Far East J. Dynamical Systems **6**(2) (2004), 129-138

SHIFT-INVARIANT SETS AND THE SHADOWING PROPERTY OF HYPERBOLIC ITERATED FUNCTION SYSTEMS

KAZUHISA KATO

Department of Mathematics, Faculty of Science
Kochi University, Kochi, 780-8520, Japan
e-mail: kato@math.kochi-u.ac.jp

Abstract

We define the shift-invariant subsets for hyperbolic iterated functions systems on metric spaces, and discuss the shadowing property for the dynamical systems that corresponds to subshifts on symbol spaces.

1. Introduction

The purpose of this paper is to investigate the dynamics of hyperbolic iterated function systems (see Definition 1.1). In [5], Hutchinson proves that a hyperbolic iterated function system on a complete metric space has the attractor. Also, it is known that the attractor is a factor of symbolic dynamics (Theorem 1.4). Therefore, we can consider the dynamical system which corresponds to subshifts. In Section 2, we shall give the definition of a shift-invariant subset for a hyperbolic iterated function system on a metric space (Definition 2.1), and show the existence of the subshift which is corresponded to the shift-invariant set (Theorem 2.3). In Section 3 we shall discuss the shadowing property of the dynamical system on a shift-invariant subset (Theorem 3.4).

2000 Mathematics Subject Classification: Primary 37B99, 37C50.

Key words and phrases: shift-invariant sets, hyperbolic iterated function systems, shadowing property.

Received August 25, 2004

In this section we review the notion of attractors for iterated function systems.

Definition 1.1. Let (X, d) be a metric space, and let $f_i : X \to X$ be continuous maps $(i = 1, 2, ..., m)$. Then we call $(X; f_1, f_2, ..., f_m)$ an *iterated function system* (abbrev. IFS). This system is said to be *hyperbolic* if there is a constant $0 \leq r < 1$ such that for $i = 1, ..., m$,

$$d(f_i(x), f_i(y)) \leq rd(x, y), \; x, \; y \in X.$$

Such a constant r is called a *contractivity factor* for $(X; f_1, f_2, ..., f_m)$.

Definition 1.2 [4]. Let $(X; f_1, f_2, ..., f_m)$ be an IFS on a metric space (X, d). A subset A of X is called an *attractor* for $(X; f_1, f_2, ..., f_m)$ provided that

(i) A is nonempty and compact;

(ii) $f_i(A) \subset A$ for $i = 1, ..., m$;

(iii) A is minimal with respect to (i) and (ii).

An attractor A satisfies

$$A = f_1(A) \cup \cdots \cup f_m(A). \tag{1}$$

We can easily prove the following lemma (see [4, 6]).

Lemma 1.3. *Let* $(X; f_1, f_2, ..., f_m)$ *be a hyperbolic IFS on a metric space* (X, d). *Let B be a nonempty compact subset of X which satisfies*

$$B \subset f_1(B) \cup \cdots \cup f_m(B).$$

Then for each $b_0 \in B$ *there are a sequence* $\{b_n\}_{n=1}^{\infty}$ *of points in B and a sequence* $\{i_n\}_{n=1}^{\infty}$ *with* $i_n \in \{1, ..., m\}$ *so that* $b_n = f_{i_{n+1}}(b_{n+1})$ *for all* $n \geq 0$ *and* $b_0 = \lim_{n \to \infty} f_{i_1} \circ \cdots \circ f_{i_n}(x)$ *for all* $x \in X$.

In case that $(X; f_1, f_2, ..., f_m)$ is hyperbolic, by making use of Lemma 1.3 we can show that as long as an attractor exists, it is unique, and that a nonempty compact subset A of X is the attractor if and only if it satisfies (1). In case that X is compact, from a standard Zorn's argument it follows that an IFS on X has an attractor.

