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A GENERALIZED FRACTIONAL POWER SERIES FOR SOLVING NONLINEAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

In this paper, an analytical solution to nonlinear fractional integro-differential equations based on a generalized fractional power series expansion is presented. The fractional derivatives are of the conformable type. The new approach is a modified form of the well-known Taylor series expansion. Illustrative examples are presented to demonstrate the accuracy and effectiveness of the proposed method.

1. Introduction

Fractional calculus and differential equations have been widely explored due to their great importance in scientific and engineering problems. For example, fractional calculus is applied in fluid-dynamic traffic modeling

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[10], signal processing [19], control theory [5], and economics [3]. For more details and applications of fractional derivatives, we refer the reader to [21, 15, 7, 6]. Several types of fractional derivatives have been introduced to date. The most popular of which are Riemann-Liouville and Caputo fractional derivatives, but these two kinds of derivatives do not satisfy the product rule. Recently, Khalil et al. [13] introduced a new definition of fractional derivative, called *conformable fractional derivative*, which satisfies the product rule. The basic properties of the conformable fractional derivative have been obtained [1, 22]. Real-world phenomena often are modeled by the linear and nonlinear fractional differential equations [4, 25]. Many mathematical formulations contain nonlinear integro-differential equations with fractional order. However, integro-differential equations are usually difficult to solve analytically, so it is necessary to obtain an efficient approximate solution. Rawashdeh [20] applied a collocation method to study the integro-differential equations of fractional order, and the authors of [24] applied a spectral collocation method to solve stochastic fractional integro-differential equations. Momani and Noor [16] applied the Adomian decomposition method (ADM) to approximate solutions for fourth-order integro-differential equations of fractional order. Nawaz [17] applied the variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations, and the authors of [26] presented a computational method based on the second kind Chebyshev wavelet to solve fractional nonlinear Fredholm integro-differential equations. In [11], an approximated solution of fractional integro-differential equations using the Taylor expansion method is presented. Among these methods, the Taylor expansion method is the most attractive. To date, several fractional power series expansions have been presented in the literature [8, 9, 18, 2, 23, 14, 12]. In [2], a new algorithm for obtaining a series solution for a class of fractional differential equations was presented. Syam [23] investigated a numerical solution of fractional Lienard's equation by using the residual power series method. In [14], a new method, called the *restricted fractional differential transform metho*d (*RFDTM*), was developed to solve rational- or irrational-order fractional differential

equations. Recently, Jaradat et al. [12] proposed a new method based on a Taylor series expansion to solve fractional (integro-)differential equations and compared numerical solutions with exact solutions. A new series expansion was proposed to obtain closed-form solutions of fractional (integro-)differential equations of the Caputo type. This expansion provides a more integrated representation of the fractional power series with a related convergence theorem called a *generalized fractional power series* (*GFPS*).

In this paper, we adopt the conformable fractional derivative with GFPS and apply it to solve nonlinear integro-differential equations:

$$T^{\alpha}[y(t)] = h(t) + \int_0^1 k(t, \tau)[y(\tau)]^q d\tau, \quad q \geq 1 \tag{1}$$

subject to the initial condition

$$y(0) = y_0, \tag{2}$$

where $0 < \alpha \leq 1$, $k(t, \tau)$ and $h(t)$ are smooth functions. The derivative used is the conformable fractional derivative. The paper is organized as follows. In Section 2, some preliminaries used in this work details the proposed method, which is the GFPS in the conformable fractional derivative are presented. Some analytical and numerical results are presented in Section 3. Section 4 gives the conclusions of the paper.

2. The Generalized Conformable Fractional Power Series Method

In this section, definitions and properties of the conformable fractional derivative and the GFPS are presented. The derivative in equation (1) is the conformable fractional derivative which was defined in [13]. Throughout the rest of this section, it is assumed that $\alpha \in (0, 1]$.

