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CONTRIBUTIONS OF INVERSE AUTOCORRELATION FUNCTION IN TRANSFER FUNCTION MODELLING

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Abstract

To examine the relationship between Inverse Autocorrelation Function (IACF) and Autocorrelation Function (ACF) as well as its effects in Transfer Function (TF) modelling we considered TF models with and without outlier input series. Transfer function models could be useful in investigating series with causal relationship, that is, where an output series is related to one or more input series. Here, a TF of order q, r, TF (q, r) is defined for the two observable stationary time series and the contribution of outlier input series as well as its effects on output series generated was examined. In this paper, a precise mathematical relationship between the inverse Autocorrelation Function and Autocorrelation is derived. The usefulness of TF models for investigating causal relationship is indicated. The theoretical basis of the relative efficiency of using IACF in TF modelling was developed. A variety of simulated and real life data were used to demonstrate the applicability and efficiency of the model.

1. Introduction

Cleveland [4] introduced the concept of Inverse Autocorrelations for discrete stationary time series, as being the autocorrelations associated with the reciprocal of the series spectrum.

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The autocovariances γ_k of a stationary series $\{X_t\}$ of zero mean are defined by

$$\gamma_k = \text{cov}(X_t, X_{t+k})$$

and the autocorrelations

$$\rho_k = \gamma_k / \gamma_0.$$

The autocovariance generating function of $\{X_t\}$ is

$$\gamma(z) = \sum_{k=-\infty}^{\infty} \gamma_k Z^k, \tag{1}$$

where Z denotes a dummy variable.

The autocorrelation generating function is given by

$$\rho(z) = \frac{\gamma(z)}{\gamma(0)}$$

$$= \sum_{k=-\infty}^{\infty} \rho(k) z^k. \tag{2}$$

The inverse autocovariances of $\{X_t\}$ are defined in such a way that their generating function is the reciprocal of (1). Thus the inverse autocovariance generating function $\gamma_i(z)$ is defined by

$$\gamma(z)\gamma_i(z) = 1$$

and $\gamma_i(k)$ is the coefficient of z^k in the expansion of $\gamma_i(z)$ in positive and negative powers of Z.

The inverse autocorrelation coefficient at lag K denoted by $\rho_i(k)$ and defined by

$$\rho_i(k) = \gamma_i(z)/\gamma_i(0), \quad k = 0, \pm 1. \tag{3}$$

Cleveland [4] and Harter [8] have described the use of inverse autocorrelations in the identification of Box and Jenkins models. It has been shown by Cleveland [4] and Chatfield [3] that the IACF behaves for moving average processes in exactly the same way as the ACF behaves for autoregressive processes. Oyetunji [12] used the method of subset selection proposed by Haggan and Oyetunji [7] to replace autocorrelations with inverse autocorrelations for the purpose of demonstrating that moving average modelling can be as straightforward as autoregressive modelling.

Aberrant observations are often encountered in data analysis. Outliers in time series depending on their nature may have a moderate to significant impact on the effectiveness of the standard methodology for time series analysis. TF models could be useful in investigating series with causal relationship. The purposes of TF modelling are to identify and estimate the TF and the noise model based on the available information of the input series X_t and the output series Y_t.

A TF (q, r) is defined as a linear regression equation between $(Y_t, Y_{t-1}, ..., Y_{t-q})$ and $(X_{t-1}, X_{t-2}, ..., X_{t-r})$, where $\{Y_t\}$ and $\{X_t\}$ are assumed to be two observable stationary time series.

In this paper, the attention is focussed on deriving a precise mathematical relationship between IACF and ACF and thereby illustrating its importance in TF modelling in the outlier free and contaminated series.