Let Σ_m denote the *symbol space* on symbols 1, 2, ..., m. That is, Σ_m consists of all one-sided infinite sequences of symbols chosen from $\{1, 2, ..., m\}$:

$$\Sigma_m = \{(i_1, i_2, ...) : i_n \in \{1, 2, ..., m\} \text{ for all } n \geq 1\}.$$

We define a metric d on Σ_m as follows: Let $s = (i_1, i_2, ...)$, $t = (j_1, j_2, ...)$ $\in \Sigma_m$. In case $s = t$, $d(s, t) = 0$. In case $s \neq t$, $d(s, t) = 2^{-k}$, where $k + 1$ $= \inf\{n : i_n \neq j_n\}$. The symbol space (Σ_m, d) is a compact metric space. The *shift map* $\sigma : \Sigma_m \to \Sigma_m$ is defined by

$$\sigma(i_1, i_2, ...) = (i_2, i_3, ...)$$

for $(i_1, i_2, ...) \in \Sigma_m$. Also, for each $1 \leq i \leq m$ we define a map $\tau_i : \Sigma_m$ $\to \Sigma_m$ by

$$\tau_i(i_1, i_2, ...) = (i, i_1, i_2, ...).$$

We can easily check that $d(\sigma(s), \sigma(t)) \leq 2d(s, t)$, $d(\tau_i(s), \tau_i(t)) = 2^{-1}d(s, t)$ for all $s, t \in \Sigma_m$. Therefore, $(\Sigma_m; \tau_1, ..., \tau_m)$ is a hyperbolic IFS and its attractor is Σ_m.

Under notations above we have the following theorem (cf. [3, 4, 6]).

Theorem 1.4. *Let* $(X; f_1, f_2, ..., f_m)$ *be a hyperbolic IFS on a metric space* (X, d). *Suppose that it has the attractor, A. Then, there is a unique map* $h : \Sigma_m \to A$ *such that for* $i = 1, ..., m$ *the following diagrams are commutative*:

$$\begin{tikzcd} A \arrow[r, "f_i"] & A \\ \Sigma_m \arrow[u, "h"] \arrow[r, "\tau_i"'] & \Sigma_m \arrow[u, "h"'] \end{tikzcd}$$

Moreover, h is continuous and onto, and satisfies

$$h(s) = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(x) \tag{2}$$

for all $s = (i_1, i_2, \ldots) \in \sum_m$ *and* $x \in X$.

The map $h : \sum_m \to A$ defined by (2) is called the *code map*.

2. Shift-invariant Subsets

Let E be a nonempty compact subset of $\sum_m$ satisfying $\sigma(E) = E$. Then we call the dynamical system (E, σ) a *subshift*.

For a subset E of a symbol space $\sum_m$, $\sigma(E) \subset E$ if and only if $(*)$ $E \subset \tau_1(E) \cup \cdots \cup \tau_m(E)$; $\sigma(E) \supset E$ if and only if $(**)$ $E \subset \tau_1^{-1}(E) \cup \cdots \cup \tau_m^{-1}(E)$.

Here we define a shift-invariant subset for an IFS as follows:

Definition 2.1. Let $(X; f_1, f_2, \ldots, f_m)$ be an IFS on a metric space (X, d). A nonempty compact subset B of X is said to be *shift-invariant* if $(*)$ $B \subset f_1(B) \cup \cdots \cup f_m(B)$ and $(**)$ $B \subset f_1^{-1}(B) \cup \cdots \cup f_m^{-1}(B)$.

Note that if a subset A of X satisfies $A = f_1(A) \cup \cdots \cup f_m(A)$, then $(*)$ and $(**)$ hold for $B = A$.

Theorem 2.2 [1]**.** *Let* $(X; f_1, f_2, \ldots, f_m)$ *be a hyperbolic IFS on a complete metric space* (X, d) *with attractor* A, *and let* $h : \sum_m \to A$ *denote the code map. Let E be a compact subset of X.*

(a) *There is a compact subset* E *of* $\sum_m$ *with* $\sigma(E) \subset E$ *and* $B = h(E)$ *if and only if* $(*)$ $B \subset f_1(B) \cup \cdots \cup f_m(B)$.

