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MEASURING ASYMPTOTIC DEPENDENCE OF EXTREMES AND TESTS BASED ON TAIL INDEXES

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Abstract

Let $\xi_1, \xi_2, \ldots$ be i.i.d. random vectors on $\Omega \subset R^d$ and let f and g be real-valued functions on Ω. Define $X_j = f(\xi_j)$ and $Y_j = g(\xi_j)$ and assume that the joint distribution of (X_j, Y_j) belongs to the domain of attraction of the bivariate maximal extreme value distribution $\mathcal{G}$ with marginals $\mathcal{G}_{\gamma_f}$ and $\mathcal{G}_{\gamma_g}$, where $\gamma_f > 0$ and $\gamma_g > 0$ are the corresponding tail indexes. We introduce probability measures on the set of extreme points Ω_f and Ω_g and derive tests for asymptotic dependence of $X_{(n)} = \max\{X_1, \ldots, X_n\}$ and $Y_{(n)} = \max\{Y_1, \ldots, Y_n\}$ based on Hill's estimates of γ_f and γ_g.

1. Introduction

Distributions with fat-tailed series are common in financial data and extreme value theory has become an important tool to analyse the extreme price movements during highly volatile periods corresponding to

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financial crisis. The univariate theory is a well-documented area and shows that the statistical behavior of extremes observed over a long time period can be modelled by three types of distributions: the Fréchet, the Weibull and the Gumbel distributions. These distributions, in the case of normalized maxima, can be summarized by the Generalized Extreme Value distribution

$$\mathcal{G}_\gamma(x) = \exp\{-(1+\gamma x)^{-1/\gamma}\} \tag{1}$$

defined on $\{x : 1 + \gamma x > 0\}$. Corresponding to $\gamma > 0$ we have the Fréchet distribution, $\gamma < 0$, the Weibull distribution and $\gamma = 0$, Gumbel, taken as the limit $\gamma \to 0$. The parameter γ, known as the tail index, represents the tail behavior of $\mathcal{G}_\gamma$. The Fréchet distribution, that corresponds to fat-tailed distributions, has been found to be the most appropriate for financial data.

In the multivariate case, no natural parametric family exists to summarize these distributions. Their study are usually done by modelling the dependence structure. Since for multivariate extreme value distribution pairwise independence is equivalent to mutual independence, enough to consider the bivariate case.

Let $(X_1, Y_1), (X_2, Y_2), \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables with a common distribution $F \in \mathcal{M}(F_X, F_Y)$, where F_X and F_Y denote the marginal distributions. We say that F belongs to the domain of attraction of the maximal bivariate extreme distribution $\mathcal{G}$, in short $F \in \mathcal{D}_{\max}(\mathcal{G})$, if there are sequences of constants $a_n > 0, b_n, c_n > 0$ and d_n such that

$$P\left(\frac{X_{(n)} - b_n}{a_n} \le x, \frac{Y_{(n)} - d_n}{c_n} \le y\right) \underset{n\to\infty}{\to} \mathcal{G}(x, y), \tag{2}$$

where $X_{(n)} = \max\{X_1, \ldots, X_n\}$, $Y_{(n)} = \max\{Y_1, \ldots, Y_n\}$ and $\mathcal{G}$ is a non-degenerate distribution. Then, clearly we must have $\mathcal{G} \in M(\mathcal{G}_\gamma, \mathcal{G}_{\gamma'})$, where $\mathcal{G}_\gamma$ and $\mathcal{G}_{\gamma'}$ are members of family (1) for some γ and γ'. Also, we have $F_X \in \mathcal{D}_{\max}(\mathcal{G}_\gamma)$ and $F_Y \in \mathcal{D}_{\max}(\mathcal{G}_{\gamma'})$. The general structure of $\mathcal{G}$ has been known since the works of Tiago de Oliveira [7] and Sibuya [6]. It makes use of the dependence function $d(\cdot)$, which in the case of Gumbel marginals, $\mathcal{G} \in \mathcal{M}(\mathcal{G}_0, \mathcal{G}_0)$, has the form

$$\mathcal{G}(x, y) = (\mathcal{G}_0(x)\mathcal{G}_0(y))^{d(x, y)} \tag{3}$$

with

$$\frac{1 \vee e^{-(x-y)}}{1 + e^{-(x-y)}} \leq d(x, y) \leq 1 \tag{4}$$

(see, for example, Galambos [2], $a \vee b = \max\{a, b\}$). In the general case a representation of $d(\cdot)$ in terms of the tail indexes γ and γ' can be obtained.

Next, we consider a more restrictive setting that will allow us to derive tests for the asymptotic dependence of the maxima $X_{(n)}$ and $Y_{(n)}$. Let $\xi_1, \xi_2, \ldots$ be i.i.d. random vectors with a common distribution H on $\Omega \subset R^d$ and let f and g be real-valued functions defined on Ω. For $X_j = f(\xi_j)$ and $Y_j = g(\xi_j)$ assume that the common distribution F of $(X_1, Y_1), (X_2, Y_2), \ldots$ satisfies (2), that is, $F \in \mathcal{D}_{\max}(\mathcal{G})$, with $F_X \in \mathcal{D}_{\max}(\mathcal{G}_{\gamma_f})$ and $F_Y \in \mathcal{D}_{\max}(\mathcal{G}_{\gamma_g})$, where the tail indexes $\gamma_f > 0$ and $\gamma_g > 0$. Under this model we may think of $\xi_1, \xi_2, \ldots$ as random factors that might affect the markets X and Y and the functions f and g may be regarded as the markets' response to these random factors. By allowing f and g be functions depending on distinct random factors or eventually the same random factors, this setting is not as restrictive as it seems. Our Theorem 1 gives an explicit representation of $\mathcal{G}$ and Corollary 1 shows that the statistics $\hat{\gamma}_n$ can be used to test the dependence between $X_{(n)} = \max\{f(\xi_1), \ldots, f(\xi_n)\}$ and $Y_{(n)} = \max\{g(\xi_1), \ldots, g(\xi_n)\}$. More specifically, for the order statistics $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$, $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)}$ and $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(n)}$ from $Z_j = X_jY_j$, consider the Hill's estimators

$$\hat{\gamma}_f = \frac{1}{k}\sum_{j=1}^{k}[\log X_{(n-j+1)} - \log X_{(n-k)}],$$

$$\hat{\gamma}_g = \frac{1}{k}\sum_{j=1}^{k}[\log Y_{(n-j+1)} - \log Y_{(n-k)}],$$

$$\hat{\gamma}_{f_g} = \frac{1}{k}\sum_{j=1}^{k}[\log Z_{(n-j+1)} - \log Z_{(n-k)}] \tag{5}$$

and define $\hat{\gamma}_n = \hat{\gamma}_{f_g}/\hat{\gamma}_f + \hat{\gamma}_g$. We will show that when $X_{(n)}$ and $Y_{(n)}$ are dependent we necessarily have $\hat{\gamma}_n \underset{n}{\rightarrow} 1$.