Definition 2.1. Given a function $f : [0, \infty) \to \mathbb{R}$, the *conformable fractional derivative* of f order α is defined by

$$T^{\alpha}[f(t)] = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{3}$$

for all $t > 0$.

Theorem 2.2. *If f and g are α-differentiable at a point where* $t > 0$, *then*

$$T^{\alpha}[af + bg] = aT^{\alpha}[f] + bT^{\alpha}[g],$$

for all $a, b \in \mathbb{R}$.

The power rule of the conformable fractional derivative is as follows.

Theorem 2.3. *The conformable fractional derivative of the power function is given by*

$$T^{\alpha}[t^{p}] = pt^{p-\alpha},$$

for all $p \in \mathbb{R}$.

The generalized fractional power series (GFPS) [12] was implemented to solve equation (1), starting with the following definition and properties related to the GFPS.

Definition 2.4. A generalized fractional power series of the form

$$\sum_{i+j=0}^{\infty} c_{ij}t^{i\alpha+j} = c_{00} + c_{01}t^{1} + c_{10}t^{\alpha} + c_{02}t^{2} + c_{11}t^{\alpha+1} + c_{20}t^{2\alpha} + \cdots \quad (4)$$

was used, where $t > 0$, is called the *generalized fractional power series* (*GFPS*) about $t = 0$. c_{ij} denotes the coefficients of the series, where $i, j \in \mathbb{N} \cup \{0\}$.

Moreover, the GFPS can be naturally obtained as a Cauchy product of a fractional power series and a power series as follows:

$$\sum_{i+j=0}^{\infty} c_{ij}t^{i\alpha+j} = \left(\sum_{i=0}^{\infty} a_{i}t^{i\alpha}\right)\left(\sum_{j=0}^{\infty} b_{j}t^{j}\right), \quad (5)$$

where $c_{ij} = a_{i}b_{j}$.

Proposition 2.5. *If* $\sum_{k=0}^{\infty} a_k t^{k\alpha}$ *converges for some* $t = a > 0$, *then it converges absolutely for* $t \in (0, a)$.

Proof. See [12].

Corollary 2.6. *If* $\sum_{k=0}^{\infty} b_k t^k$ *converges for some* $t = b > 0$, *then it converges absolutely for* $t \in (0, b)$.

Proof. See [12].

Theorem 2.7. *Consider the two power series* $A = \sum_{k=0}^{\infty} a_k t^{k\alpha}$ *and* $B = \sum_{k=0}^{\infty} b_k t^k$ *such that A converges absolutely to a for* $t = t_a > 0$, *and B converges to b for* $t = t_b > 0$. *Then the Cauchy product of A and B converges to ab for* $t = t_c > 0$, *where* $t_c = \min\{t_a, t_b\}$.

Proof. See [12].

Theorem 2.8. *If* $y(t)$ *is a generalized fractional power series*, $y(t) = \sum_{i+j=0}^{\infty} c_{ij} t^{i\alpha+j}$, *then the conformable fractional derivative of* $y(t)$ *of order* α *within the interval of convergence of* $t > 0$ *is given by*

$$T^{\alpha}[y(t)] = \sum_{i=1}^{\infty} c_{i0}(i\alpha)t^{(i-1)\alpha} + \sum_{j=1}^{\infty} c_{0j}(j)t^{j-\alpha}$$

$$+ \sum_{i+j=0}^{\infty} c_{i+1,\, j+1}[(i+1)\alpha + j + 1]t^{i\alpha+j+1}. \tag{6}$$

Proof. Since $y(t)$ converges, the conformable fractional derivative of order α can be operated term-by-term within the interval of convergence of $t > 0$. Then equation (6) is obtained.

To solve problems (1) and (2), we assumed that solution $y(t)$ takes the form

$$y(t) = \sum_{i+j=0}^{\infty} c_{ij} t^{i\alpha+j}, \tag{7}$$

where $y(0) = y_0$ and c_{ij} are constants to be determined. Clearly, $c_{00} = y_0$.