2. The Mathematical Relationship between IACF and ACF

Let $\{X_t\}$ be a real-valued stationary process with absolutely summable autocovariance sequence γ_k and spectral density function $f(w)$ such that

$$f(w) = 1/2\pi \sum_{k=-\infty}^{\infty} \gamma_k e^{-iwk}. \tag{4}$$

The sequence γ_k can be recovered from $f(w)$ through the inverse Fourier transform

$$\gamma_k = \int_{-\pi}^{\pi} f(w) e^{iwk} dw. \tag{5}$$

Cleveland [4] denotes the reciprocal of $f(w)$ by

$$f_i(w) = 1/f(w). \tag{6}$$

He defined the inverse autocovariance function of $\{X_t\}$ by

$$\gamma_i(k) = \int_{-\pi}^{\pi} f_i(w) e^{iwk} dw. \tag{7}$$

Considering (3) and (7), the inverse transform is given by

$$f_i(w) = 2\pi \sum_{k=-\infty}^{\infty} e^{iwk} \gamma_i(k) \tag{8}$$

it follows that

$$\gamma_i(k) = 2\pi \sum_{k=-\infty}^{\infty} (1/k)(1/\gamma_k) \sin 2k\pi. \tag{9}$$

To express $\gamma_i(k)$, we consider these two cases:

Case I. When $k = 0$, (9) fails to exist and (7) becomes

$$\begin{aligned}\gamma_i(0) &= 2\pi \int_{-\pi}^{\pi} 1/\gamma_0 dw \\ &\geq 2\pi \int_{-\pi}^{\pi} 1/\gamma_0 dw = 2\pi^2/\gamma_0.\end{aligned} \tag{10}$$

Case II. When $k \neq 0$, (9) becomes

$$\gamma_i(k) = 4\pi \sum_{j=i}^{k} (1/j)(1/\gamma_j) \sin 2j\pi. \tag{11}$$

Hence, the inverse autocovariance function is given by

$$\gamma_i(k) = \begin{cases} 2\pi^2/\gamma_0, & k = 0, \\ 4\pi \sum_{j=1}^{k} (1/j)(1/\gamma_j) \sin 2j\pi, & k \neq 0. \end{cases} \tag{12}$$

3. IACF in Transfer Function Modelling

Box and Jenkins [1] express mathematically a basic transfer relationship for a bivariate stochastic process for X_t and Y_t as

$$\phi(B)Y_t = \theta(B)X_t + \varepsilon_t, \tag{13}$$

where

$$\phi(B) = 1 - \sum_{j=i}^{r} \phi_j B^j,$$

$$\theta(B) = 1 - \sum_{j=i}^{r} \theta_j B^j$$

and ε_t is a white noise process.

Equation (13) is said to be a *transfer function model* denoted by $\mathrm{TF}(q, r, d)$ in which $\phi(B)$ is of order r, $\theta(B)$ is of order q and the series in the model requires differencing d times to attain stationarity.

Suppose we have two stochastic processes $\{X_t\}$ and $\{Y_t\}$; a TF model of order (q, r) abbreviated as $\mathrm{TF}(q, r)$ is defined by

$$(1 - \delta_1 B - \dots - \delta_q B^q)Y_t = (w_0 + w_1 B + \dots + w_r B^r)X_{t-b} + N_t, \tag{14}$$

where B is the backward shift operator, b is the delay in the response of Y to a change in X, N_t is the error process which is assumed to follow an ARMA process.

The main identification and fitting of a TF model defined by (14) is discussed in Box and Jenkins [1].

In this paper, we consider a particular class of TF models in which the error process is assumed to be white noise, and define a $\mathrm{TF}(q, r)$ as

$$Y_t - \delta_1 Y_{t-1} - \dots - \delta_q Y_{t-q} = w_1 X_{t-1} + \dots + w_r X_{t-r} + e_t, \tag{15}$$

where $\{e_t\}$ is assumed to be a white noise process with variance σ^2 and, $\{X_t\}$ and $\{Y_t\}$ are marginally and jointly stationary.

With $q + r$ parameters in the model, we are likely to have more parameters than in the corresponding univariate autoregressive models.

Oyetunji [11], reduced the number of parameters in the TF model, that is, by constraining some parameters to zero.

