ of degree less than or equal to 2. Consider the map ρ and its evaluation map*

$$\rho : \mathbb{R}^p \to C^r(M \times M \setminus \Delta, \mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1})$$

$$ev_{\rho}(\alpha, x, y) = \left(\sum_{i=1}^{p} \alpha_i f_i(x), \sum_{i=1}^{p} \alpha_i f_i(y) \right),$$

where Δ denotes the diagonal of $M \times M$ and $\hat{\Delta}$ the diagonal of $\mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1}$.

Then ev_{ρ} is transverse to $\hat{\Delta}$.

Proof. Consider the map

$$T_{\alpha,x,y}ev_{\rho} : T_{\alpha}\mathbb{R}^p \times T_{x,y}(M \times M \setminus \Delta) \to T_{z,z}(\mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1})$$

$$T_{\alpha,x,y}ev_{\rho}(\overline{a}, u, v) = \left(\sum_{i=1}^{p} a_i f_i(x) + \sum_{i=1}^{p} \alpha_i T_x f(u), \sum_{i=1}^{p} a_i f_i(y) + \sum_{i=1}^{p} \alpha_i T_y f(v) \right),$$

where $ev_{\rho}(\alpha, x, y) \in \hat{\Delta}$ and $z = \sum_{i=1}^{p} \alpha_i f_i(x) = \sum_{i=1}^{p} \alpha_i f_i(y)$.

We need to prove that the image of $T_{\alpha,x,y}ev_{\rho}$ contains a basis vector of $T_{z,z}(\mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1})$. For some i let $x_i \neq y_i$ and consider the polynomial of degree one and two in the variable i for each component of $\mathbb{R}^{2m+1}$, and such that the rank of $((f_{2m+2}(x), f_{2m+2}(y)), ..., (f_p(x), f_p(y)))$ is $4m + 2$.

Now find a certain $\overline{a} = (0, ..., 0, a_{2m+2}, ..., a_p) \in \mathbb{R}^p$ with $T_{\alpha, x, y} ev_\rho(\overline{a}, 0, 0) = (0, ..., e_i, ..., 0)$.

Then the map ev_ρ is transverse to $\hat{\Delta}$.

Lemma 5. *Let M be a compact smooth manifold of dimension m contained in $\mathbb{R}^k$ and let $\{f_1, f_2, ..., f_p\}$ be a basis for the finite-dimensional space of polynomial functions from $\mathbb{R}^k$ to $\mathbb{R}^{2m+1}$ of degree less than or equal to 2. Consider the map σ and its evaluation*

$$\sigma : \mathbb{R}^p \to C^r(\widetilde{T}M, T\mathbb{R}^{2m+1})$$

$$ev_\sigma(\alpha, v) = \sum_{i=1}^{p} \alpha_i T f_i(v).$$

Then ev_σ is transverse to every submanifold of $T\mathbb{R}^{2m+1}$, for $r \geq 1$.

Proof. Considering the map

$$T_{\alpha, v} ev_\sigma : T_\alpha \mathbb{R}^p \times T_v \widetilde{T}M \to T_{\sum_{i=1}^{p} \alpha_i T f_i(v)} T\mathbb{R}^{2m+1}$$

given by

$$T_{\alpha, v} ev_\sigma(\overline{a}, u) = \sum_{i=1}^{p} \alpha_i T_v T f_i(u) + \sum_{i=1}^{p} \alpha_i T f_i(v)$$

we are proving that it is surjective.

The number of homogeneous polynomial of degree 1 existing in $\{f_1, f_2, ..., f_p\}$ is $(2m+1)k$ and the rank of

$$(T f_1(v), ..., T f_p(v))$$

is $(2m+1)$ provided $v \neq 0$. For every $w \in T_{\sum_{i=1}^{p} \alpha_i T f_i(v)} T\mathbb{R}^{2m+1}$ there exist real numbers such that $w = \sum_{i=1}^{p} a_i T_x f_i(v) = T_{\alpha, v} ev_\sigma(\overline{a}, 0)$ and therefore ev_σ is transverse to every submanifold of $T\mathbb{R}^{2m+1}$.

When we are considering a property held by a subset of an infinite dimensional linear metric space, we will use the term *almost every* to mean that the subset is *prevalent* with respect to such property. *Prevalence* is an appropriate condition when one desires a probabilistic result on the likelihood of a given property on a space of functions. The notion of prevalent set was introduced by Hunt et al. (see [4]). The use of this term was introduced as an extension of (*Lebesgue*) *almost every* to the infinite dimensional case and allow us to establish similar results to those made on manifolds in the setting of compact sets which are not manifolds.