2. The Univariate and Uniform Case

Let $\xi, \xi_1, \xi_2, \ldots$ be independent and uniformly distributed over $\Omega = [0, 1]$. For $X = f(\xi)$ and $Y = g(\xi)$ assume that (X, Y) possesses a distribution $F \in \mathcal{D}_{\max}(\mathcal{G})$ with $\mathcal{G} \in \mathcal{M}(\mathcal{G}_{\gamma_f}, \mathcal{G}_{\gamma_g})$, where $\gamma_f > 0$ and $\gamma_g > 0$. Note that for

$$x_{\gamma_f} = (1 + \gamma_f x)^{-1/\gamma_f}, \qquad y_{\gamma_g} = (1 + \gamma_g y)^{-1/\gamma_g} \tag{6}$$

we can write (3) and (4) as

$$\mathcal{G}(x, y) = [\mathcal{G}_0(-\log x_{\gamma_f})\mathcal{G}_0(-\log y_{\gamma_g})]^{\hat{d}(\log y_{\gamma_g} - \log x_{\gamma_f})}$$

with

$$d(x_{\gamma_f}, y_{\gamma_g}) = \hat{d}(\log y_{\gamma_g} - \log x_{\gamma_f})$$

$$= \frac{1 \vee (x_{\gamma_f}/y_{\gamma_g})}{1 + (x_{\gamma_f}/y_{\gamma_g})} = \frac{x_{\gamma_f} \vee y_{\gamma_g}}{x_{\gamma_f} + y_{\gamma_g}}.$$

Since for $\gamma > 0$ we have $\mathcal{G}_\gamma(z) = \mathcal{G}_0(-\log(1 + \gamma z)^{-1/\gamma})$, it follows that

$$\mathcal{G}(x, y) = (\mathcal{G}_{\gamma_f}(x)\mathcal{G}_{\gamma_g}(y))^{d(x_{\gamma_f}, y_{\gamma_g})},$$

where

$$\frac{x_{\gamma_f} \vee y_{\gamma_g}}{x_{\gamma_f} + y_{\gamma_g}} \le d(x_{\gamma_f}, y_{\gamma_g}) \le 1. \tag{7}$$

Clearly (4) can be obtained from (7) by letting $\gamma_f \to 0$ and $\gamma_g \to 0$. This suggests that tests based on the tail indexes γ_f and γ_g could be obtained. Next, we derive a representation for $\mathcal{G}$ that displays the iteration of the maximum points of the coordinate functions f and g. To build such representation we need to introduce measures of maximality contact μ_f and μ_g (for related work see Dorea [1]). These measures will be defined on the set of points where f or g is unbounded,

$$\Omega_f = \{x : x \in \Omega, f(x^-) = \lim_{x' \uparrow x} f(x') = \infty \text{ or } f(x^+) = \lim_{x' \downarrow x} f(x') = \infty\} \tag{8}$$

and

$$\Omega_g = \{y : y \in \Omega, g(y^-) = \lim_{y' \uparrow y} g(y') = \infty \text{ or } g(y^+) = \lim_{y' \downarrow y} g(y') = \infty\}. \tag{9}$$

The notion of regularly varying function will also be needed: for $R^+ = (0, \infty)$, we say that a function $v : R^+ \to R^+$ is δ-varying if $\lim_{t \to \infty} \frac{v(tx)}{v(t)} = x^\delta$ for all $x > 0$.

Condition 1. Given $\gamma_f > 0$ and $\gamma_g > 0$, let v_f and v_g be γ_f-varying and γ_g-varying functions such that for all $x_* \in \Omega_f$ and $y_* \in \Omega_g$ the following limits exist and are positive and finite (possibly 0):

$$R_f(x_*, x) = \lim_{t \to \infty} \frac{f(x_* + t^{-1}x)}{v_f(t)}, \quad x \in R,\ x \ne 0 \tag{10}$$

and

$$R_g(y_*, y) = \lim_{t \to \infty} \frac{g(y_* + t^{-1}y)}{v_g(t)}, \quad y \in R,\ y \ne 0. \tag{11}$$

Write $R_f(x_*^{\pm}) = R_f(x_*, \pm 1)$, $R_g(y_*^{\pm}) = R_g(y_*, \pm 1)$ and assume that for some $x_* \in \Omega_f$ and some $y_* \in \Omega_g$ we have $R_f(x_*) = R_f(x_*^+) + R_f(x_*^-) > 0$ and $R_g(y_*) = R_g(y_*^+) + R_g(y_*^-) > 0$.

Theorem 1 below shows that, under Condition l, we can define the following probability measures (maximality contact) on Ω_f and Ω_g:

$$\mu_f(x_*^{\pm}) = \frac{(R_f(x_*^{\pm}))^{1/\gamma_f}}{R_f}, \quad R_f = \sum_{x \in \Omega_f} [(R_f(x^+))^{1/\gamma_f} + (R_f(x^-))^{1/\gamma_f}] \tag{12}$$

and

$$\mu_g(y_*^{\pm}) = \frac{(R_g(y_*^{\pm}))^{1/\gamma_g}}{R_g}, \quad R_g = \sum_{y \in \Omega_g} [(R_g(y^+))^{1/\gamma_g} + (R_g(y^+))^{1/\gamma_g}]. \tag{13}$$

Similarly, $\mu_f(x_*) = \mu_f(x_*^+) + \mu_f(x_*^-)$ and $\mu_g(y_*) = \mu_g(y_*^+) + \mu_g(y_*^-)$.

To derive the representation for $\mathcal{G}$ we partition Ω_f and Ω_g into dependent and independent parts

$$D = \Omega_f \cap \Omega_g, \; I_f = \Omega_f \backslash D \text{ and } I_g = \Omega_g \backslash D \tag{14}$$

and we get representation (15).