Shangodoyin [13] considered the estimation of the dynamic parameter space and the analysis of the residuals of TF model in the presence of outlier input series and noticed that the magnitudes of outliers have a bandwagon effect on the autocovariances generated in the series.

Rewriting (15) as outlier free (OF) and outlier contaminated (OC) transfer function models respectively, we have

$$Y_t = \delta_1 Y_{t-1} + \dots + \delta_q Y_{t-q} + w_1 X_{t-1} + \dots + w_r X_{t-r} + e_t \tag{16}$$

and

$$\begin{aligned} Y_t &= \delta_1 Y_{t-1} + \dots + \delta_q Y_{t-q} + w_1 Z_{t-1} + \dots + w_r Z_{t-r} \\ &\quad + w_1 D_{t-1} + \dots + w_r D_{t-r} + e_t. \end{aligned} \tag{17}$$

Multiplying (16) and (17) by Y_{t-k} and taking expectations for $k = 1, 2, \dots, q$ and also assuming that Y_{t-k} is uncorrelated with e_t, we obtain a set of q linear simultaneous equations for both the OF and OC models and express the inverse autocovariance function $\gamma_i(k)$ as

$$\left\{ \begin{array}{l} \gamma_{iy}(1) = \sum_{j=1}^{q} \delta_j \gamma_{iy}^{(1-j)} + \sum_{j=1}^{r} w_j \gamma_{ixy}^{(1-j)} \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ \gamma_{iy}(q) = \sum_{j=1}^{q} \delta_j \gamma_{iy}^{(q-j)} + \sum_{j=1}^{r} w_j \gamma_{iy}^{(q-j)} \end{array} \right\} \tag{18}$$

for the OF model and

$$\left\{\begin{array}{l} \gamma_{iy}(1) = \sum_{j=1}^{q} \delta_j \gamma_{iy}^{(1-j)} + \sum_{j=1}^{r} w_j [\gamma_{izy}^{(1-j)} + \gamma_{iDY}^{(1-j)}] \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \gamma_{iy}(q) = \sum_{j=1}^{q} \delta_j \gamma_{iy}^{(q-j)} + \sum_{j=1}^{r} w_j [\gamma_{iy} + \gamma_{iy}^{(q-j)}] \end{array}\right\} \tag{19}$$

for the OC model, where γ_{iy} and γ_{ixy} denote the inverse autocovariance function of the series $\{Y_t\}$ and the inverse cross covariance function of X and Y.

Multiplying (16) by X_{t-k} and (17) by $(Z_{t-k} + D_{t-k})$ and taking expectations for $k = 1, 2, \ldots, r$ results to a set of r linear simultaneous equations for both the OF and OC models respectively and we have

$$\left\{\begin{array}{l} \gamma_{ixy}(1) = \sum_{j=1}^{q} \delta_j \gamma_{ixy}^{(1-j)} + \sum_{j=1}^{r} w_j \gamma_{ix}^{(1-j)} \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \gamma_{ixy}(r) = \sum_{j=1}^{q} \delta_j \gamma_{ixy}^{(r-j)} + \sum_{j=1}^{r} w_j \gamma_{ix}^{(r-j)} \end{array}\right\} \tag{20}$$

and

$$\left\{\begin{array}{l} \gamma_{izy}(1) + \gamma_{iDY} = \sum_{j=1}^{q} \delta_j [\gamma_{izy}^{(1-j)} + \gamma_{iDY}^{(1-j)}] + \sum_{j=1}^{r} w_j [\gamma_{iz}^{(1-j)} + \gamma_{iD}^{(1-j)} + 2\gamma_{iZD}] \\ \vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ \gamma_{izy}(r) + \gamma_{iDY}^{(r)} = \sum_{j=i}^{q} \delta_j [\gamma_{izy}^{(r-j)} + \gamma_{iDY}^{(r-j)}] + \sum_{j=i}^{r} w_j [\gamma_{iz}^{(r-j)} + \gamma_{iD}^{(r-j)} + 2\gamma_{iZD}^{(r-j)}] \end{array}\right\}. \tag{21}$$