Definition 5. A Borel subset S of a normed linear space V is *prevalent* if there is a finite-dimensional subspace $E \subset V$ such that for each $v \in V$, $v + e \in S$ for (Lebesgue) almost every $e \in E$ (see [11]).

A prevalent set means that if we start at any point in the linear space V and explore along the finite dimensional $E \subset V$, then almost every point encountered will lie in this set. For subsets of finite-dimensional spaces the term prevalent is synonymous with almost every, in the sense of outside a set of measure zero. When there is no possibility of confusion, we will say that *almost every map* satisfies a property when the set of such maps is prevalent.

Now we are recalling when a compact set $A \subset \mathbb{R}^k$ has *box-counting* dimension d.

Let $\varepsilon > 0$ and $N(\varepsilon)$ be the number of subsets $A_\varepsilon = \{x \in \mathbb{R}^k : |x - a| \leq \varepsilon,$ $a \in A\}$ such that $A \subseteq \cup A_\varepsilon$. When

$$\lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{-\log(\varepsilon)}$$

exists and takes the value d, the box-counting dimension of A is d.

In what follows we are proving some technical results necessary to prove a prevalent version of Whitney's theorem for compact sets.

Lemma 6. *Let $A \subset \mathbb{R}^k$ be a compact set. Then there is a map from $\mathbb{R}^k$ to $\mathbb{R}^n$ injective on A, for some n.*

Proof. Since $A \subset \mathbb{R}^k$ is compact, it can be covered by a finite number of open set $\{V_i\}_{i=1}^m$ such that $A \subseteq \bigcup_{i=1}^m V_i$. Let U_i $(i = 1, ..., m)$ be a finite number of open sets holding $V_i \subset \overline{V_i} \subset U_i$ and $\lambda_i : \mathbb{R}^k \to [0, 1]$ a C^∞-map equal to 1 on $\overline{V_i}$ and to 0 on $\mathbb{R}^k \setminus \overline{U_i}$ for $i = 1, ..., m$ (see [3]). Let the C^r-map $f_i : \mathbb{R}^k \to \mathbb{R}^k$ given by

$$f_i(x) = \lambda_i(x)x.$$

Put $g_i = (f_i, \lambda_i) : \mathbb{R}^k \to \mathbb{R}^{k+1}$ and

$$g = (g_1, ..., g_m) : \mathbb{R}^k \to \mathbb{R}^{m(k+1)},$$

then g is a C^r-map.

To see that g is injective on A, let $x, y \in A$ with $x \neq y$ and $y \in \overline{V_i}$ for some i. If $x \notin \overline{V_i}$, then $\lambda_i(x) \neq 1 = \lambda_i(y)$ and $g_i(x) \neq g_j(y)$. If $x \in \overline{V_i}$, then $\lambda_i(x) = \lambda_i(y)$ but $f_i(x) \neq f_i(y)$.

Lemma 7. *Let $A \subset \mathbb{R}^k$ be a compact set of box-counting dimension $d < k$ and $f : \mathbb{R}^k \to \mathbb{R}^k$ be a C^1-map. Then $f(A)$ has Lebesgue measure zero.*

Proof. Let $A \subset \mathbb{R}^k$ be a compact set of box-counting dimension $d < k$. Then $N(\varepsilon) \approx \varepsilon^{-d}$, where $N(\varepsilon)$ is the number of cubes covering A of side ε. The measure of each cube is $\mu(C) = \varepsilon^k$ and the measure of A is $\mu(A) = \varepsilon^k \varepsilon^{-d} = \varepsilon^{k-d}$, then if $d < k$ the measure of this set is zero. Therefore $f(A)$ has Lebesgue measure zero (see [3]).

Lemma 8. *Let $A \subset \mathbb{R}^k$ be a compact set of box-counting dimension $d < k$ and $f : \mathbb{R}^k \to \mathbb{R}^q$ be a C^1-map. If $q > d$, then $f(A)$ has Lebesgue measure zero in $\mathbb{R}^q$.*

Proof. Let $\overline{U}(\varepsilon)$ be a cube of side ε such that $A \subseteq \bigcup \overline{U}(\varepsilon)$ and $N(\varepsilon) \approx \varepsilon^{-d}$ because the box counting dimension of A is d. Let

$A_\varepsilon = \overline{U}(\varepsilon) \cap A$ be such that $A = \cup A_\varepsilon$. We know that $f(A) \subseteq \cup f(A_\varepsilon) \subseteq \cup f(\overline{U}(\varepsilon))$, then $f(A)$ is covered by ε^{-d} sets. If $x, y \in \overline{U}(\varepsilon)$, then

$$\| f(x) - f(y) \| \leq k \| x - y \| \leq k\varepsilon,$$

where k is the maximum value taken by the differential of f on A. Then $\mu(f(\overline{U}(\varepsilon))) \leq k^q \varepsilon^q$ and $f(A)$ has measure zero.