(b) *The set* $E = h^{-1}(B)$ *satisfies* $\sigma(E) \supset E$ *if and only if* $(**)$ $B \subset f_1^{-1}(B) \cup \cdots \cup f_m^{-1}(B)$.

(c) *B is the image set of a subshift of* $\sum_m$ *under the code map if and only if B satisfies* $(*)$ *and* $(**)$.

We can generalize Theorems 1.4 and 2.2 as follows:

Theorem 2.3. *Let* $(X; f_1, f_2, ..., f_m)$ *be an IFS on a metric space* (X, d), *and let a shift-invariant subset B of X be given. We put*

$$E_B = \left\{ (i_1, i_2, ...) \in \Sigma_m : \begin{array}{l} \text{There is a sequence } \{b_n\}_{n=0}^{\infty} \text{ of points in } B \\ \text{such that } b_n = f_{i_{n+1}}(b_{n+1}) \text{ for all } n \geq 0. \end{array} \right\}.$$

Then we have the following:

(1) E_B *is shift-invariant.*

(2) *Moreover, if* $(X; f_1, f_2, ..., f_m)$ *is hyperbolic, then there is a unique continuous onto map* $h : E_B \to B$ *such that* $h(s) = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(x)$ *for all* $s = (i_1, i_2, ...) \in E_B$ *and* $x \in X$. *Also, for* $i = 1, ..., m$ *the following diagrams are commutative*:

$$\begin{tikzcd} B \arrow[r, "f_i"] & B \\ E_B \arrow[u, "h"] \arrow[r, "\tau_i"'] & E_B \arrow[u, "h"'] \end{tikzcd}$$

Proof. (1) By definition we can easily check that E_B is nonempty and $\sigma(E_B) = E_B$. We shall prove that E_B is closed. Let $s = (i_1, i_2, ...) \in \overline{E_B}$, where $\overline{E_B}$ denotes the closure of E_B. Then, for each $n \geq 0$ we can take a finite sequence $\{b_0^{(n)}, ..., b_n^{(n)}\}$ of points in B such that $b_k^{(n)} = f_{i_{k+1}}(b_{k+1}^{(n)})$ for $k = 0, ..., n-1$. Here we may assume that for each $k \geq 0$ the sequence $\{b_k^{(n)}\}_{n=k}^{\infty}$ converges: $b_k = \lim_{n\to\infty} b_k^{(n)} (\in B)$. While, $b_k^{(n)} = f_{i_{k+1}}(b_{k+1}^{(n)})$ for all $0 \leq k < n$. Letting $n \to \infty$ we have $b_k = f_{i_{k+1}}(b_{k+1})$ for all $k \geq 0$. So $s = (i_1, i_2, ...) \in E_B$.

(2) Suppose that $(X; f_1, f_2, ..., f_m)$ is a hyperbolic IFS with contractivity factor r. We shall define the code map $h : E_B \to B$. Take $s = (i_1, i_2, ...) \in E_B$ and $x \in X$. By definition of E_B there is a sequence

$\{b_n\}_{n=0}^{\infty}$ of points in B such that $b_n = f_{i_{n+1}}(b_{n+1})$ for all $n \geq 0$. Since B is compact, we can take a constant $K \geq 0$ so that $d(b, x) \leq K$ for all $b \in B$. Then we have $d(b_0, f_{i_1} \circ \cdots \circ f_{i_n}(x)) = d(f_{i_1} \circ \cdots \circ f_{i_n}(b_n), f_{i_1} \circ \cdots \circ f_{i_n}(x)) \leq r^n K$. So $b_0 = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(x)$. Therefore, we define

$$h(s) = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(x) (\in B).$$

From Lemma 1.3 we see that h is onto. Also, from definition it follows that $d(h(s), h(t)) \leq r^n \operatorname{diam} B$, where $s = (i_1, i_2, \ldots)$, $t = (j_1, j_2, \ldots) \in E_B$ and $i_1 = j_1, \ldots, i_n = j_n$, and $\operatorname{diam} B = \sup\{d(x, y) : x, y \in B\}$.