Theorem 1. *Assume that Condition* 1 *holds and that* $F \in \mathcal{D}_{\max}(\mathcal{G})$. *Then, if f and g are continuous in some neighborhood of* Ω_f *and* Ω_g *respectively, we have*

$$\Omega_f^+ = \{x : x \in \Omega_f, \, \mu_f(x) > 0\} \text{ and } \Omega_g^+ = \{y : y \in \Omega_g, \, \mu_g(y) > 0\}$$

both finite, μ_f *and* μ_g *probability measures,* $F_X \in \mathcal{D}_{\max}(\mathcal{G}_{\gamma_f})$ *and* $F_Y \in \mathcal{D}_{\max}(\mathcal{G}_{\gamma_g})$. *Moreover,*

$$G(x, y) = \exp\left\{- \mu_f(I_f)x_{\gamma_f} - \mu_g(I_g)y_{\gamma_g} - \sum_{z \in D} d_{fg}(z, x, y)\right\}, \tag{15}$$

where

$$d_{fg}(z, x, y) = \{[(\mu_f(z^+)x_{\gamma_f}) \vee (\mu_g(z^+)y_{\gamma_g})] + [(\mu_f(z^-)x_{\gamma_f}) \vee (\mu_g(z^-)y_{\gamma_g})]\}.$$

Proof. (i) To see that Ω_f^+ is countable, enough to show that for each $x_* \in \Omega_f^+$ there exists an open interval (a, b) such that $(a, b) \cap \Omega_f^+ = \{x_*\}$. Assume no such interval exists, $0 < x_* < 1$ and $\mu_f(x_*^+) > 0$. Then given $\varepsilon_n \downarrow 0$ there exists $x_n \in \Omega_f^+$, $x_n \neq x_*$ and $x_n \in (x_* - \varepsilon_n, x_* + \varepsilon_n)$. We may assume $x_n \to x_*$, otherwise there is a subsequence $x_{n'} \to x_*$. In particular, we may take $x_n \downarrow x_*$. Let $t_n = (x_n - x_*)^{-1}$. Then $f(x_* + t_n^{-1}) = f(x_n) = \infty$ and

$$R(x_*, 1) = \lim_{n\to\infty} \frac{f(x_* + t_n^{-1})}{v_f(t_n)} = \infty, \tag{16}$$

a contradiction. The endpoints 0 and 1 can be similarly treated. To see that Ω_f^+ is finite, suppose that there are infinitely many $\{x_n\} \subset \Omega_f^+$. Then since $\{x_n\} \subset [0, 1]$ there will be a convergent subsequence $x_{n'} \downarrow \hat{x} \in [0, 1]$ (or $x_{n'} \uparrow \hat{x}$). Since f is continuous in a neighborhood of Ω_f we must have $\hat{x} \in \Omega_f$. By (16) we have a contradiction. It follows that (10) is well-defined and that μ_f is a probability measure.

Clearly, Ω_g^+ is also a finite set and μ_g is a probability measure.

(ii) Let $\Omega_f^+ = \{x_1, \dots, x_N\}$ and let $I_1, \dots, I_N$ be disjoint intervals such that $I_1 \cup \cdots \cup I_N = [0, 1]$, $I_j \cap \Omega_f^+ = \{x_j\}$ and, except for the endpoints 0 and 1, x_j is an interior point of I_j. Define $b_n = v_f(n) R_f^{\gamma_f}$ and $a_n = \gamma_f b_n$. Then

$$P(f(\xi) > a_n x + b_n) = \sum_{j=1}^{N} m\{z : z \in I_j,\ f(z) > a_n x + b_n\}$$

$$= \frac{1}{n}\left[\sum_{j=1}^{N} m\left\{u : \left(x_j + \frac{u}{n}\right) \in I_j,\ f\left(x_j + \frac{u}{n}\right) > a_n x + b_n\right\}\right]$$

$$= \frac{1}{n}\left[\sum_{j=1}^{N} m(C_n(x_j, R_f^{\gamma_f}(1 + \gamma_f x)))\right], \tag{17}$$

where m stands for the Lebesgue measure and

$$C_n(x_j, x) = \left\{u : \left(x_j + \frac{u}{n}\right) \in I_j, \frac{f\left(x_j + \frac{u}{n}\right)}{v_f(n)} > x\right\}. \tag{18}$$

Using (10) and the fact that v_f is γ_f-varying we have for $u > 0$

$$\lim_{n\to\infty} \frac{f\left(x_j + \frac{u}{n}\right)}{v_f\left(\frac{n}{u}\right)} \frac{v_f\left(\frac{n}{u}\right)}{v_f(n)} = R_f(x_j^+)u^{-\gamma_f}.$$

It follows that for $u > 0$

$$R_f(x_j, u) = \lim_{n\to\infty} \frac{f\left(x_j + \frac{u}{n}\right)}{v_f(n)} = R_f(x_j^+)u^{-\gamma_f}.$$

Similarly, for $u < 0$ we have $R_f(x_j, u) = R_f(x_j^-)|u|^{-\gamma_f}$.

From (12) and (18) we have

$$\lim_{n\to\infty} m(C_n(x_j, x))$$

$$= m\{(u > 0, R_f(x_j^+)u^{-\gamma_f} > x) \cup (u < 0, R_f(x_j^-)|u|^{-\gamma_f} > x)\}$$

$$= [(R_f(x_j^+))^{1/\gamma_f} + (R_f(x_j^-))^{1/\gamma_f}]x^{-1/\gamma_f}$$

$$= [\mu_f(x_j^+) + \mu_f(x_j^-)]R_f x^{-1/\gamma_f} = \mu_f(x_j)R_f x^{-1/\gamma_f}.$$

And from (17)

$$\lim_{n\to\infty} nP(f(\xi) > a_n x + b_n) = \sum_{j=1}^{N} \mu_f(x_j)R_f[R_f^{\gamma_f}(1 + \gamma_f x)]^{-1/\gamma_f}$$

$$= \sum_{x_j \in \Omega_f} \mu_f(x_j)(1 + \gamma_f x)^{-1/\gamma_f} = x_{\gamma_f}.$$

Since

$$\lim_{n\to\infty} n \log F_X(a_n x + b_n) = -\lim_{n\to\infty} nP(f(\xi) > a_n x + b_n) = -x_{\gamma_f}$$

we have $F_X \in \mathcal{D}_{\max}(\mathcal{G}_{\gamma_f})$. Similarly, $F_Y \in \mathcal{D}_{\max}(\mathcal{G}_{\gamma_g})$.