Lemma 9. *Let $A \subset \mathbb{R}^k$ be a compact set of box-counting dimension $d < k$. Then there is a C^r-map from $\mathbb{R}^k$ to $\mathbb{R}^n$ injective on A, where $n = \min\{j \in \mathbb{N} : j \geq 2d + 1\}$ and $r \geq 1$.*

Proof. By Lemma 6 there exists a map from $\mathbb{R}^k$ to $\mathbb{R}^q$ injective on A, for some q. If q is an integer less than or equal to $2d + 1$, then we have finished. Let us suppose q is an integer greater than n. To finish the prove it is sufficient to prove that we are able to find a map from $\mathbb{R}^k$ to $\mathbb{R}^{q-1}$ injective on A.

Identifying $\mathbb{R}^{q-1}$ with $\{x \in \mathbb{R}^q : x_q = 0\}$ let $v \in \mathbb{R}^q \setminus \mathbb{R}^{q-1}$ and $f_v : \mathbb{R}^q \to \mathbb{R}^{q-1}$ be the parallel projection to v. If f denotes the injective map from $\mathbb{R}^k$ to $\mathbb{R}^q$, then it is sufficient to prove that $f_v/_{f(A)}$ is injective. To get it, we need that v is not parallel to any secant of $f(A)$. That is, if x, y are any two distinct points of A, then we need $v \neq \dfrac{f(x) - f(y)}{|f(x) - f(y)|}$.

Consider the map

$$\sigma : \mathbb{R}^k \times \mathbb{R}^k - \Delta \to S^{q-1}$$

given by

$$\sigma(x, y) = \frac{f(x) - f(y)}{|f(x) - f(y)|},$$

where Δ denotes the diagonal of $\mathbb{R}^k \times \mathbb{R}^k$. It is clear that v holds the conditions needed for f_v to be injective on A if and only if $v \notin \sigma(f(A) \times f(A)) - ((f(A) \times f(A)) \cap \Delta)$.

But it is possible to find a vector v holding the above conditions since the set $A \times A$ has box counting dimension $2d$ and being $k > d$, the set $(A \times A) \setminus \Delta$ has Lebesgue measure zero in $(\mathbb{R}^k \times \mathbb{R}^k) \setminus \Delta$ and its image by a C^r-map $(r \geq 1)$ is again a zero Lebesgue measure, if $q - 1 > 2d$ by Lemma 7.

Let (M, f) be a discrete dynamical system, where $M \subset \mathbb{R}^n$ is a C^r-manifold and $f : M \to M$ is a continuous map. Recall that a point $x \in M$ is *periodic* of *period* p of the system if $f^p(x) = x$ and $f^j(x) \neq x$ for $j < p$.

Takens (see [15]) gave the first theoretical justification of data embedding techniques used by experimentalists to reconstruct dynamical information from time series. To this end, he dealt with a class of maps called *delay coordinate maps* because they are more accessible to scientists making experimental work than other maps.

Definition 6. Let f be a diffeomorphism of an open set $U \subset \mathbb{R}^k$ and $\alpha : U \to \mathbb{R}$ be a map. Then the delay coordinate map $\Phi_{f,\alpha} : U \to \mathbb{R}^n$ is

$$\Phi_{f,\alpha}(x) = (\alpha(x), \alpha(f(x)), \ldots, \alpha(f^{n-1}(x))).$$

Now we recall a Takens's theorem (see [13] and [14]) from which we will prove a prevalent version.

Theorem 4. *Suppose that* $f \in C^r(\mathbb{R}^k, \mathbb{R}^k)$ *is a diffeomorphism on* $M \subset \mathbb{R}^k$ *an m-dimensional compact manifold such that f has only a finite number of periodic points of period less than* $2m + 1$, *and such that the eigenvalues at all periodic points are distinct.*

Then there is an open and dense set of $\alpha \in C^r(\mathbb{R}^k, \mathbb{R})$ *for which* $\Phi_{f,\alpha}$ *is an embedding on M.*

Finally we recall an easy result following necessary to prove Theorem 8 (see [13]).

Lemma 10. *Let V be a vector space with* $\dim V = n$. *Let* $A : V \to V$ *be an invertible linear map with distinct eigenvalues, and* $a : V \to \mathbb{R}$ *be any linear map holding* $a(v) \neq 0$ *for all eigenvectors v of A.*

Then vectors $a, a \circ A, \ldots, a \circ A^{n-1}$ *are linearly independent.*

3. The Theorems

Since $C^r(M, \mathbb{R}^n)$ is dense in $C^s(M, \mathbb{R}^n)$ for all $r > s$, for r sufficiently large we obtain the following results (see [3]).

