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MONOTONICITY AND INEQUALITIES FOR THE GAMMA FUNCTION

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Abstract

The authors discuss the monotonicity of the ratio of gamma functions and obtain double inequalities which extend Guo and Qi's result. A complete monotonicity result of a function involving the gamma function is proved in this note.

1. Introduction

For real and positive values of x the classical gamma function is usually defined as

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt. \tag{1.1}$$

The psi or digamma function, the logarithmic derivative of the gamma function, can be expressed as [10],

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$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt, \quad x > 0, \tag{1.2}$$

where $\gamma = 0.57721566490 \cdots$ is the Euler-Mascheroni constant.

The gamma function has been investigated intensively by many authors in recent past years. Over the past half century many authors have obtained numerous interesting inequalities for these important functions (see [1]-[9]).

In [5], Guo and Qi obtained the following result: The function

$$\frac{[\Gamma(x + y + 1)/\Gamma(y + 1)]^{1/x}}{x + y + 1} \tag{1.3}$$

is decreasing in $x \geq 1$ for fixed $y \geq 0$. In this note, we shall extend this result to $x > 0$ and obtain another similar result from which double inequalities for the ratio of the gamma functions follow. At the same time, we are about to prove a complete monotonicity result of a function involving the gamma function.

2. Main Results

The following double inequalities of the second logarithmic derivative of the gamma function belong to M. Merkle:

Lemma 2.1 [6]. *For* $x \in (0, \infty)$,

$$\frac{1}{x+1} + \frac{1}{x^2} + \frac{1}{2(x+1)^2} < \Psi'(x) < \frac{1}{x+1} + \frac{1}{x^2} + \frac{1}{(x+1)^2}. \tag{2.1}$$

Theorem 2.2. *The function*

$$f(x) = \frac{[\Gamma(x + y + 1)/\Gamma(y + 1)]^{1/x}}{x + y + 1}$$

is strictly decreasing in $x > 0$ *for fixed* $y \geq 0$, *and the function*

$$g(x) = \frac{[\Gamma(x + y + 1)/\Gamma(y + 1)]^{1/x}}{\sqrt{x + y + 1}}$$

is strictly increasing in $x > 0$ *for fixed* $y \geq 0$. *In particular, for all*

$x \in (0, \infty)$ *and* $y \in [0, \infty)$, *we have*

$$\frac{x+y+1}{x+y+2} < \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}} < \sqrt{\frac{x+y+1}{x+y+2}}. \tag{2.2}$$

Proof. Taking the logarithm yields

$$\log f(x) = \frac{1}{x}[\log\Gamma(x+y+1) - \log\Gamma(y+1)] - \log(x+y+1). \tag{2.3}$$

Differentiating with respect to x on both sides of (2.3) gives

$$\begin{aligned} x^2\frac{f'(x)}{f(x)} &= -\log\Gamma(x+y+1) + \log\Gamma(y+1) + x\Psi(x+y+1) - \frac{x^2}{x+y+1} \\ &\equiv f_1(x). \end{aligned}$$

By direct computation and using inequalities (2.1), we have

$$\begin{aligned} \frac{1}{x}f_1'(x) &= \Psi'(x+y+1) - \frac{x+2y+2}{(x+y+1)^2} \\ &\le \frac{1}{x+y+2} + \frac{1}{(x+y+1)^2} + \frac{1}{(x+y+2)^2} - \frac{x+2y+2}{(x+y+1)^2} \\ &= \frac{1}{t+1} + \frac{1}{t^2} + \frac{1}{(t+1)^2} - \frac{t+y+1}{t^2} \quad (\text{Let } t = x+y+1) \\ &= \frac{t^2(t+1) + t^2 - (t+y)(t+1)^2}{t^2(t+1)^2} \\ &< \frac{t^2(t+1) + t^2 - t(t+1)^2}{t^2(t+1)^2} \\ &= -\frac{1}{t(t+1)^2} \\ &< 0. \end{aligned}$$

Thus, the function $f_1(x)$ is strictly decreasing, and then $f_1(x) < f_1(0) = 0$. Therefore $f'(x) < 0$ and $f(x)$ is strictly decreasing on $(0, \infty)$.

