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ON THE STRONG LAW OF LARGE NUMBERS FOR ARRAYS OF *NA* RANDOM VARIABLES

SUNG MO CHUNG, JEONG YEOL CHOI, BYUNG YOON CHO, KYUNG JAE LEE, GIL HWAN LEE and DAE HEE RU*

Division of Mathematical Science and Institute of Basic Natural Science
Wonkwang University, Iksan 570-749, Republic of Korea

*Department of Computer Science, Chungwoon University
Republic of Korea

Abstract

Let $\{X_{ni} \,|\, 1 \le i \le n,\, n \ge 1\}$ be an array of rowwise negatively associated random variables under some suitable conditions. Then it is shown that for some $0 < t < \frac{1}{2}$, $n^{-1/t} \max_{1\le k\le n} \sum_{i=1}^{k} X_{ni} \to 0$ completely as $n \to \infty$ if and only if $E|X|^{2t} < \infty$ and $E|X_{ni}| = 0$ and $\frac{1}{r_n} \max_{1\le j\le k_n} \left| \sum_{i=1}^{j} X_{ni} \right| \to 0$ completely as $n \to \infty$ implies $E|X|^{\frac{k+1}{r}} < \infty$.

1. Introduction

The concept of negatively associated random variables was introduced by Joag-Dev and Proschan [7] although a very special case was first introduced by Lehmann [9]. Many authors derived several important properties about negatively associated (NA) sequences and also discussed some applications in the area of statistics, probability, reliability and

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multivariate analysis. Compared to positively associated random variables, the study of *NA* random variables has received less attention in the literature. Readers may refer to Karlin and Rinott [8], Ebrahimi and Ghosh [3], Block et al. [2], Newman [12], Joag-Dev [6], Joag-Dev and Proschan [7], Matula [11] and Roussas [13] among others.

Recently, some authors focused on the problem of limiting behavior of partial sums of *NA* sequences. Su et al. [15] derived some moment inequalities of partial sums and a weak convergence for a strongly stationary *NA* sequence. Su and Qin [14] studied some limiting results for *NA* sequences. More recently, Liang and Su [10], and Baek et al. [1] considered some complete convergence for weighted sums of *NA* sequences.

Let $\{X_{nk}\}$ be an array of random variables with $EX_{nk} = 0$ for all n and k and let $1 \le p < 2$. Then

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{nk} \to 0 \text{ completely as } n \to \infty \tag{1.1}$$

and where complete convergence is defined (Hsu and Robbins [4]) by

$$\sum_{n=1}^{\infty} P\left(\left| \frac{1}{n^{1/p}} \sum X_{nk} \right| > \varepsilon \right) < \infty \text{ for each } \varepsilon > 0. \tag{1.2}$$

Hu et al. [5] showed that for an array of i.i.d. random variables $\{X_{nk}\}$, (1.1) holds if and only if $E|X_{11}|^{2p} < \infty$.

The main purpose of this paper is to extend a similar results above to rowwise *NA* random variables, since independent and identically random variables are a special case of *NA* random variables. That is, we investigate the necessary and sufficient condition for $\frac{1}{n^{1/t}} \max_{1 \le k \le n} \left| \sum_{i=1}^{k} X_{ni} \right| \to 0$ completely as $n \to \infty$, where $0 < t < \frac{1}{2}$ and let $\{k_n\}$ and $\{r_n\}$ be two increasing positive sequences satisfying some conditions, then, we show that $\frac{1}{r_n} \max_{1 \le j \le k_n} \left| \sum_{i=1}^{j} X_{ni} \right| \to 0$ completely as $n \to \infty$ implies $E|X|^{\frac{k+1}{r}} < \infty$ in *NA* setting.

Finally, in order to prove the strong law of large numbers for array of NA random variables, we give an important definition and some lemmas which will be used in obtaining the strong law of large numbers in the next section.

Definition 1.1 [7]. Random variables $X_1, \ldots, X_n$ are said to be *negatively associated* (NA) if for any two disjoint nonempty subsets A_1 and A_2 of $\{1, \ldots, n\}$ and f_1 and f_2 are any two coordinatewise nondecreasing functions,

$$Cov(f_1(X_i,\, i \in A_1),\, f_2(X_j,\, j \in A_2)) \leq 0,$$

whenever the covariance is finite. An infinite family of random variables is NA. If every finite subfamily is NA.

Lemma 1.2 [10]. *Let* $\{X_i \mid i \geq 1\}$ *be a sequence of NA random variables and* $\{a_{ni} \mid 1 \leq i \leq n,\, n \geq 1\}$ *be an array of real numbers. If* $P(\max_{1 \leq j \leq n} |a_{nj} X_j| > \varepsilon) < \delta$ *for* δ *small enough and n large enough, then*

$$\sum_{j=1}^{n} P(|a_{nj} X_j| > \varepsilon) = O(1) P(\max_{1 \leq j \leq n} |a_{nj} X_j| > \varepsilon)$$

for sufficient large n.