The first result is a version of the well known Whitney theorem [18]. Originally this result was proved in the setting of analytical manifold and for analytical maps.

Theorem 5. *Let M be a compact smooth manifold of dimension m. Then for* $r \geq 1$, *there is an open and dense set of* $C^r(M, \mathbb{R}^{2m+1})$ *composed of embeddings of M.*

Proof. We first prove that the set of immersive maps from M to $\mathbb{R}^{2m+1}$ is an open and dense set of $C^r(M, \mathbb{R}^{2m+1})$. For that, consider the maps

$$\sigma : C^r(M, \mathbb{R}^{2m+1}) \to C^{r-1}(\widetilde{T}M, T\mathbb{R}^{2m+1})$$
$$\sigma(f) = Tf$$
$$ev_\sigma : C^r(M, \mathbb{R}^{2m+1}) \times \widetilde{T}M \to T\mathbb{R}^{2m+1}$$
$$ev_\sigma(f, v) = Tf(v).$$

By Lemma 3, ev_σ is transverse to every submanifold of $T\mathbb{R}^{2m+1}$, thus is transverse to $L = \{0_y \in T\mathbb{R}^{2m+1}\}$, with $\dim L = 2m + 1$. This evaluation map is C^1, if $r \geq 3$ and $\dim \widetilde{T}M - co\dim L < 0$, then we apply the Parametric Transversality Theorem (Theorem 2) to σ and obtain that the set of f's belonging to $C^r(M, \mathbb{R}^{2m+1})$ and holding that $\sigma(f)$ is transverse to L is open and dense in $C^r(M, \mathbb{R}^{2m+1})$.

For any $v \in \widetilde{T}M$ the dimension of $T_v(\widetilde{T}M)$ is $2m-1$ and then the dimension of $T_v(Tf)(T_v(\widetilde{T}M))$ is at most $2m-1$ and the dimension of T_0L is $2m+1$. If $Tf(v) \in L$, $T_v(Tf)(T_v(\widetilde{T}M)) + T_0L$ has dimension at most $4m < 2(2m+1) = \dim T_0T\mathbb{R}^{2m+1}$, then $T_v(Tf)(T_v(\widetilde{T}M)) + T_0L$ cannot span $T_0T\mathbb{R}^{2m+1}$. Thus the unique way for Tf to be transverse to L is that its image does not intersect L. In such case f is an immersion.

We denote by I the open and dense set of $C^r(M, \mathbb{R}^{2m+1})$ holding that Tf is an immersion for every $f \in I$.

Now we prove that the set of injective maps from M to $\mathbb{R}^{2m+1}$ which are immersive is an open and dense set in I. To this aim, consider the maps

$$\rho : I \to C^r(M \times M \setminus \Delta) \to \mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1}$$

$$ev_\rho : I \times (M \times M \setminus \Delta) \to \mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1}$$

$$ev_\rho(f, x, y) = (f(x), f(y)),$$

where Δ is the diagonal of $M \times M$, with $z = f(x) = f(y)$ and

$$T_{f,x,y}ev_\rho : T_fC^r(M, \mathbb{R}^{2m+1}) \times T_{x,y}(M \times M \setminus \Delta) \to T_{z,z}\mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1}$$

$$T_{f,x,y}ev_\rho(\eta, u, v) = (\eta(x) + T_xf(u), \eta(y) + T_yf(v)).$$

Using Lemma 4, for all $f, x, y \in C^r(M, \mathbb{R}^{2m+1}) \times (M \times M \setminus \Delta)$ and every $(v, 0) \in T_{z,z}\mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1}$, we can find $\eta \in T_fC^r(M, \mathbb{R}^{2m+1})$ such that $\eta(x) = v$ and $\eta(y) = 0$. Hence $T_{f,x}ev_\rho(\eta, 0, 0) = (v, 0)$, which implies the map ev_ρ is transverse to $\hat{\Delta}$ the diagonal of $\mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1}$.

Since ev_ρ is C^1, $r \geq 2$ and $\dim M \times M \setminus \Delta - co\dim \hat{\Delta} = 2m - 2m - 1 < 1$, we can apply the Parametric Transversality Theorem to obtain that the set of f's in I such that $\rho(f)$ is transverse to $\hat{\Delta}$ is open and dense in I.

The dimension of $T_{x,y}M \times M \setminus \Delta$ is $2m$, then the dimension of $T_{x,y}\rho(f)(T_{x,y}M \times M \setminus \Delta)$ is less than or equal to $2m$. Since the dimension of $T_{z,z}\hat{\Delta}$ is $2m+1$, the transversality of $\rho(f)$ to $\hat{\Delta}$ implies that the image of $\rho(f)$ cannot intersect $\hat{\Delta}$. This is the condition for f to be injective, and hence the injective maps form an open and dense set in I.