Remark 2.4. Let $s = (i_1, i_2, \ldots) \in E_B$. We put $b_n = h(\sigma^n(s))$ for $n \geq 0$. Then $\{b_n\}$ is a sequence of points in B with $b_n = f_{i_{n+1}}(b_{n+1})$ for all $n \geq 0$.

Conversely, let $\{b_n\}_{n=0}^{\infty}$ be a sequence of points in B, and $s = (i_1, i_2, \ldots) \in \Sigma_m$ satisfies $b_n = f_{i_{n+1}}(b_{n+1})$ for all $n \geq 0$. Then, $s \in E_B$ and $b_n = h(\sigma^n(s))$ for all $n \geq 0$.

Remark 2.5. The code map $h : E_B \to B$ is one-to-one if and only if the $B \cap f_i(B)$ is pairwise disjoint and each $f_i : f_i^{-1}(B) \cap B \to B$ is one-to-one.

Remark 2.6. If $(X; f_1, f_2, \ldots, f_m)$ be a hyperbolic IFS with attractor A, then $E_A = \Sigma_m$.

3. The Shadowing Property

Definition 3.1. Let f be a continuous map from a metric space (X, d) into itself. For $\delta > 0$, a sequence $\{x_n\}_{n=0}^{\infty}$ of points in X is called a δ-*pseudo-orbit* for f provided that $d(f(x_n), x_{n+1}) < \delta$ for all $n \geq 0$. Given $\varepsilon > 0$, a sequence $\{x_n\}_{n=0}^{\infty}$ is said to be ε-*traced* by a point $x \in X$ provided that $d(x_n, f^n(x)) < \varepsilon$ for all $n \geq 0$. We say that f has the *shadowing property* if for any $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit for f can be ε-traced by some point in X.

Definition 3.2. For a nonempty subset W of $\{1, ..., m\}^k (k \geq 1)$ we put

$$\Sigma_W = \{(i_1, i_2, ...) \in \Sigma_m : (i_{n+1}, ..., i_{n+k}) \in W, n = 0, 1, ...\}.$$

Let E be a shift-invariant subset of Σ_m. The dynamical system (E, σ) is called a *subshift of finite type* provided that $E = \Sigma_W$ for some nonempty subset W of $\{1, ..., m\}^k$.

Let $M = (M_{ij})$ be an $m \times m$ matrix with entries 0 and 1. We put

$$\Sigma_M = \{(i_1, i_2, ...) \in \Sigma_m : M_{i_n i_{n+1}} = 1, n = 1, 2, ...\}.$$

If Σ_M is shift-invariant, then (Σ_M, σ) is a subshift of finite type. We call M the *structure matrix.*

It is well-known that a subshift (E, σ) has the shadowing property if and only if it is of finite type ([7]).

Definition 3.3. Let $(X; f_1, f_2, ..., f_m)$ be an IFS on a metric space (X, d), and let a shift-invariant subset B be given. A sequence $\{x_n\}_{n=0}^{\infty}$ of points in B is called an *orbit* if there is $(i_1, i_2, ...) \in E_B$ such that $x_n = f_{i_{n+1}}(x_{n+1})$ for all $n \geq 0$. For $\delta > 0$, a sequence $\{\tilde{x}_n\}_{n=0}^{\infty}$ of points in B is called a δ-*pseudo-orbit* if there are $(i_1, i_2, ...) \in E_B$ and a sequence $\{x'_n\}_{n=1}^{\infty}$ of points in B such that $d(\tilde{x}_{n+1}, x'_{n+1}) < \delta$ and $\tilde{x}_n = f_{i_{n+1}}(x'_{n+1})$ for all $n \geq 0$.