Combining (18) and (20) for the OF model and (19) and (21) for the OC model, we obtain sets of $q + r$ linear simultaneous equations and using the relationship

$$\gamma_{YZ}(-k) = \gamma_{ZY}(k)$$

we express them in matrix form as

$$\underline{\gamma} = \underline{\Sigma}\,\underline{\theta}, \tag{22}$$

where

$$\underline{\gamma} = \begin{bmatrix} \gamma_{iY} \\ \gamma_{ixY} \end{bmatrix},$$

$$\underline{\Sigma} = \begin{bmatrix} \Sigma_{YY} & \Sigma_{YX} \\ \Sigma_{XY} & \Sigma_{XX} \end{bmatrix},$$

$$\underline{\theta} = \begin{bmatrix} \underline{\delta} \\ \underline{w} \end{bmatrix},$$

$$\gamma_{iy} = [\gamma_{iy}, \ldots, \gamma_{iy}]^T,$$

$$\underline{\delta} = [\delta_1, \delta_2, \ldots, \delta_q]^T,$$

$$\underline{w} = [w_1, w_2, \ldots, w_r]^T,$$

$$\gamma_{ixy} = [\gamma_{ixy}, \ldots, \gamma_{ixy}]^T,$$

for the OF model and

$$[\gamma_{ixy} = \gamma_{izy} + \gamma_{iDY}, \ldots, \gamma_{izy} + \gamma_{iDY}]^T$$

for the OC model.

Considering the errors in (16) and (17) with appropriate starting values, the least squares estimates of

$$U = (\delta, w) \tag{23}$$

are those values U which minimize the sum of squares

$$S(U) = \sum e_t \text{ as a function of } U.$$

Hartley [9] and, Draper and Smith [5] discussed the estimation methods of (23) and more specifically for time series models by Box and Jenkins [1].

Following Tsay [14], the proposed method only uses the least squares method to obtain parameter estimates.

Shangodoyin [13] compared two well-known test statistics in assessing the existence of and how to distinguish between the additive and innovation outliers.

Empirical Illustration

We present a variety of examples to illustrate the feasibility of the method. We assumed an outlier input series and used Tsay [14] detection technique to observe the timings of the outliers. The estimates of the coefficients are computed with their standard errors. The model was checked for adequacy and a comparison is made between contaminated and uncontaminated TF models.

Series A is the stack-loss data (Brownlee [2]), Series B is the first word-Gessel adaptive score (Mickey et al. [10]), Series C is the Gas Furnace data (Box and Jenkins [1]), Series S_1 and S_2 are the simulated series of sizes 25 and 100.

Series A

We estimated the TF $(2, 2)$ for both the OF and OC input series

$$Y_t = 0.477Y_{t-1} + 0.280Y_{t-2} + 0.640X_{t-1} - 0.587X_{t-2} + e_t$$

for the OF series and

$$Y_t = 0.314Y_{t-1} + 0.305Y_{t-2} + 0.172X_{t-1} - 0.085X_{t-2} + e_t$$

for the OC series.

The model standard error with outliers is about 2.123 multiple of the model standard error without outliers.

The magnitude of the standard error of estimates of δ_1, δ_2 and w_1 is smaller for OF than what was obtained for the OC model. The test criterion indicates that both models fitted adequately.

Series B

When the input is outlier free and contaminated we fitted a TF $(2, 2)$

and obtained

$$Y_t = 0.331Y_{t-1} + 0.480Y_{t-2} + 0.945X_{t-1} + 0.312X_{t-2} + e_t$$

for the OF model and

$$Y_t = 0.709Y_{t-1} + 0.108Y_{t-2} + 0.866X_{t-1} + 0.038X_{t-2} + e_t$$

for the OC model.

The standard error of the model reveals that the OC model is about 1.624 multiple of that obtained for the OF model.

Series C

We obtained a TF $(2, 1)$ which is

$$Y_t = 1.766Y_{t-1} - 0.766Y_{t-2} - 0.253X_{t-1} + e_t$$

for the OF model and

$$Y_t = 0.554Y_{t-1} + 0.439Y_{t-2} - 0.101X_{t-1} + e_t$$

for the OC model.