(iii) Note that for n large

$$P(X_{(n)} \leq a_n x + b_n, Y_{(n)} \leq c_n y + d_n) \approx -n(1 - F(a_n x + b_n, c_n y + d_n))$$

and

$$1 - F(a_n x + b_n, c_n y + d_n) = P(f(\xi) > a_n x + b_n \text{ or } g(\xi) > c_n y + d_n).$$

Thus, to prove (15) enough to show that for some $a_n > 0$, b_n, $c_n > 0$ and d_n

$$nP(f(\xi) > a_n x + b_n \text{ or } g(\xi) > c_n y + d_n)$$

$$\to \mu_f(I_f)x_{\gamma_f} + \mu_g(I_g)y_{\gamma_g} + \sum_{z \in D} d_{fg}(z, x, y). \tag{19}$$

Since Ω_f^+ and Ω_g^+ are finite, let $I_1, \ldots, I_M$ be a partition such that $I_j \cap (\Omega_f^+ \cup \Omega_g^+) = \{z_j\}$ for $j = 1, \ldots, M$. Proceeding as in (ii) define $b_n = v_f(n) R_f^{\gamma_f}$, $a_n = \gamma_f b_n$, $d_n = v_g(n) R_g^{\gamma_g}$, $c_n = \gamma_g d_n$ and write

$$P(f(\xi) > a_n x + b_n \text{ or } g(\xi) > c_n y + d_n)$$

$$= \frac{1}{n} \sum_{j=1}^{M} [m(D_n(z_j^+, x, y) \cup D_n(z_j^-, x, y))], \tag{20}$$

where

$$D_n(z_j^+, x, y) = \left\{u > 0, \left(z_j + \frac{u}{n}\right) \in I_j, f(\xi) > a_n x + b_n \text{ or } g(\xi) > c_n y + d_n\right\}$$

$$= \left\{u > 0, \left(z_j + \frac{u}{n}\right) \in I_j, \frac{f\left(z_j + \frac{u}{n}\right)}{v_f(n)} > R_f^{\gamma_f} x_{\gamma_f} \text{ or } \frac{g\left(z_j + \frac{u}{n}\right)}{v_g(n)} > R_g^{\gamma_g} y_{\gamma_g}\right\}$$

and for $D_n(z_j^-, x, y)$ we replace $u > 0$ by $u < 0$. Then

$$D_n(z_j^+, x, y) \to \{u > 0, u < \mu_f(z_j^+)x_{\gamma_f} \text{ or } u < \mu_g(z_j^+)y_{\gamma_g}\}.$$

It follows that

$$\lim_{n\to\infty} m(D_n(z_j^+, x, y)) = (\mu_f(z_j^+)x_{\gamma_f}) \vee (\mu_g(z_j^+)y_{\gamma_g})$$

and similarly

$$\lim_{n\to\infty} m(D_n(z_j^-, x, y)) = (\mu_f(z_j^-)x_{\gamma_f}) \vee (\mu_g(z_j^-)y_{\gamma_g}).$$

From the definition of d_{fg} and (20)

$$nP(f(\xi) > a_n x + b_n \text{ or } g(\xi) > c_n y + d_n) \to \sum_{z\in\Omega_f^+\cup\Omega_g^+} d_{fg}(z, x, y).$$

Using notation (14), if $z \in I_f$, then $\mu_g(z^{\pm}) = 0$ and if $z \in I_g$ we have $\mu_f(z^{\pm}) = 0$. Hence

$$\sum_{z\in\Omega_f^+\cup\Omega_g^+} d_{f_g}(z, x, y) = \mu_f(I_f)x_{\gamma_f} + \mu_g(I_g)y_{\gamma_g} + \sum_{z\in D} d_{fg}(z, x, y)$$

and (19) follows.

Example 1. (a) Let $f(x) = \left|x - \frac{1}{2}\right|^{-1}$ and $g(x) = 2\left|x - \frac{1}{4}\right|^{-2}$ if $x \in \left[0, \frac{1}{2}\right]$ and $g(x) = 2\left|x - \frac{3}{4}\right|^{-2}$ if $x \in \left(\frac{1}{2}, 1\right]$. We have $I_f = \Omega_f = \left\{\frac{1}{2}\right\}$, $I_g = \Omega_g = \left\{\frac{1}{4}, \frac{3}{4}\right\}$ and $D = \varnothing$. Let $v_f(t) = t$ and $v_g(t) = t^2$. Then Condition 1 is satisfied with $\mu_f\left(\frac{1}{2}\right) = 1$ and $\mu_g\left(\frac{1}{4}\right) = \mu_g\left(\frac{3}{4}\right) = \frac{1}{2}$. And we have $\mathcal{G} \in \mathcal{M}(\mathcal{G}_1, \mathcal{G}_2)$ with

$$\mathcal{G}(x, y) = \exp\{-(1 + x)^{-1} - (1 + 2y)^{-1/2}\} = \mathcal{G}_1(x)\mathcal{G}_2(y).$$

Note that for $a_n = b_n = 2n$, $c_n = 16n^2$ and $d_n = 8n^2$ we have

$$P\left(\frac{X_{(n)} - 2n}{2n} \leq x, \frac{Y_{(n)} - 8n^2}{16n^2} \leq y\right) \underset{n\to\infty}{\to} \mathcal{G}_1(x)\mathcal{G}_2(y).$$

(b) Let $f(x) = \left(\frac{1}{3} - x\right)^{-1}$ if $x \in \left[0, \frac{1}{3}\right)$, $f(x) = \left|x - \frac{1}{2}\right|^{-1}$ if $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$ and $f(x) = \left|x - \frac{5}{6}\right|^{-1}$ if $x \in \left(\frac{2}{3}, 1\right]$. Let $g(x) = x^{-2}$ if $x \in \left[0, \frac{1}{4}\right]$ and $g(x) = \left|x - \frac{1}{3}\right|^{-2}$ if $x \in \left(\frac{1}{4}, 1\right]$. Then $I_f = \left\{\frac{1}{2}, \frac{5}{6}\right\}$, $I_g = \{0\}$ and $D = \left\{\frac{1}{3}\right\}$. By taking v_f and v_g as in (a) we have $\mu_f(x_*) = \frac{1}{5}$ for $x_* = \left(\frac{1}{3}\right)^-, \left(\frac{1}{2}\right)^\pm, \left(\frac{5}{6}\right)^\pm$, $\mu_f\left(\left(\frac{1}{3}\right)^+\right) = 0$ and $\mu_g(0^+) = \mu_g\left(\left(\frac{1}{3}\right)^\pm\right) = \frac{1}{3}$. A direct verification shows that we do not have asymptotic independency and

$$\mathcal{G}(x, y) = \exp\left\{-\frac{4}{5}(1+x)^{-1} - \frac{2}{3}(1+2y)^{-1/2} - \left[\frac{1}{5}(1+x)^{-1}\right] \vee \left[\frac{1}{3}(1+2y)^{-1/2}\right]\right\}.$$

To derive the dependence test, let $Z_j = f(\xi_j)g(\xi_j)$ and let F_Z denote the common distribution of the i.i.d. sequence $\{Z_n\}_{n\geq 1}$.