Then the set of injective and immersive maps from M to $\mathbb{R}^{2m+1}$ is an open and dense set in $C^r(M, \mathbb{R}^{2m+1})$, for $r \geq 3$. But $C^r(M, \mathbb{R}^{2m+1})$ is an open and dense set in $C^1(M, \mathbb{R}^{2m+1})$ for $r \geq 1$, and the statement is proved.

Theorem 6. *Let M be a compact smooth manifold of dimension m contained in $\mathbb{R}^k$. Then almost every $f \in C^r(\mathbb{R}^k, \mathbb{R}^{2m+1})$ is an embedding of M, for $r \geq 1$.*

Proof. Let $S = \{f \in C^r(\mathbb{R}^k, \mathbb{R}^{2m+1}) : f$ is an embedding of $M, r \geq 1\}$. We want to prove that this set is prevalent.

To this end, we take the finite-dimensional space of polynomial functions from $\mathbb{R}^k$ to $\mathbb{R}^{2m+1}$ of degree less than or equal to 2. Let $\{f_1, f_2, ..., f_p\}$ be a basis for this space.

Let $f_0 \in C^r(\mathbb{R}^k, \mathbb{R}^{2m+1})$ and $f_\alpha = f_0 + \sum_{i=1}^{p} \alpha_i f_i$. Consider the maps

$$\sigma : \mathbb{R}^p \to C^r(\widetilde{T}M, T\mathbb{R}^{2m+1})$$

given by

$$\sigma(\alpha) = Tf_\alpha$$

and

$$ev_\sigma : \mathbb{R}^p \times \widetilde{T}M \to T\mathbb{R}^{2m+1}$$

given by

$$ev_\sigma(\alpha, v) = Tf_\alpha(v) = Tf_0(v) + \sum_{i=1}^{p} \alpha_i Tf_i(v).$$

By Lemma 5, the map ev_σ is transverse to every submanifold of $T\mathbb{R}^{2m+1}$, and then is transverse to $L = \{0_y \in T\mathbb{R}^{2m+1}\}$. Since $\dim(\widetilde{T}M) - co\dim L < 0$, we apply the Morse-Sard's theorem (see [3], [13]) to obtain that the set $\{\alpha \in \mathbb{R}^p : \sigma(\alpha)$ is transverse to $L\}$ has full measure and is an open set. Now we count dimensions: $\dim(T_v\widetilde{T}M) = 2m-1$, the dimension of its image $T_v(Tf_\alpha)(T_v\widetilde{T}M)$ is at most $2m-1$ and the dimension of T_0L is $2m+1$. Hence $T_v(Tf_\alpha)(T_v\widetilde{T}M) + T_0L$ cannot span $T_0T\mathbb{R}^{2m+1}$. Therefore the map Tf_α is transverse to L when its image does not intersect L. Then the set $\{\alpha \in \mathbb{R}^p : f_\alpha$ is an immersion$\}$ has full measure and is an open set. We denote this set by $\mathbb{I}$.

Now consider the map

$$\rho : \mathbb{I} \to C^r(M \times M \setminus \Delta, \mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1})$$

$$ev_\rho : \mathbb{I} \times (M \times M \setminus \Delta) \to \mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1}$$

$$ev_\rho(\alpha, x, y) = \left(f_0(x) + \sum_{i=1}^{p} \alpha_i f_i(x), f_0(y) + \sum_{i=1}^{p} \alpha_i f_i(y) \right).$$

By Lemma 4, the map ev_ρ is C^1 and is transverse to $\hat{\Delta}$. Since $\dim(M \times M \setminus \Delta) - co\dim \hat{\Delta} < 0$, we apply the Morse-Sard's theorem to obtain that the set $\{\alpha \in \mathbb{I} : \rho(\alpha)$ is transverse to $\hat{\Delta}\}$ has full measure and is an open set in $\mathbb{I}$. We count dimensions: for any $x, y \in (M \times M \setminus \Delta)$, if the dimension of $T_{x,y}(M \times M \setminus \Delta)$ is $2m$, then the dimension of $T_{x,y}\rho(\alpha)$ $(T_{x,y}(M \times M \setminus \Delta))$ is less than or equal to $2m$ and if the dimension of $T_{z,z}\hat{\Delta}$ is $2m+1$, then $T_{x,y}(M \times M \setminus \Delta) + T_{z,z}\hat{\Delta}$ cannot span $T_{z,z}(\mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1})$. Thus $\rho(\alpha)(x, y) \notin \hat{\Delta}$, that is, $\rho(\alpha)$ is injective for almost every $\alpha \in \mathbb{I}$.