The standard error of the model is higher in the OC model than in the OF model. It is about 3.67 multiple of that obtained for the OF model.

Series S_1

This is the first simulated series of size 25. We fitted the TF $(1, 1)$ models to both series. For the OF, we fitted

$$Y_t = 0.214Y_{t-1} + 0.074X_{t-1} + e_t$$

and for the OC, we fitted

$$Y_t = 0.133Y_{t-1} - 0.809X_{t-1} + e_t.$$

This reveals that the standard error for the OC model is about 3.67 multiple of that obtained for the OF model.

Series S_2

We fitted the TF $(2, 2)$ models for both the OF and OC as

$$Y_t = -0.078Y_{t-1} - 0.026Y_{t-2} + 0.044X_{t-1} + 0.012X_{t-2} + e_t$$

and

$$Y_t = -0.011Y_{t-1} + 0.036Y_{t-2} + 0.005X_{t-1} + 0.051X_{t-2} + e_t.$$

The model standard error shows that the OC model is about 1.76 multiple of that obtained for the OF model.

Implications of the result

We have defined a class of Transfer Function model, which has a lot to recommend on the contributions of outliers in empirical work on TF modelling. It was noticed that outliers affect the inverse autocorrelation structure of a time series and hence the distribution of the estimates of TF models. They also bias the estimated inverse autocorrelation function. The standard error of the model is also affected. The existence of outliers may cause substantive biases in TF modelling and can seriously jeopardize the function as model identification tool. We suggest that with the existence of outliers a check is very necessary whether the TF models can be used in investigating one-way dependence between two stationary processes. This will be achieved by looking at the order of the fitted TF model. If r is much greater than q, then this will suggest that the series $\{Y_t\}$ is heavily dependent on $\{X_t\}$. For further research work, maximum likelihood estimates given by Fox [6] can be compared with the least squares estimates to measure the nearness of the magnitude of outliers to the true value.

References

[1] G. E. P. Box and G. M. Jenkins, Time Series Analysis, Forecasting and Control, Holden-Day, San Francisco, 1976.

[2] K. A. Brownlee, Statistical Theory and Methodology in Science and Engineering, Wiley, New York, 1965.

[3] C. Chatfield, Inverse autocorrelations, J. Roy. Statist. Soc. Ser. A 142(3) (1979), 363-377.

[4] W. S. Cleveland, The inverse autocorrelations of a time series and their applications, Technometrics 14 (1972), 277-293.

[5] N. R. Draper and H. Smith, Applied Regression Analysis, Wiley, New York, 1966.

[6] A. J. Fox, Outliers in time series, J. Roy. Statist. Soc. Ser. B 34 (1972), 350-363.

[7] V. Haggan and O. B. Oyetunji, On the selection of subset autoregressive time series models, J. Time Series Anal. 5(2) (1984), 103-244.

[8] H. L. Harter, The method of least squares and some alternatives, Internat. Statist. Review 44 (1977), 113-159.

[9] H. O. Hartley, The modified Gauss-Newton method, for the fitted of non-linear regression function by least squares, Technometrics 3 (1961), 269-280.

[10] M. R. Mickey et al., Note on use of stepwise regression in detecting outliers, Compt. Biomed. Res. 1 (1967), 105-111.

[11] O. B. Oyetunji, Subset transfer function model fitting, Time Series Analysis: Theory and Practice 4, O. D. Anderson, ed., North-Holland, 1983.

[12] O. B. Oyetunji, Inverse autocorrelations and moving average time series modelling, J. Official Statist. 1(3) (1985), 315-322.

[13] D. K. Shangodoyin, On the specification of time series models in the presence of aberrant observations, Ph.D. Thesis, Department of Statistics, University of Ibadan, Nigeria, 1994.

[14] R. S. Tsay, Time series model specification in the presence of outliers, J. Amer. Stat. Assoc. 81 (1986), 132-141.

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