Corollary 1. *Under the hypotheses of Theorem* 1 *if* $X_{(n)}$ *and* $Y_{(n)}$ *are asymptotically dependent, then*

$$\hat{\gamma}_n = \sum_{j=1}^{k_n} \log\left(\frac{Z_{(n-j+1)}}{Z_{(n-k_n)}}\right) \Bigg/ \sum_{j=1}^{k_n} \log\left(\frac{X_{(n-j+1)}}{X_{(n-k_n)}} \frac{Y_{(n-j+1)}}{Y_{(n-k_n)}}\right) \underset{n}{\overset{p}{\to}} 1, \tag{21}$$

where $\frac{k_n}{n} \to 0$ *and* $\overset{p}{\to}$ *stands for convergence in probability. Moreover, if* $F_Z \in \mathcal{D}(\mathcal{G}_\gamma)$ *for some* $\gamma > 0$ *and* (21) *holds, then* $X_{(n)}$ *and* $Y_{(n)}$ *are asymptotically dependent.*

Proof. (a) First, note that if $x_0 \notin \Omega_f$, then $f(x_0) < \infty$ and $\lim_{t\to\infty} f(x_0 + t^{-1}x) < \infty$. Since $\lim_{t\to\infty} v_f(t) = \infty$, by (10) we have $R_f(x_0, x) = 0$. It follows that for $v_{f_g} = v_f v_g$ we have

$$R_{fg}(x_0, x) = \frac{f(x_0 + t^{-1}x)}{v_f(t)} \frac{g(x_0 + t^{-1}x)}{v_g(t)} = 0.$$

Similarly, if $x_0 \notin \Omega_g$, then we have $R_g(x_0, x) = 0$. And, in either case, we have $R_{fg}(x_0, x) = 0$. It follows that for

$$\Omega_{fg} = \{x : x \in \Omega, \lim_{x' \to x} f(x')g(x') = \infty\}$$

we have

$$D = \Omega_f \bigcap \Omega_g \subset \Omega_{fg} \subset \Omega_f \bigcup \Omega_g. \tag{22}$$

(b) If $X_{(n)}$ and $Y_{(n)}$ are asymptotically dependent, then by (12), (13), (15), and (22) we have $\mu_f(D) > 0$, $\mu_g(D) > 0$, $R_{fg}(z_0, z) \geq 0$ for $z_0 \in \Omega_{fg}$ and there exist $z_* \in \Omega_{fg}$ with $R_{fg}(z_*) > 0$. Since v_{fg} is $(\gamma_f + \gamma_g)$-varying and Condition 1 is satisfied, from Theorem 1 we must have $F_Z \in \mathcal{D}_{\max}(G_{\gamma_f+\gamma_g})$. On the other hand, $\hat{\gamma}_n = \frac{\hat{\gamma}_{fg}}{\hat{\gamma}_g + \hat{\gamma}_g}$, where $\hat{\gamma}_f = \frac{1}{k_n}\sum_{j=1}^{k_n} \log\left(\frac{X_{(n-j+1)}}{X_{(n-k_n)}}\right)$, $\hat{\gamma}_g$ and $\hat{\gamma}_{fg}$ are just Hill's [3] estimators of γ_f, γ_g and $\gamma_f + \gamma_g$ respectively. And by Mason [5] if $F_Z \in \mathcal{D}_{\max}(G_{\gamma_f+\gamma_g})$ and $\gamma_f + \gamma_g > 0$, then $\hat{\gamma}_{fg} \xrightarrow{p} \gamma_f + \gamma_g$. Clearly, we also have $\hat{\gamma}_f \xrightarrow{p} \gamma_f$ and $\hat{\gamma}_g \xrightarrow{p} \gamma_g$. And (21) follows.

(c) Assume that (21) holds and that $F_Z \in \mathcal{D}_{\max}(\mathcal{G}_\gamma)$ then we must have $\gamma = \gamma_f + \gamma_g$. Using the same type of arguments as in (a) we necessarily have $D \neq \varnothing$ and by (15) we have X_n and $Y_{(n)}$ asymptotically dependent.

Example 2. Consider the setting of Example 1.

(a) We have $\Omega_{fg} = \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}$ and for $v_{fg}(t) = t^2$ we have $R_{fg}\left(\frac{1}{4}\right) > 0$, $R_{fg}\left(\frac{3}{4}\right) > 0$ and $R_{fg}\left(\frac{1}{2}\right) = 0$. From Theorem 1 we have $F_Z \in \mathcal{D}_{\max}(\mathcal{G}_2)$. Thus $\hat{\gamma}_{fg} \to 2 \neq \gamma_f + \gamma_g = 3$. Indeed, $X_{(n)}$ and $Y_{(n)}$ are asymptotically independent.

(b) We have $\Omega_{fg} = \left\{0, \frac{1}{3}, \frac{1}{2}, \frac{5}{6}\right\}$ and for $v_{fg}(t) = t^3$ we have $R_{fg}\left(\frac{1}{3}\right) > 0$ and $R_{fg}(z) = 0$ for $z \neq \frac{1}{3}$. In this case, $\hat{\gamma}_{fg} \to 3 = \gamma_f + \gamma_g$ and asymptotic dependence follows.

Remark 1. Consider the particular situation where μ_f and μ_g are the uniform measure on Ω_f^+ and Ω_g^+, respectively, and $\gamma_f = \gamma_g = \gamma$. Then (15) can be expressed as

$$\mathcal{G}(x, y) = \exp\{-(1-\alpha)x_\gamma - (1-\alpha)y_\gamma - \alpha(x_\gamma \vee y_\gamma)\},$$

where $x_\gamma = (1+\gamma x)^{-1/\gamma}$, $y_\gamma = (1+\gamma y)^{-1/\gamma}$ and $0 \leq \alpha \leq 1$. Equivalently,

$$\mathcal{G}(x, y) = (\mathcal{G}_\gamma(x)\mathcal{G}_\gamma(y))^{d(x_\gamma,\, y_\gamma)},$$

where

$$d(x_\gamma, y_\gamma) = \frac{(1-\alpha)(x_\gamma + y_\gamma) + \alpha(x_\gamma \vee y_\gamma)}{x_\gamma + y_\gamma}$$

$$= \alpha\frac{x_\gamma \vee y_\gamma}{x_\gamma + y_\gamma} + (1-\alpha). \tag{23}$$

By noting that $\lim_{\gamma \to 0} \frac{x_\gamma \vee y_\gamma}{x_\gamma + y_\gamma} = \frac{\max\{1, e^{x-y}\}}{1+e^{x-y}}$, (23) is just the dependence function considered by Longin [4].