Theorem 7. *Let A be a compact subset of box counting dimension d contained in $\mathbb{R}^k$. Then almost every $f \in C^r(\mathbb{R}^k, \mathbb{R}^n)$, with $r \geq 1$ and*

$n = \min\{j \in \mathbb{N} : j \geq 2d + 1\}$, *is injective on* A *and immersive on each compact subset of a smooth manifold contained in* A.

Proof. Let $S = \{f \in C^r(\mathbb{R}^k, \mathbb{R}^n) : f$ is injective on $A \subset \mathbb{R}^k\}$. We will prove that this set is prevalent in $C(\mathbb{R}^k, \mathbb{R}^n)$.

An ε-*map* f is a map such that $\Delta_A(f) = \sup\{\dim f^{-1}(\{z\}) : z \in f(A)\} < \varepsilon$. The set of all those maps consists on the maps on A which deviate from being injective by less than ε.

Consider the set of those maps on A which deviate from being injective by less than ε and $U_\varepsilon = \{f \in C(\mathbb{R}^k, \mathbb{R}^n) : \Delta_A(f) < \varepsilon\}$. It is open in the topology of compact convergence in $C(\mathbb{R}^k, \mathbb{R}^n)$.

By Lemma 9 there is a map from $\mathbb{R}^k$ to $\mathbb{R}^n$ injective on A with $n = \min\{j \in \mathbb{N} : j \geq 2d + 1\}$. Then it is held

$$\{f \in C(\mathbb{R}^k, \mathbb{R}^n) : f \text{ is injective on } A \subset \mathbb{R}^k\} = \bigcap_{n=1}^{\infty} U_{1/n} \neq \emptyset.$$

Let $\{f_1, f_2, \ldots, f_q\}$ be a basis of the space of linear transformation from $\mathbb{R}^k$ to $\mathbb{R}^n$. Consider the map

$$\psi : \mathbb{R}^k \times \mathbb{R}^q \to \mathbb{R}^n$$

given by

$$\psi(x, \alpha) = f_0(x) + \sum_{i=1}^{p} \alpha_i f_i(x),$$

where $f_0 \in C(\mathbb{R}^k, \mathbb{R}^n)$ and $\alpha = (\alpha_1, \ldots, \alpha_q)$. This map is continuous and the same for the map

$$\Psi : \mathbb{R}^q \to C(\mathbb{R}^k, \mathbb{R}^n)$$

given by

$$\Psi(\alpha) = f_0 + \sum_{i=1}^{q} \alpha_i f_i$$

see [8].

If $n \geq k$, then there exists some $\alpha \in \mathbb{R}^q$ such that $\Psi(\alpha)$ is injective on $\mathbb{R}^k$ and then injective on A.

If $n < k$, then by Lemma 9 we know that A is in a subspace of $\mathbb{R}^k$ of dimension n and also there exists an injective map $\Psi(\alpha)$ on this subspace and then injective on A.

Then the subset of $\{\alpha \in \mathbb{R}^q$ holding $\Psi(\alpha) \in \bigcap_{n=1}^{\infty} U_{1/n}\}$ is an open set of $\mathbb{R}^q$.

Take $\alpha \in \mathbb{R}^q$ and the set $B(\alpha, \xi), \xi > 0$. Then the map

$$\psi/_B : \mathbb{R}^k \times B(\alpha, \xi) \to \mathbb{R}^n$$

given by

$$\psi/_B(x, \beta) = f_0(x) + \sum_{i=1}^{q} \beta_i f_i(x)$$

is continuous and $\Psi/_B = f_0 + \sum_{i=1}^{q} \beta_i f_i$ is also continuous [8] and then the subset of $\{\beta : \beta \in B(\alpha, \xi)$ such that $\Psi/_B(\beta) \in \bigcap_{n=1}^{\infty} U_{1/n}$ is an open set of $B(\alpha, \xi)\}$. We have then proved that for almost every $\alpha \in \mathbb{R}^q$, where the map $f_\alpha = f_0 + \sum_{i=1}^{q} \alpha_i f_i$ is injective on A.

The second part of the statement is immediate using Theorem 6.

Theorem 8. *Let M be a compact smooth manifold of dimension m contained in $\mathbb{R}^k$ and let $f \in C^r(\mathbb{R}^k, \mathbb{R}^k)$ be a diffeomorphism on M with only a finite number of periodic orbits of period less than $4m + 2$, and eigenvalues distinct. Then for almost every $\alpha \in C^r(M, \mathbb{R})$, the map $\Phi_{f, \alpha} : M \to \mathbb{R}^{2m+1}$ given by*

$$\Phi_{f, \alpha}(x) = (\alpha(x), \alpha(f(x)), \ldots, \alpha(f^{2m}(x)))$$

is an embedding on M, with $r \geq 1$.