Lemma 1.3 [5]. *For any* $r \geq 1$, $E|X|^r < \infty$ *if and only if*

$$\sum_{n=1}^{\infty} n^{r-1} P(|X| > n) < \infty.$$

More precisely,

$$r 2^{-r} \sum_{n=1}^{\infty} n^{r-1} P(|X| > n) \leq E|X|^r \leq 1 + r 2^r \sum_{n=1}^{\infty} n^{r-1} P(|X| > n).$$

Lemma 1.4 [5]. *If* $r \geq 1$ *and* $t > 0$, *then*

$$E|X|^r I(|X| \leq n^{1/t}) \leq r \int_0^{n^{1/t}} t^{r-1} P(|X| > t) dt$$

and

$$E|X|I(|X| > n^{1/t}) = n^{1/t}P(|X| > n^{1/t}) + \int_{n^{1/t}}^{\infty} P(|X| > t)dt.$$

2. Main Results

Theorem 2.1. *Let* $0 < t < \frac{1}{2}$ *and let* $\{X_{ni} \mid 1 \le i \le n, n \ge 1\}$ *be an array of rowwise NA random variables such that* $EX_{ni} = 0$ *and* $P(|X_{ni}| > x) = O(1)P(|X| > x)$ *for all* $x \ge 0$. *If* $E|X|^{2t} < \infty$, *then*

$$\frac{1}{n^{1/t}} \max_{1\le k\le n}\left|\sum_{i=1}^{k} X_{ni}\right| \to 0 \text{ completely as } n \to \infty.$$

Proof. We define that for $1 \le i \le n$, $n \ge 1$ and $0 < t < \frac{1}{2}$,

$$Y_{ni} = X_{ni}I(|X_{ni}| \le n^{1/t}) + n^{1/t}I(X_{ni} > n^{1/t}) - n^{1/t}I(X_{ni} < -n^{1/t}).$$

To prove Theorem 2.1, it suffices to show that

$$\sum_{n=1}^{\infty} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_{ni} - \sum_{i=1}^{k} Y_{ni}\right| \ge \varepsilon n^{1/t}\right) < \infty \text{ for all } \varepsilon > 0, \tag{2.1}$$

$$\frac{1}{n^{1/t}} \max_{1\le k\le n}\left|\sum_{i=1}^{k} EY_{ni}\right| \to 0, \tag{2.2}$$

$$\sum_{n=1}^{\infty} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} Y_{ni}\right| \ge \varepsilon n^{1/t}\right) < \infty, \text{ for all } \varepsilon > 0. \tag{2.3}$$

The proofs of (2.1)-(2.3) can be found in the following Lemmas 2.1-2.3.

Lemma 2.1. *If* $E|X|^{2t} < \infty$, *then* (2.1) *holds.*

Proof.

$$\sum_{n=1}^{\infty} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_{ni} - \sum_{i=1}^{k} Y_{ni}\right| \ge \varepsilon n^{1/t}\right)$$

$$\leq \sum_{n=1}^{\infty} P\left(\bigcup_{i=1}^{n} X_{ni} \neq Y_{ni}\right)$$

$$\leq \sum_{n=1}^{\infty} \sum_{i=1}^{n} P(|X_{ni}| > n^{1/t})$$

$$= \sum_{n=1}^{\infty} O(1) n P(|X| > n^{1/t})$$

$$\leq O(1) E|X|^{2t} < \infty,$$

when $0 < t < \frac{1}{2}$ since $E|X|^{2t} < \infty$.

Lemma 2.2. *If* $E|X|^{2t} < \infty$ *and* $EX_{ni} = 0$, *then*

$$\frac{1}{n^{1/t}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} EY_{ni} \right| \to 0.$$

Proof. To prove $\frac{1}{n^{1/t}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} EY_{ni} \right| \to 0$, it suffices to show that $\sum_{n=1}^{\infty} \frac{1}{n^{1/t}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} EY_{ni} \right| < \infty$. Note that by $EX_{ni} = 0$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/t}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} EY_{ni} \right|$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n^{1/t}} \sum_{i=1}^{n} E|X_{ni}| I(|X_{ni}| > n^{1/t})$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n^{1/t}} \sum_{i=1}^{n} n^{1/t} P(|X_{ni}| > n^{1/t})$$

$$=: I_1 + I_2 \text{ (say).}$$

First, to estimate I_1, by using Lemma 1.4,

$$I_1 = \sum_{n=1}^{\infty} \frac{1}{n^{1/t}} \sum_{i=1}^{n} \left[n^{1/t} P(|X_{ni}| > n^{1/t}) + \int_{n^{1/t}}^{\infty} P(|X_{ni}| > x) dx \right]$$

$$= O(1) \sum_{n=1}^{\infty} nP(|X| > n^{1/t}) + O(1) \sum_{n=1}^{\infty} \frac{n}{n^{1/t}} \int_{n^{1/t}}^{\infty} P(|X| > x) dx =: I_1' \text{ (say)}.$$