3. The General Case

Though Theorem 1 and Corollary 1 were restricted to the special case when H is the uniform distribution on $\Omega = [0, 1]$ it is not difficult to extend these results to a more general framework. Assume that $H = H_1 \cdot H_2 \cdots H_d$, where H_j's are continuous marginal distributions. For $\mathbf{u} = (u_1, \ldots, u_d) \in (0, 1)^d$ define

$$H^{-1}(\mathbf{u}) = (H_1^{-1}(u_1), \ldots, H_d^{-1}(u_d)), \quad H_j^{-1}(u_j) = \inf\{z : H_j(z) \geq u_j\}.$$

Then if $\boldsymbol{U}$ is uniformly distributed over $[0, 1]^d$ we have $H^{-1}(\boldsymbol{U})$ with distribution H on Ω. In this case, to analyse the behavior of the sequences $\{f(\xi_n)\}_{n\geq 1}$ and $\{g(\xi_n)\}_{n\geq 1}$ where the ξ_n has distribution H enough to analyse the sequences $\{\hat{f}(\boldsymbol{U}_n)\}_{n\geq 1}$ and $\{\hat{g}(\boldsymbol{U}_n)\}_{n\geq 1}$, where $\hat{f} = f(H^{-1})$, $\hat{g} = g(H^{-1})$ and $\boldsymbol{U}_n$ is uniformly distributed over $[0, 1]^d$. Thus, it suffices to extend the results of Section 2 to the multivariate uniform distribution. For $\boldsymbol{x} \in R^d$ and $\boldsymbol{y} \in R^d$ denote by $\boldsymbol{x} + \boldsymbol{y}$ the coordinatewise sum and $\boldsymbol{xy}$ the coordinatewise product (similarly, $\boldsymbol{x} > \boldsymbol{y}$ or $\boldsymbol{x} \geq \boldsymbol{y}$).

Condition 2. Let Ω_f and Ω_g denote the sets, where f and g are unbounded and assume that both Ω_f and Ω_g are finite sets. For some $\gamma_f > 0$ and $\gamma_g > 0$, let v_f and v_g be $d\gamma_f$-varying and $d\gamma_g$-varying functions such that for all $\boldsymbol{x}_* \in \Omega_f$ and all $\boldsymbol{y}_* \in \Omega_g$ the following limits exist and are positive and finite (possibly 0):

$$R_f(\boldsymbol{x}_*, \boldsymbol{x}) = \lim_{t\to\infty} \frac{f(\boldsymbol{x}_* + t^{-1}\boldsymbol{x})}{v_f(t)} \quad \text{and} \quad R_g(\boldsymbol{y}_*, \boldsymbol{y}) = \lim_{t\to\infty} \frac{g(\boldsymbol{y}_* + t^{-1}\boldsymbol{y})}{v_g(t)} \tag{24}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in R^d$, $\boldsymbol{x}, \boldsymbol{y} \neq \mathbf{0}$. For $\boldsymbol{i} = (i_1, \ldots, i_d) \in L^d = \{-1, 1\}^d$, if $R_f(\boldsymbol{x}_*, \boldsymbol{i}) > 0$ assume that

$$\lim_{t\to\infty} \frac{f(\boldsymbol{x}_* + t^{-1}\boldsymbol{x}\boldsymbol{i})}{f(\boldsymbol{x}_* + t^{-1}\boldsymbol{i})} = \lambda_f(\boldsymbol{x}) > 0, \quad \boldsymbol{x} > \mathbf{0} \tag{25}$$

and if $R_g(\boldsymbol{y}_*, \boldsymbol{i}) > 0$

$$\lim_{t\to\infty} \frac{g(\boldsymbol{y}_* + t^{-1}\boldsymbol{y}\boldsymbol{i})}{g(\boldsymbol{y}_* + t^{-1}\boldsymbol{i})} = \lambda_g(\boldsymbol{y}) > 0, \quad \boldsymbol{y} > \boldsymbol{0}. \tag{26}$$

Moreover, for

$$R_f(\boldsymbol{x}_*) = \sum_{\boldsymbol{i}\in L^d} R_f^{1/\gamma_f}(\boldsymbol{x}_*, \boldsymbol{i}) \text{ and } R_g(\boldsymbol{y}_*) = \sum_{\boldsymbol{i}\in L^d} R_g^{1/\gamma_g}(\boldsymbol{y}_*, \boldsymbol{i}) \tag{27}$$

assume that

$$\Omega_f^+ = \{\boldsymbol{x}_* : R_f(\boldsymbol{x}_*) > 0\} \neq \varnothing \text{ and } \Omega_g^+ = \{\boldsymbol{y}_* : R_g(\boldsymbol{y}_*) > 0\} \neq \varnothing. \tag{28}$$

Remark 2. Note that under the hypotheses of Theorem 1 we have Condition 2 satisfied. The first part of the proof shows that Ω_f^+ and Ω_g^+ are finite. Also, if $R_f(x_*) > 0$, then either $R_f(x_*^+) > 0$ or $R_f(x_*^-) > 0$. If $R_f(x_*^+) > 0$, then for $x > 0$ write

$$\frac{f(x_* + t^{-1}x)}{f(x_* + t^{-1})} = \frac{f(x_* + t^{-1}x)}{v_f(tx^{-1})} \frac{v_f(t)}{f(x_* + t^{-1})} \frac{v_f(tx^{-1})}{v_f(t)}.$$

Using (10) and the fact that v_f is γ_f-varying we have (25) satisfied with $\lambda_f(x) = x^{-\gamma_f}$. Similarly, (26) is satisfied with $\lambda_g(y) = y^{-\gamma_g}$.

As in the univariate case we can define the maximality contact measures on Ω_f and Ω_g,

$$\mu_f(\boldsymbol{x}_*) = \sum_{\boldsymbol{i}\in L^d} \mu_f(\boldsymbol{x}_*, \boldsymbol{i}), \quad \mu_g(\boldsymbol{y}_*) = \sum_{\boldsymbol{i}\in L^d} \mu_g(\boldsymbol{y}_*, \boldsymbol{i})$$

$$\mu_f(\boldsymbol{x}_*, \boldsymbol{i}) = \frac{R_f^{1/\gamma_f}(\boldsymbol{x}_*, \boldsymbol{i})}{R_f}, \quad \mu_g(\boldsymbol{y}_*, \boldsymbol{i}) = \frac{R_g^{1/\gamma_g}(\boldsymbol{y}_*, \boldsymbol{i})}{R_g}$$

$$R_f = \sum_{\boldsymbol{x}_*\in\Omega_f^+} \sum_{\boldsymbol{i}\in L^d} R_f^{1/\gamma_f}(\boldsymbol{x}_*, \boldsymbol{i}) \text{ and } R_g = \sum_{\boldsymbol{y}_*\in\Omega_g^+} \sum_{\boldsymbol{i}\in L^d} R_g^{1/\gamma_g}(\boldsymbol{y}_*, \boldsymbol{i}). \tag{29}$$

For $l > 0$ define

$$B_f(l) = \{\boldsymbol{u} : \boldsymbol{u} > \boldsymbol{0}, \lambda_f(\boldsymbol{u}) > l\} \text{ and } B_g(l) = \{\boldsymbol{u} : \boldsymbol{u} > \boldsymbol{0}, \lambda_g(\boldsymbol{u}) > l\}. \quad (30)$$

And let

$$c_f = m_d^{\gamma_f}(B_f(1)) \text{ and } c_g = m_d^{\gamma_g}(B_g(1)), \quad (31)$$

where m_d is the Lebesgue measure on R^d.