Let $(X; f_1, f_2, ..., f_m)$ be a hyperbolic IFS with attractor A. In the case that the code map $h : \Sigma_m \to A$ is one-to-one, we can consider a continuous onto map $S = h \circ \sigma \circ h^{-1} : A \to A$. The system (A, S) is called the *associated shift dynamical system*. Since (A, S) is topologically conjugate to (Σ_m, σ), (A, S) has the shadowing property. Barnsley directly shows that the dynamical system (A, S) has the shadowing property ([2, Theorem 7.1]). We shall generalize his result as follows:

Theorem 3.4. *Let* $(X; f_1, f_2, \ldots, f_m)$ *be a hyperbolic IFS on a metric space* (X, d) *with contractivity factor r, and let B be a shift-invariant subset of X. Then, for* $\delta > 0$ *and a* δ*-pseudo-orbit* $\{\widetilde{x}_n\}_{n=0}^{\infty}$ *there is an orbit* $\{x_n\}_{n=0}^{\infty}$ *such that* $d(x_n, \widetilde{x}_n) \leq r\delta(1-r)^{-1}$ *for all* $n \geq 0$.

Proof. Let a δ-pseudo-orbit $\{\widetilde{x}_n\}_{n=0}^{\infty}$ be given. From definition there are $s = (i_1, i_2, \ldots) \in E_B$ and a sequence $\{x'_n\}_{n=1}^{\infty}$ of points in B such that $d(\widetilde{x}_{n+1}, x'_{n+1}) < \delta$ and $\widetilde{x}_n = f_{i_{n+1}}(x'_{n+1})$ for all $n \geq 0$. We define a sequence $\{x_n\}_{n=0}^{\infty}$ by $x_n = h(\sigma^n(s))$, where $h: E_B \to B$ denotes the code map. Then, by Remark 2.4, $x_n = f_{i_{n+1}}(x_{n+1})$ for all $n \geq 0$, and so $\{x_n\}_{n=0}^{\infty}$ it is an orbit.

Now we shall prove that for $1 \leq k \leq N$,

$$d(x_{N-k}, \widetilde{x}_{N-k}) \leq r\delta + \cdots + r^{k-1}\delta + r^k K, \tag{3}$$

where $K = \operatorname{diam} B$. We fix N and prove (3) by induction on k. For $k = 1$, $d(x_{N-1}, \widetilde{x}_{N-1}) = d(f_{i_N}(x_N), f_{i_N}(x'_N)) \leq rK$. Thus (3) holds for $k = 1$. Suppose that (3) holds for a given value of k $(1 \leq k < N)$, and consider $k+1$. Then, we have

$$\begin{aligned} d(x_{N-(k+1)}, \widetilde{x}_{N-(k+1)}) &\leq rd(x_{N-k}, x'_{N-k}) \\ &\leq r(d(x_{N-k}, \widetilde{x}_{N-k}) + d(\widetilde{x}_{N-k}, x'_{N-k})) \\ &\leq r(r\delta + \cdots + r^{k-1}\delta + r^k K + \delta) \end{aligned}$$

which shows that (3) holds for $k+1$. This completes the induction.

Let $n \geq 0$ be given. By (3), $d(x_n, \widetilde{x}_n) \leq r\delta(1-r)^{-1} + r^{N-n}K$ for all $N > n$. Let $N \to \infty$ to obtain $d(x_n, \widetilde{x}_n) \leq r\delta(1-r)^{-1}$.

Proposition 3.5. *In Theorem* 3.4 *assume that* (E_B, σ) *is a subshift of finite type. Then, for any* $\delta > 0$ *there is* $\eta > 0$ *such that if* $\{s^{(n)}\}_{n=0}^{\infty}$ *is an* η*-pseudo-orbit for the dynamical system* (E_B, σ), *then* $\{h(s^{(n)})\}_{n=0}^{\infty}$ *is a* δ*-pseudo-orbit.*