Proof. Let $\{\alpha_1, \ldots, \alpha_q\}$ be a basis of the vectorial space of polynomial in k variables of degree less than or equal to $4m+2$. Denote by $P_f = \{x_1, \ldots, x_p\}$ the set of periodic points of f in M.

It is proved in [13] that the set of maps $\alpha \in C^r(M, \mathbb{R})$ holding $\Phi_{f,\alpha}$ is injective and immersive on P_f is an open and dense set of $C^r(M, \mathbb{R})$.

Consider the map

$$ev_\sigma : \mathbb{R}^q \times M \to \mathbb{R}$$

given by

$$ev_\sigma(a, x) = \alpha_0(x) + \sum_{i=1}^{q} a_i \alpha_i(x),$$

where $\alpha_0 \in C^r(M, \mathbb{R})$. It is continuous and then $\sigma : \mathbb{R}^q \to C^r(M, \mathbb{R})$ with $\sigma(a) = \alpha_0 + \sum_{i=1}^{q} a_i \alpha_i$ is also continuous, provided $r \geq 1$.

Following [11] we construct a polynomial in k variables of degree at most $p-1$ such that $\Phi_{f,\alpha}$ is injective on P_f and a polynomial in k variables of degree at most p such that $\Phi_{f,\alpha}$ is immersive on P_f.

Then we have an open and dense set of $\mathbb{R}^q$, denoted by O, such that for all $a \in O$ the map $\Phi_{f,\alpha}$ is injective and immersive on P_f.

Consider the map

$$ev_\rho : O \times \widetilde{T}M \to T\mathbb{R}^{2m+1}$$

given by

$$ev_\rho(a, v) = T\Phi_{f,\alpha_0}(v) + \sum_{i=1}^{q} a_i T\Phi_{f,\alpha_i}(v),$$

where $a = (a_1, \ldots, a_q)$ and $L = \{0_y \in T\mathbb{R}^{2m+1}\}$. We claim that this map is transverse to L.

To see it, if $ev_\rho(a, v) \in L$ with $a \in O$, $v \in T_x M$, then $x \notin P_f$, since if $x \in P_f$, then $T\Phi_{f,\alpha_0}(v) + \sum_{i=1}^{q} a_i T\Phi_{f,\alpha_i}(v) \neq 0$ and as consequence ev_ρ would be transverse to L at such periodic point x.

If $x \notin P_f$, then $x, x_1, \ldots, x_{2m}$ are distinct and it is true that

$$T_{a,v} ev_\rho(T_a \mathbb{R}^q \times T_v \widetilde{T}M) + T_0 L = T_0 T\mathbb{R}^{2m+1},$$

since given any $u \in T_0 T\mathbb{R}^{2m+1}$ and $v_i = Tf^i(v) \neq 0$ there would exist a polynomial in k variables of degree at most $2m+1$ (see [11]) such that $T_{a,v} ev_\rho(b, 0) = u$.

The map ev_ρ is C^1 and $\dim \widetilde{T}M - co\dim L < 1$, then applying Theorem 3 we have that thus the set of $a \in O$ has full measure in O. Now we count dimensions: $\dim T_v(\widetilde{T}M)$ is $2m-1$, the dimension of $T_v\rho(a)(T_v(\widetilde{T}M))$ is at most $2m-1$ and $\dim T_0 L = 2m+1$, then $T_v\rho(a)(T_v(\widetilde{T}M)) + T_0 L$ has dimension at most $4m$ and this is strictly less than $\dim T\mathbb{R}^{2m+1} = 4m+2$. Therefore $\rho(a)$ is transverse to L whenever its image does not intersect L, that is, when $\rho(a)$ be an immersion.

Consider now the map

$$ev_\theta : O \times (M \times M \setminus \Delta) \to \mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1}$$

given by

$$ev_\theta(a, x, y) = \left(\Phi_{f,\alpha_0}(x) + \sum_{i=1}^{q} a_i \Phi_{f,\alpha_i}(x),\ \Phi_{f,\alpha_0}(y) + \sum_{i=1}^{q} a_i \Phi_{f,\alpha_i}(y)\right)$$

we now claim that this map is transverse to the diagonal of $\mathbb{R}^{2m+1} \times \mathbb{R}^{2m+1}$. Following a similar procedure than in Stark [14], we distinguish the following cases:

(a) If $x, y \in P_f$, then it is immediate that for almost every element of $\mathbb{R}^q$ the map is transverse.