Letting $x = n^{1/t}s$ and applying Lemma 1.3, we have

$$I_1' = O(1) \sum_{n=1}^{\infty} nP(|X| > n^{1/t}) + O(1) \sum_{n=1}^{\infty} n \int_1^{\infty} P(|X| > n^{1/t}s) ds$$

$$\leq O(1) E|X|^{2t} + O(1) \int_1^{\infty} \sum_{n=1}^{\infty} nP(|s^{-1}X|^t > n) ds$$

$$\leq O(1) E|X|^{2t} + O(1) E|X|^{2t} \int_1^{\infty} s^{-2t} ds$$

$$= O(1) E|X|^{2t} < \infty.$$

As to I_2, we have

$$I_2 = \sum_{n=1}^{\infty} \frac{1}{n^{1/t}} \sum_{i=1}^{n} n^{1/t} P(|X_{ni}| > n^{1/t})$$

$$= \sum_{n=1}^{\infty} \sum_{i=1}^{n} P(|X_{ni}| > n^{1/t})$$

$$= O(1) \sum_{n=1}^{\infty} nP(|X| > n^{1/t})$$

$$\leq O(1) E|X|^{2t} < \infty.$$

Lemma 2.3. *If* $E|X|^{2t} < \infty$, *then* $\sum_{n=1}^{\infty} P\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} Y_{ni} \right| \geq \varepsilon n^{1/t} \right) < \infty$ *for all* $\varepsilon > 0$.

Proof. From the definition of NA random variables, we know that $\{Y_{ni} \mid 1 \le i \le k,\ n \ge 1\}$ is still an array of rowwise NA random variables. Thus, we obtain that

$$\sum_{n=1}^{\infty} P\left(\max_{1 \le k \le n}\left|\sum_{i=1}^{k} Y_{ni}\right| \ge \varepsilon n^{1/t}\right)$$

$$\le O(1)\sum_{n=1}^{\infty}\frac{1}{n} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} Y_{ni}\right|\right)^{t}$$

$$\le O(1)\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n} E| Y_{ni} |^{t}$$

$$\le O(1)\left[\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n} E| X_{ni} |^{t} I(| X_{ni} | \le n^{1/t}) + \sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n} P(| X_{ni} | > n^{1/t})\right]$$

$$=: I_3 + I_4 \text{ (say).}$$

First, we prove that $I_3 < \infty$. Let $G_{ni}(x) = P(| X_{ni} | \le x)$. Then we have

$$I_3 = O(1)\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n} E| X_{ni} |^{t} I(| X_{ni} | \le n^{1/t})$$

$$\le O(1)\sum_{n=1}^{\infty}\sum_{i=1}^{n}\int_{0}^{n^{1/t}}\left(\frac{x}{n^{1/t}}\right)^{t} dG_{ni}(x)$$

$$= O(1)\sum_{n=1}^{\infty}\sum_{i=1}^{n}\int_{0}^{1}\int_{(ns)^{1/t}}^{n^{1/t}} dG_{ni}(x)ds$$

$$= O(1)\sum_{n=1}^{\infty}\sum_{i=1}^{n}\int_{0}^{1} P((ns)^{1/t} < | X_{ni} | < n^{1/t})ds$$

$$\le O(1)\int_{0}^{1}\sum_{n=1}^{\infty} nP(| X | > (ns)^{1/t})ds$$

$$\le O(1)E| X |^{2t} < \infty.$$

Also, the proof of I_4 is similar to that of Lemma 2.3.

Corollary 1 below is a corresponding result for a sequence of rowwise *NA* random variables.

Corollary 1. *Let* $0 < t < \frac{1}{2}$ *and let* $\{X_i \mid i \geq 1\}$ *be a sequence of NA random variables such that* $EX_i = 0$ *for all* i *and* $P(|X_i| > x) = O(1)P(|X| > x)$ *for all* $x \geq 0$. *If* $E|X|^{2t} < \infty$, *then*

$$\frac{1}{n^{1/t}} \max_{1 \leq k \leq n} \sum_{i=1}^{k} X_i \to 0 \text{ completely as } n \to \infty.$$

Theorem 2.2. *Let* $0 < t < \frac{1}{2}$ *and let* $\{X_{ni} \mid 1 \leq i \leq n, n \geq 1\}$ *be an array of rowwise NA random variables such that* $P(|X| > 0) = O(1)P(|X_{ni}| > x)$ *for all* $x \geq 0$. *Assume that* $\frac{1}{n^{1/t}} \max_{1 \leq k \leq n} \sum_{i=1}^{k} X_{ni}$ $\to 0$ *completely as* $n \to \infty$, *then* $E|X|^{2t} < \infty$ *and* $EX_{ni} = 0$.