The proposed expansion (6) is utilized to introduce a parallel scheme to the power series solution method. Illustrative examples are presented to demonstrate the accuracy and effectiveness of the proposed method in Section 3.

3. Numerical Results

In this section, three examples of the nonlinear integro-differential equations are presented to exhibit the usefulness of the expansion (6). It should be noted here that all the necessary calculations and graphical analyses were done with MATLAB 2017a.

Example 3.1. Consider the nonlinear Fredholm fractional integro-differential equation

$$T^{\alpha}[y(t)] = te^{t} + e^{t} - t + \int_{0}^{1} ty(\tau)d\tau, \quad 0 \leq t < 1, \, 0 < \alpha \leq 1 \tag{8}$$

subject to the initial condition, $y(0) = 0$.

In accordance with the previous discussion and using the initial condition, the proposed generalized fractional power series solution to equation (8) has the form

$$y(t) = \sum_{i+j=1}^{\infty} c_{ij} t^{i\alpha+j}. \tag{9}$$

By substituting equation (9) into equation (8), the coefficients c_{ij}, $i + j \geq 1$, are determined by equating the coefficients of like powers of t through determining a formal recurrence relation. This obtains

$$c_{11} = \frac{\alpha + 2}{\alpha^2 + 3\alpha + 1}\left[\frac{\alpha(\alpha+1)+1}{\alpha(\alpha+1)} + \sum_{j=2}^{\infty} \frac{j+1}{(\alpha+j)(\alpha+j+1)\,j!}\right], \tag{10}$$

$$c_{1j} = \frac{j+1}{(\alpha+j)\,j!} \text{ for } j = 0, 2, 3, 4, ..., \tag{11}$$

and $c_{ij} = 0$ otherwise. Therefore, the exact solution of equation (8) is

$$y(t) = c_{11}t^{\alpha+1} + \frac{t^{\alpha}}{\alpha} + \sum_{j=2}^{\infty} \frac{j+1}{(\alpha+j)\,j!} t^{\alpha+j}, \tag{12}$$

with c_{11} as in equation (10). Particularly, with $\alpha = 1$, the exact solution for the classical version of equation (8) is thus obtained as

$$y(t) = t^2 + t + \sum_{j=2}^{\infty} \frac{t^{j+1}}{j!} = te^t. \tag{13}$$

Figure 1 illustrates the approximate solutions for $\alpha = 0.25, 0.5, 0.75, 1$ in $I \in [0, 1)$.

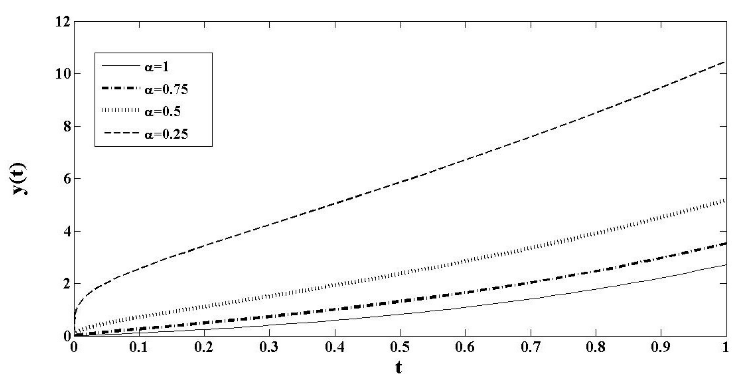

Figure 1. The approximate solution of Example 3.1 for $\alpha = 0.25, 0.5, 0.75, 1$.