Theorem 2. *Let* $\xi, \xi_1, \xi_2, \ldots$ *be independent and uniformly distributed over* $[0, 1]^d$. *Let* F *be the distribution of* $(f(\xi), g(\xi))$ *and assume that* $F \in \mathcal{D}_{\max}(\mathcal{G})$. *If Condition* 2 *holds and* f *and* g *are continuous in some neighborhood of* Ω_f *and* Ω_g *respectively, then we have* (15) *with*

$$d_{fg}(\boldsymbol{z}, x_{\gamma_f}, y_{\gamma_g}) = \sum_{\boldsymbol{i} \in L^d} [(\mu_f(\boldsymbol{z}, \boldsymbol{i}) x_{\gamma_f}) \vee (\mu_g(\boldsymbol{z}, \boldsymbol{i}) y_{\gamma_g})] \widetilde{d}(\boldsymbol{z}, \boldsymbol{i}), \quad (32)$$

where

$$\widetilde{d}(\boldsymbol{z}, \boldsymbol{i}) = m_d \left\{ \left(\left[\frac{\mu_f(\boldsymbol{z}, \boldsymbol{i}) x_{\gamma_f}}{\mu_f(\boldsymbol{z}, \boldsymbol{i}) x_{\gamma_f} \vee \mu_g(\boldsymbol{z}, \boldsymbol{i}) y_{\gamma_g}} \right]^{1/d} B_f(c_f) \right) \right.$$
$$\left. \cup \left(\left[\frac{\mu_g(\boldsymbol{z}, \boldsymbol{i}) y_{\gamma_g}}{\mu_f(\boldsymbol{z}, \boldsymbol{i}) x_{\gamma_f} \vee \mu_g(\boldsymbol{z}, \boldsymbol{i}) y_{\gamma_g}} \right]^{1/d} B_g(c_g) \right) \right\}. \quad (33)$$

Proof. The proof is similar to that of Theorem 1 and we will sketch the differences.

(i) First, we show that c_f and c_g defined by (31) are finite and strictly positive. Notice that if $R_f(\boldsymbol{x}_*, \boldsymbol{i}) > 0$, then for $\boldsymbol{u} > \boldsymbol{0}$ and $s > 0$ write

$$\frac{f(\boldsymbol{x}_* + t^{-1} s \boldsymbol{u} \boldsymbol{i})}{f(\boldsymbol{x}_* + t^{-1} \boldsymbol{i})} = \frac{f(\boldsymbol{x}_* + t^{-1} s \boldsymbol{u} \boldsymbol{i})}{f(\boldsymbol{x}_* + t^{-1} s \boldsymbol{i})} \frac{f(\boldsymbol{x}_* + t^{-1} s \boldsymbol{i})}{v_f(t s^{-1})} \frac{v_f(t)}{f(\boldsymbol{x}_* + t^{-1} \boldsymbol{i})} \frac{v_f(t s^{-1})}{v_f(t)}.$$

Using (24) and (25) we get $\lambda_f(s\boldsymbol{u}) = s^{-\gamma_f d}\lambda_f(\boldsymbol{u})$. It follows that $\lim_{s\to\infty}\lambda_f(s\boldsymbol{u}) = 0$, so that $m_d(B_f(l)) < \infty$ for $l > 0$. On the other hand,

$$\begin{aligned} sB_f(l) &= \{s\boldsymbol{u} : s\boldsymbol{u} > \boldsymbol{0},\ s^{-\gamma_f d}\lambda_f(\boldsymbol{u}) > s^{-\gamma_f d} l\} \\ &= \{s\boldsymbol{u} : s\boldsymbol{u} > \boldsymbol{0},\ \lambda_f(s\boldsymbol{u}) > s^{-\gamma_f d} l\} \\ &= B_f(s^{-\gamma_f d} l). \end{aligned} \tag{34}$$

Since $\lambda_f(\boldsymbol{u}) > 0$ we have $m_d(B_f(1)) > 0$. Similarly,

$$\lambda_g(s\boldsymbol{u}) = s^{-\gamma_g d}\lambda_g(\boldsymbol{u}),\quad sB_g(l) = B_g(s^{-\gamma_g d} l) \text{ and } 0 < c_g < \infty.$$

(ii) By taking $a_n = \gamma_f b_n$ and $b_n = c_f R_f^{\gamma_f} v_f(n^{1/d})$ we have

$$a_n x + b_n = c_f R_f^{\gamma_f} v_f(n^{1/d})(1 + \gamma_f x) = c_f R_f^{\gamma_f} x_{\gamma_f}^{-\gamma_f} v_f(n^{1/d}).$$

Let $\boldsymbol{x}_* \in \Omega_f^+$ and let I be an open cube such that $I \cap \Omega_f^+ = \{\boldsymbol{x}_*\}$. Then using (30) , (31) and (34) we get

$$nm_d\left\{b u : \boldsymbol{u} > \boldsymbol{0},\ (\boldsymbol{x}_* + n^{-1/d}\boldsymbol{u}\boldsymbol{i}) \in I,\ \frac{f(\underline{x}_* + n^{-1/d}\boldsymbol{u})}{v_f(n^{1/d})} > c_f R_f^{\gamma_f} x_{\gamma_f}^{-\gamma_f}\right\}$$

$$= \sum_{\boldsymbol{i}\in L^d} m_d\left\{\boldsymbol{u} : \boldsymbol{u} > \boldsymbol{0},\ (\boldsymbol{x}_* + n^{-1/d}\boldsymbol{u}) \in I, \frac{f(\boldsymbol{x}_* + n^{-1/d}\boldsymbol{u})}{v_f(n^{1/d})} > c_f R_f^{\gamma_f} x_{\gamma_f}^{-\gamma_f}\right\}$$

$$\to \sum_{\boldsymbol{i}\in L^d} m_d\{B_f(c_f R_f^{\gamma_f} R_f^{-1}(\boldsymbol{x}_*, \boldsymbol{i}) x_{\gamma_f}^{-\gamma_f})\}$$

$$= \sum_{\boldsymbol{i}\in L^d} m_d\{[c_f R_f^{\gamma_f} R_f^{-1}(\boldsymbol{x}_*, \boldsymbol{i}) x_{\gamma_f}^{-\gamma_f}]^{-1/\gamma_f d} B_f(1)\}$$

$$= \sum_{\boldsymbol{i}\in L^d} c_f^{1/\gamma_f} \frac{R_f^{1/\gamma_f}(\boldsymbol{x}_*, \boldsymbol{i})}{R_f} x_{\gamma_f} m_d(B_f(1)) = \sum_{\boldsymbol{i}\in L^d} \mu_f(\boldsymbol{x}_*, \boldsymbol{i}) x_{\gamma_f}.$$