(b) Let $x \neq y$ and suppose that some of them are not periodic points of *f*. Then without loss of generality, we may assume that the points $x_i = f^i(x)$, for $i = 0, \ldots, 2m$ are distinct. Let $z = \Phi_{f, \alpha_0}(x) + \sum_{i=1}^{q} \Phi_{f, \alpha_i}(x) = \Phi_{f, \alpha_0}(y) + \sum_{i=1}^{q} \Phi_{f, \alpha_i}(y)$. Then $T_{z, z}\hat{\Delta} = \{(u, u) : u \in T_z\mathbb{R}^{2m+1}\}$. We need to show that the image of $T_{a, x, y}ev_\theta$ contains a complement of this space. Let $e^{(i)} = (0, \ldots, e_i, \ldots, 0) \in T_z\mathbb{R}^{2m+1}$ and $e^{(0)}, \ldots, e^{(2m)}$ form a basis of $T_z\mathbb{R}^{2m+1}$. We know that

$$\begin{aligned} T_{a, x, y}ev_\theta(\overline{a}, 0, 0) &= \left(\sum_{i=1}^{q} \overline{a_i}\Phi_{f, \alpha_i}(x), \sum_{i=1}^{q} \overline{a_i}\Phi_{f, \alpha_i}(y) \right) \\ &= \left(\sum_{i=1}^{q} \overline{a_i}(\alpha_i(x), \alpha_i(x_1), \ldots, \alpha_i(x_{2m})), \right. \\ &\qquad \left. \sum_{i=1}^{q} \overline{a_i}(\alpha_i(y), \alpha_i(y_1), \ldots, \alpha_i(y_{2m})) \right). \end{aligned}$$

First suppose that $x, x_1, \ldots, x_{2m}, y, y_1, \ldots, y_{2m}$ are distinct and again using [11] we can find a polynomial with k variables of degree $4m + 2$ such that $T_{a, x, y}ev_\theta(\overline{a}, 0, 0) = (e^{(i)}, 0)$ and therefore the image of $T_{a, x, y}ev_\theta$ contains the space $T_z\mathbb{R}^{2m+1} \times \{0\}$ which is clearly a complement of $T_{z, z}\hat{\Delta}$.

It remains to consider the case $y = f^j(x)$, for some $-2m \leq j \leq 2m$, $j \neq 0$. Since then y cannot be a periodic orbit of period less than $4m + 2$, we can assume without loss of generality that $y = f^j(x)$, $0 < j \leq 2m$. The points $x, x_1, \ldots, x_{j-1}, y, y_1, \ldots, y_{2m}$ are distinct. Hence for $0 \leq i < j$ we construct a polynomial with k variables of degree $2m + j + 1$ such that $T_{a, x, y}ev_\theta(\overline{a}, 0, 0) = (e^{(i)}, 0)$. Now we proceed by induction. Our inductive hypothesis on k is that for all $0 \leq i < k$ there exist a polynomial and a $u_i \in T_z\mathbb{R}^{2m+1}$ such that $T_{a, x, y}ev_\theta(\overline{a}, 0, 0) = (e^{(i)} + u_i, u_i)$. By above this holds for $k = j - 1$ (with $u_i = 0$).

Now we proceed as follows: since $f^{i-j}(y) = f^i(x)$ the points

$$x, x_1, \ldots, x_{2m}, y_{2m-j}, y_{2m-j+1}, \ldots, y_{2m}$$

are distinct, so we can find a polynomial in k variables of degree $2m + j + 1$ such that $T_{a,x,y}ev_\theta(\overline{b_k}, 0, 0) = (e^{(k)}, e^{(k-j)})$. By inductive hypothesis there exists a polynomial such that $T_{a,x,y}ev_\theta(\overline{a_{k-j}}, 0, 0) = (e^{(k-j)} + u_{k-j}, u_{k-j})$. Thus $T_{a,x,y}ev_\theta(\overline{a_{k-j}} + \overline{b_k}, 0, 0) = (e^{(k)} + e^{(k-j)} + u_{k-j}, e^{(k-j)} + u_{k-j})$. This complete the inductive process with $u_k = e^{(k-j)} + u_{k-j}$.

We have thus shown that $Im(T_{a,x,y}ev_\theta) + T_{z,z}\hat{\Delta}$ contains the space $T_z\mathbb{R}^{2m+1} \times \{0\}$ which is clearly a complement of $T_{z,z}\hat{\Delta}$.

The preimage of an open and dense set of α's $\in C^r(M, \mathbb{R})$ holding $\Phi_{f,\alpha}$ is injective and immersive on P_f, therefore it is an open set of $\mathbb{R}^q$.

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