Straightforward calculation for $x > 0$ produces

$$\log g(x) = \frac{1}{x}[\log \Gamma(x+y+1) - \log \Gamma(y+1)]$$

$$- \frac{1}{2}\log(x+y+1). \tag{2.4}$$

By differentiation with respect to x on both sides of (2.4) and using inequalities (2.1), we have that

$$x^2 \frac{g'(x)}{g(x)} = -\log \Gamma(x+y+1) + \log \Gamma(y+1)$$

$$+ x\Psi(x+y+1) - \frac{1}{2}\frac{x^2}{x+y+1}$$

$$\equiv g_1(x)$$

and

$$\frac{1}{x} g_1'(x) = \Psi'(x+y+1) - \frac{1}{2}\frac{x+2y+2}{(x+y+1)^2}$$

$$\geq \frac{1}{x+y+2} + \frac{1}{(x+y+1)^2} + \frac{1}{2(x+y+2)^2} - \frac{x+2y+2}{2(x+y+1)^2}$$

$$= \frac{1}{t+1} + \frac{1}{t^2} + \frac{1}{2(t+1)^2} - \frac{t+y+1}{2t^2} \quad (\text{Let } t = x+y+1)$$

$$= \frac{t^2(t+1) + (t+1)^2 - y(t+1)^2 - t}{2t^2(t+1)^2}$$

$$> \frac{t^2(t+1) + (t+1)^2 - t(t+1)^2 - t}{2t^2(t+1)^2}$$

$$= \frac{1}{2t^2(t+1)^2}$$

$$> 0.$$

Therefore, the function $g_1(x)$ is strictly increasing, and $g_1(x) > g_1(0) = 0$. Thus $g'(x) > 0$ and then $g(x)$ is strictly increasing on $(0, \infty)$.

The double inequalities (2.2) follow from the monotonicity of the functions f and g. This completes the proof.

Next, we shall discuss the complete monotonicity of a function involving the gamma function. The following lemma is needed in our discussion.

Lemma 2.3. *For any* $t > 0$,

$$\frac{1}{2} < \frac{1}{1 - e^{-t}} - \frac{1}{t} < 1. \tag{2.5}$$

Proof. Let $h(t) = \dfrac{1}{1 - e^{-t}} - \dfrac{1}{t}$.

By L'Hôpital's Rule, we have $h(0^+) = \frac{1}{2}$. $h(\infty) = 1$ is clear. Straightforward calculation produces

$$\begin{aligned} t^2(1 - e^{-t})^2 h'(t) &= (1 - e^{-t})^2 - t^2 e^{-t} \equiv h_1(t), \\ e^t h_1'(t) &= t^2 - 2t + 2(1 - e^{-t}) \\ &> t^2 - 2t + 2\left(t - \frac{t^2}{2}\right) \\ &= 0. \end{aligned}$$

Therefore, the function $h_1(t)$ is strictly increasing, and $h_1(t) > h_1(0) = 0$. Thus $h'(t) > 0$ and then $h(t)$ is strictly increasing on $(0, \infty)$. The inequalities (2.5) follow from the monotonicity and the limiting values of $h(t)$.

Theorem 2.4. *The function*

$$F(x) = \frac{1}{x} \log \Gamma(x) - \log x + \frac{1}{2x} \log \frac{x}{2\pi} + 1$$

is strictly completely monotonic on $(0, \infty)$, *with* $\lim_{x \to \infty} F(x) = 0$ *and* $\lim_{x \to 0} F(x) = \infty$. (*Here, strictly completely monotone means* $(-1)^n f^{(n)}(x) > 0$ *for all* $x \in (0, \infty)$, $n \in N$.)

Proof. The main ingredient of the proof is Binet's integral representation (cf. [10])

$$\log \Gamma(x) = \left(x - \frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) - \int_0^\infty \left(\frac{1}{2} + \frac{1}{t} - \frac{1}{1-e^{-t}}\right)\frac{e^{-xt}}{t}\,dt. \quad (2.6)$$

We obtain

$$F(x) = \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2}\right)\frac{e^{-xt}}{xt}\,dt.$$

The function $l(x) = \frac{e^{-x}}{x} = \int_1^\infty e^{-xt}dt$ is strictly completely monotone on $(0, \infty)$. By Lemma 2.3, $\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} > 0$ for all $t > 0$, we conclude that F is strictly completely monotone. $\lim_{x\to\infty} F(x) = 0$ is clear by (2.6),

$$\lim_{x\to 0} F(x) = \lim_{x\to 0}\left[\frac{1}{x}\log\Gamma(x+1) - \frac{1}{x}\log x - \log x + \frac{1}{2x}\log\frac{x}{2\pi} + 1\right] = \infty.$$

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