Now, proceeding as in the proof of Theorem 1 we get

$$nP(f(\xi) > a_n x + b_n) \to \sum_{\boldsymbol{x}_* \in \Omega_f^+} \sum_{\boldsymbol{i} \in L^d} \mu_f(\boldsymbol{x}_*, \boldsymbol{i}) x_{\gamma_f} = x_{\gamma_f}.$$

(iii) Take $c_n = \gamma_g d_n$ and $d_n = c_g R_g^{\gamma_g} v_g(n^{1/d})$. Let $\boldsymbol{z}_* \in D = \Omega_f^+ \cap \Omega_g^+$. Then by (30), (31), (33) and (34)

$$m_d\left\{\boldsymbol{u} : \boldsymbol{u} > \boldsymbol{0}, \left(\boldsymbol{x}_* + \frac{\boldsymbol{ui}}{n^{1/d}}\right) \in I, f\left(\boldsymbol{z}_* + \frac{\boldsymbol{u}}{n^{1/d}}\right) > a_n x + b_n\right.$$

$$\left.\text{or } g\left(\boldsymbol{z}_* + \frac{\boldsymbol{u}}{n^{1/d}}\right) > c_n y + d_n\right\}$$

$$\to m_d\{B_f(c_f R_f^{\gamma_f} R_f^{-1}(\boldsymbol{z}_*, \boldsymbol{i}) x_{\gamma_f}^{-\gamma_f}) \cup B_g(c_g R_g^{\gamma_g} R_g^{-1}(\boldsymbol{z}_*, \boldsymbol{i}) y_{\gamma_g}^{-\gamma_g})\}$$

$$= m_d\{[(\mu_f(\boldsymbol{z}_*, \boldsymbol{i}) x_{\gamma_f})^{1/d} B_f(c_f)] \cup [(\mu_g(\boldsymbol{z}_*, \boldsymbol{i}) y_{\gamma_g})^{1/d} B_g(c_g)]\}$$

$$= [\mu_f(\boldsymbol{z}_*, \boldsymbol{i}) x_{\gamma_f} \vee \mu_g(\boldsymbol{z}_*, \boldsymbol{i}) y_{\gamma_g}] \tilde{d}(\boldsymbol{z}_*, \boldsymbol{i}).$$

And the proof can be completed as in Theorem 1.

Example 3. (a) Let $f(u_1, u_2) = \left(\left|u_1 - \frac{1}{3}\right| + \left|u_2 - \frac{1}{3}\right|\right)^{-1}$ for $(u_1, u_2) \in \left[0, \frac{3}{8}\right]^2$ and $f(u_1, u_2) = \left(\left|u_1 - \frac{1}{2}\right| + \left|u_2 - \frac{1}{2}\right|\right)^{-1}$ otherwise. Let $g(u_1, u_2) = (u_1^2 + u_2^2)^{-1}$ for $(u_1, u_2) \in \left[0, \frac{1}{4}\right]^2$ and $\left(\left(u_1 - \frac{1}{2}\right)^2 + \left(u_2 - \frac{1}{2}\right)^2\right)^{-1}$ otherwise. Then $I_f = \left(\frac{1}{3}, \frac{1}{3}\right)$, $I_g = (0, 0)$, $D = \left(\frac{1}{2}, \frac{1}{2}\right)$, $v_f(t) = t$, $v_g(t) = t^2$, $\gamma_f = 1/2$, $\gamma_g = 1$, $\lambda_f(u_1, u_2) = 2/(u_1 + u_2)$, $\lambda_g(u_1, u_2) = 2/(u_1^2 + u_2^2)$, $c_f = \sqrt{2}$, $c_g = \pi/2$, $\mu_f(\cdot, \cdot) = 1/8$ and $\mu_g((0, 0), (1, 1)) = \mu_g\left(\left(\frac{1}{2}, \frac{1}{2}\right), .\right) = 1/5$. Take $a_n = \sqrt{n}$, $b_n = 2\sqrt{n}$ and $c_n = d_n = 5\pi n/4$. A direct computation shows that for (U_1, U_2) uniformly distributed on $[0, 1]^2$, $x_{\gamma_f} = \left(1 + \frac{1}{2}x\right)^{-2}$, $y_{\gamma_g} = (1 + y)^{-1}$ and for n large

$$nP(f(U_1, U_2) > a_n x + b_n \text{ or } g(U_1, U_2) > c_n y + d_n)$$

$$= \frac{x_{\gamma_f}}{2} + \frac{y_{\gamma_g}}{5} + 4m_2\left\{u_1 > 0, u_2 > 0 : u_1 + u_2 < \frac{x_{\gamma_f}^{1/2}}{2} \text{ or } u_1^2 + u_2^2 < \frac{4y_{\gamma_g}}{5\pi}\right\}.$$

Then the representation (32) and (33) follow.

Remark 3. (a) By Remark 2, if $d = 1$, we have $\lambda_f(u) = u^{-\gamma_f}$ and $B_f(l) = (u < l^{-1/\gamma_f})$. It follows that $B_f(c_f) = B_g(c_g) = (u < 1)$ and $c_f = c_g = 1$. Also, for all $z \in \Omega_f^+ \cap \Omega_g^+$ and $i \in \{-1, +1\}$ we have $\widetilde{d}(z, i) = 1$. Thus Theorem 1 follows as a particular case of Theorem 2.

(b) If in Theorem 2 we had $\xi_1, \xi_2, \ldots$ i.i.d. with a common distribution H on $\Omega \subset R^d$, then enough to assume that $\hat{f} = f(H^{-1})$ and $\hat{g} = g(H^{-1})$ satisfy Condition 2. Clearly, under the hypotheses of Theorem 2 we also have Corollary 1 and the dependence test (21).

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