Proof. From the assumptions, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} X_{ni} \right| \geq \varepsilon n^{1/t}\right) < \infty, \tag{2.4}$$

By Lemma 1.2, we obtain that

$$\sum_{i=1}^{n} P(|X_{ni}| \geq \varepsilon n^{1/t}) = O(1)P\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} X_{ni} \right| \geq \varepsilon n^{1/t}\right),$$

which, together with (2.4) and assumptions, we have

$$\sum_{n=1}^{\infty} nP(|X| \geq \varepsilon n^{1/t}) < \infty$$

which is equivalent to $E|X|^{2t} < \infty$, by Lemma 1.3.

Now, under $E|X|^{2t} < \infty$, we obtain from Theorem 2.1 that

$$\sum_{n=1}^{\infty} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}(X_{ni} - EX_{ni})\right| \ge \varepsilon n^{1/t}\right) < \infty \text{ for any } \varepsilon > 0 \tag{2.5}$$

(2.4) and (2.5) yield $EX_{ni} = 0$.

Theorem 2.3. *Let* $\{X_{ni} \mid 1 \le i \le k_n, n \ge 1\}$ *be an array of rowwise NA random variables with* $C_1 P(|X| > x) \le C_1 \inf_{n,i} P(|X_{ni}| > x) \le C_2 \sup_{n,i} P(|X_{ni}| > x) \le C_2 P(|X| > x)$ *for all* $x \ge 0$. *Assume that* $\{k_n\}$ *and* $\{r_n\}$ *are two sequences satisfying* $r_n \ge b_1 n^r$, $k_n \le b_2 n^k$, *for some* $b_1, b_2, r, k > 0$. *Let*

$$\frac{1}{r_n}\max_{1\le j\le k_n}\left|\sum_{i=1}^{j} X_{ni}\right| \to 0 \textit{ completely as } n \to \infty.$$

If $k + 1 < r$, *then* $E|X|^{\frac{k+1}{r}} < \infty$.

Proof. Note that $\frac{1}{r_n}\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_{ni}\right| \to 0$ completely as $n \to \infty$, i.e.,

$$\sum_{n=1}^{\infty} P\left(\max_{1\le j\le k_n}\left|\sum_{i=1}^{j} X_{ni}\right| \ge \varepsilon r_n\right) < \infty, \text{ for all } \varepsilon > 0. \tag{2.6}$$

Since $\max_{1\le j\le k_n}|X_{nj}| \le 2\max_{1\le j\le k_n}\left|\sum_{i=1}^{j} X_{ni}\right|$, (2.6) implies

$$\sum_{n=1}^{\infty} P(\max_{1\le j\le n}|X_{nj}| \ge n) < \infty \tag{2.7}$$

and

$$P(\max_{1\le j\le k_n}|X_{nj}| \ge r_n) \to 0 \text{ as } n \to \infty. \tag{2.8}$$

By (2.7) and (2.8), and using Lemma 1.2, we obtain that

$$\sum_{i=1}^{k_n} P(|X_{ni}| > r_n) = O(1)P(\max_{1\le j\le k_n} |X_{nj}| \ge r_n),$$

which, together with (2.7), it follows that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} P(|X_{ni}| > r_n) < \infty.$$

Thus, using the assumptions of Theorem 2.3, we have

$$\sum_{n=1}^{\infty} k_n P(|X| > b_1 n^r) < \infty,$$

which is equivalent to $E|X|^{\frac{k+1}{r}} < \infty.$

Corollary 2. *Let* $\{X_{ni} | 1 \le i \le k_n,\ n \ge 1\}$ *be an array of rowwise identically distributed NA random variables. Assume that* $\{k_n\}$ *and* $\{r_n\}$ *are two sequences satisfying* $r_n \sim n^r$, $k_n \sim n^k$, *for some* $r, k > 0$, *where* $a_n \sim b_n$ *means that* $C_1 a_n \le b_n \le C_2 a_n$ *for large enough* n. *If*

(1) $k + 1 < r$, *or*

(2) $r \le k + 1 < tr$ *for some* $0 < t < \frac{1}{2}$ *and* $EX_{ni} = 0$, *then* $\frac{1}{r_n}\max_{1\le j\le k_n}\left|\sum_{i=1}^{j} X_{ni}\right| \to 0$ *completely as* $n \to \infty$ *if and only if* $E|X|^{\frac{k+1}{r}} < \infty.$

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