Proof. Let $\delta > 0$ be given. Take $k > 0$ with $r^k \operatorname{diam} B < \delta$ and put $\eta = 2^{-k}$. Now, let an η-pseudo-orbit $\{s^{(n)}\}_{n=0}^{\infty}$ for (E_B, σ) be given. Here we put $\tilde{x}_n = h(s^{(n)})$, $s^{(n)} = (i_1^{(n)}, i_2^{(n)}, \ldots)$ and $x'_{n+1} = h(\sigma(s^{(n)}))(\in B)$. Since $d(s^{(n+1)}, \sigma(s^{(n)})) < \eta = 2^{-k}$, $d(\tilde{x}_{n+1}, x'_{n+1}) = d(h(s^{(n+1)}), h(\sigma(s^{(n)}))) \leq r^k \operatorname{diam} B < \delta$. Also, $\tilde{x}_n = f_{i_1^{(n)}}(h(\sigma(s^{(n)}))) = f_{i_1^{(n)}}(x'_{n+1})$. While, since the subshift (E_B, σ) is of finite type, $(i_1^{(0)}, \ldots, i_1^{(n)}, \ldots) \in E_B$ (if $\eta > 0$ is sufficiently small). Thus, $\{\tilde{x}_n\}_{n=0}^{\infty}$ is a δ-pseudo-orbit.

4. An Example

Let X denote the closed interval $[0, 1]$. We define

$$f_1(x) = \frac{1}{2}x, \qquad f_2(x) = \frac{1}{2} + \frac{1}{2}x.$$

Then, $(X; f_1, f_2)$ is a hyperbolic IFS and its attractor is X. Let $h : \Sigma_2 \to X$ denote the code map. We consider a structure matrix $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then

$$\Sigma_M = \{(\underbrace{1, \ldots, 1}_{n \text{ times}}, 2, 2, \ldots) \in \Sigma_2 : 0 \leq n \leq \infty\},$$

and Σ_M is shift invariant. Now, we put $B = h(\Sigma_M)$. Then, by Theorem 2.2, B is shift-invariant, and we have the following:

(1) $B = \{2^{-n} : 0 \leq n \leq \infty\}$.

(2) $h : \Sigma_M \to B$ is one-to-one.

(3) $B \cap f_1(B) \cap f_2(B) = \{2^{-1}\}$.

(4) $E_B = \Sigma_M \cup \{(\underbrace{1, \ldots, 1}_{n \text{ times}}, 2, 1, \ldots) : 0 \le n < \infty\}$.

(5) (E_B, σ) does not have the shadowing property.

(6) (E_B, σ) is not of finite type.

Proof of (5). Let $0 < \varepsilon < 2^{-1}$ be given. Take a sufficiently large integer $N > 0$ and put $\delta = 2^{-(N-2)}$. Define a finite sequence $\{s^{(n)}\}_{n=0}^{N}$ of points in Σ_M as follows:

$$s^{(0)} = (2, 1, 1, \ldots),\ s^{(1)} = (\underbrace{1, \ldots, 1}_{N-1 \text{ times}}, 2, 1, 1, \ldots),$$

and $s^{(n)} = \sigma^{n-1}(s^{(1)})$ for $n = 2, \ldots, N$. Then, $s^{(N)} = s^{(0)}$ and $d(\sigma(s^{(0)}), s^{(1)}) = 2^{-(N-1)} < \delta$. Therefore, extending this sequence periodically, we obtain a δ-pseudo-orbit for the dynamical systems (E_B, σ). But it can not be ε-traced by any orbit. Also, note that the sequence $\{h(s^{(n)})\}$ is not a pseudo-orbit on B (see Definition 3.3).

References

[1] C. Bandt, Self-similar sets. I. Topological Markov chains and mixed self-similar sets, Math. Nach. 142 (1989), 107-123.

[2] M. F. Barnsley, Fractals Everywhere, 2nd ed., Academic Press, 1993.

[3] G. A. Edgar, Measure, Topology, and Fractal Geometry, Springer-Verlag, Berlin, 1990.

[4] D. B. Ellis and M. G. Branton, Non-self-similar attractors of hyperbolic iterated function systems, Lecture Notes in Math. 1342, Springer-Verlag, Berlin, 1988, pp. 158-171.

[5] J. E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30 (1981), 713-747.

[6] K. Kato, On attractors of hyperbolic iterated function systems, Mem. Fac. Sci. Kochi Univ. (Math.) 22 (2001), 29-43.

[7] P. Walters, On the pseudo orbit tracing property and its relationship to stability, Lecture Notes in Math. 668, Springer-Verlag, Berlin, 1978, pp. 231-244.

■