Example 3.2. Consider the Volterra integro-differential equation

$$T^{\alpha}[y(t)] = 1 - \int_0^t y(\tau)d\tau, \quad 0 \le t \le 1, \;\; 0 < \alpha \le 1, \;\; y(0) = 0. \tag{14}$$

Upon substituting all the relevant quantities into equation (14) and collecting powers of t, we have

$$c_{10} = \frac{1}{\alpha}, \tag{15}$$

$$c_{i+1,i} = \frac{1}{\alpha}\left[\frac{(-1)^i}{(\alpha+1)(2\alpha+1)\cdots(i\alpha+i)((i+1)\alpha+i)}\right] \text{ for } i = 1, 2, 3, \ldots, \tag{16}$$

where $c_{ij} = 0$ otherwise. Then the exact solution is

$$y(t) = \frac{1}{\alpha}t^{\alpha} + \sum_{i=1}^{\infty} c_{i+1,i}t^{(i+1)\alpha+i}, \tag{17}$$

where $c_{i+1,i}$ satisfies equation (16).

Particularly, we can see the approximate solutions for $\alpha = 1$, which are derived for different values of t. Then the exact solution in a closed form is $y(t) = \sin t$. Figure 2 shows the effect of α on the solution for $\alpha =$ $0.25, 0.5, 0.75, 1$ in $I \in [0, 1)$.

Example 3.3. Consider the nonlinear Fredholm fractional integro-differential equation

$$T^{\frac{1}{2}}[y(t)] = 2t^{\frac{3}{2}} - t^{\frac{1}{2}} - \frac{t}{1260} + \int_0^1 t\tau[y(\tau)]^4 d\tau \tag{18}$$

subject to the initial condition, $y(0) = 0$.

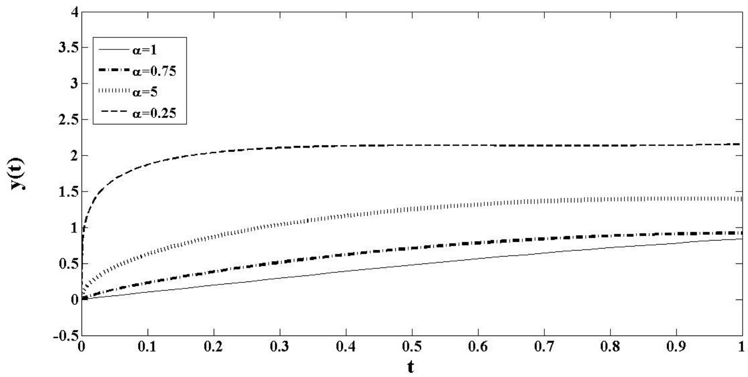

Figure 2. The approximate solution of Example 3.2 for $\alpha = 0.25, 0.5, 0.75, 1.$

Since the definite integral in equation (18) completely depends on the variable τ, the solution is spanned by the monomials $\{t, t^{\frac{3}{2}}, t^2\}.$ That is,

$$y(t) = c_{01}t + c_{11}t^{\frac{3}{2}} + c_{02}t^2 \tag{19}$$

with

$$T^{\frac{1}{2}}[y(t)] = c_{01}t^{\frac{1}{2}} + \frac{3}{2}c_{11}t + 2c_{02}t^{\frac{3}{2}}. \tag{20}$$

By substituting all the relevant quantities into equation (18) and equating the coefficients of like powers of t from both sides, we obtain $c_{01} = -1,$ $c_{02} = 1,$ and c_{11} satisfies

$$c_{11}\left(c_{11}^3 - \frac{128}{255}c_{11}^2 + \frac{4}{21}c_{11} - \left(\frac{1024}{20995} + 12\right)\right) = 0. \tag{21}$$

Subsequently, we have exact solutions of the form $y(t) = t^2 - t + c_{11}t^{\frac{3}{2}},$ where c_{11} satisfies equation (21).

4. Conclusions

In this paper, the analytical solution to nonlinear integro-differential equations based on the GFPS was demonstrated. Three numerical examples were presented. Figures 1 and 2 showed that as α increases, the approximate solution decreases. The results reveal that exact solutions are obtained in the form of a rapidly convergent series with an easily computable component. In conclusion, the proposed scheme could be used further to study identical applications. It could be extended to solve a variety of fractional differential and integral equations in the science and engineering fields.

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