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TYPE I HALF LOGISTIC POWER LINDLEY DISTRIBUTION WITH APPLICATIONS

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Abstract

In recent years, several improved and extended probability distributions have been discovered from the current distributions to facilitate their applications in many fields. A new three-parameter distribution extended from the power Lindley distribution, the so-called the type I half logistic power Lindley (TIHLPL), is introduced for modeling lifetime data. Some mathematical properties of the type I half logistic power Lindley distribution are provided. Explicit expressions for the moments, probability weighted moments and order statistics are investigated. Maximum likelihood estimation technique employed to estimate the model parameters is presented. In addition, the superiority of the subject distribution is illustrated with an application to two real data sets. Indeed, the TIHLPL model yields a better fit to these data than the other distributions.

1. Introduction

The Lindley distribution is a very well-known distribution that has been extensively used over the past decades for modeling data in reliability, biology, insurance, finance, and lifetime analysis. The Lindley distribution was introduced by Lindley [21] to analyze failure time data. The motivation for introducing the Lindley distribution arises from its ability to model failure time data with increasing, decreasing, unimodal and bathtub shaped hazard rates. This distribution represents a good alternative to the exponential failure time distributions that suffer from not exhibiting unimodal and bathtub shaped failure rates. The need for extended forms of the Lindley distribution arises in many applied areas. The emergence of such distributions in the statistics literature is only very recent. For some extended forms of the Lindley distribution and applications, the reader is referred to generalized Lindley (Nadarajah et al. [23]), a new generalized Lindley (Elbatal et al. [11]), transmuted quasi Lindley distribution (Elbatal and Elgarhy [10]), transmuted generalized Lindley distribution (Elgarhy et al. [13]) and the odd log-logistic Marshall-Olkin power Lindley distribution (Alizadeh et al. [3]). The pdf and cdf of the Lindley distribution are, respectively, given by

$$g(y, \theta) = \frac{\theta^2}{\theta + 1}(1 + y)e^{-\theta y}, \quad y > 0, \theta > 0$$

and

$$G(y, \theta) = 1 - e^{-\theta y}\left[1 + \frac{\theta y}{\theta + 1}\right], \quad y > 0, \theta > 0.$$

Using the transformation X, Ghitany et al. [15] derived the power Lindley (PL) distribution given by

$$g(x, \theta, \alpha) = \frac{\alpha\theta^2}{\theta + 1}x^{\alpha - 1}(1 + x^{\alpha})e^{-\theta x^{\alpha}}, \quad x > 0, \theta, \alpha > 0 \tag{1}$$

and

$$G(x, \theta, \alpha) = 1 - e^{-\theta x^{\alpha}}\left[1 + \frac{\theta x^{\alpha}}{\theta + 1}\right], \quad x > 0, \theta, \alpha > 0. \tag{2}$$

The PL distribution does not provide enough flexibility for analyzing different types of lifetime data. To increase the flexibility for modelling purposes, it will be useful to consider further alternatives to this distribution.

Recently, new generated families of continuous distributions have attracted several statisticians to develop new models. These families are obtained by introducing one or more additional shape parameter(s) to the baseline distribution. Some of the generated families are: the beta-G (Eugene et al. [14]), gamma-G (Zografos and Balakrishanan [26]), Kumaraswamy-G

(Cordeiro and de Castro [6]), McDonald-G (Alexander et al. [2]), transformed-transformer (Alzaatreh et al. [5]), type 1 half logistic family (TIHL-G) (Cordeiro et al. [8]), Garhy-G (Elgarhy et al. [13]), Kumaraswamy Weibull-G (Hassan and Elgarhy [17]), exponentiated Weibull-generated family (Hassan and Elgarhy [18]), type II half logistic-G (TIIHL-G) (Hassan et al. [19]), and odd Fréchet-G (Haq and Elgarhy [16]) and Muth-G by (Almarashi and Elgarhy [4]).

The cumulative distribution function (cdf) of the TIHL-G family is given by

$$F(x) = \frac{1 - [1 - G(x; \zeta)]^{\lambda}}{1 + [1 - G(x; \zeta)]^{\lambda}}, \quad x > 0, \lambda > 0, \tag{3}$$

where λ is the shape parameter. The probability density function (pdf) corresponding to (3) is given by

$$f(x) = \frac{2\lambda g(x; \zeta)[1 - G(x; \zeta)]^{\lambda - 1}}{[1 + [1 - G(x; \zeta)]^{\lambda}]^2}, \quad x > 0, \lambda > 0. \tag{4}$$

The new model is referred to as the type I half logistic power Lindley distribution. Based on the TIHL-G family, we construct the TIHLPL distribution as well as we provide the main statistical distributions. The remainder of the paper is organized as follows: In Section 2, we define the TIHLPL distribution and provide its special models. In Section 3, we derive a very useful representation for the TIHLPL density and distribution functions. Further, we derive some mathematical properties of the subject distribution. The maximum likelihood method is used to estimate the model parameters in Section 4. In Section 5, we prove the importance of the TIHLPL distribution using two real data sets. Finally, we give some concluding remarks in Section 6.

2. The New Model

In this section, we introduce the three-parameter type I half logistic power Lindley TIHLPL distribution. Using (2) in (3), the cdf of the TIHLPL distribution can be written as

$$F(x)=\frac{1-e^{-\theta\lambda x^{\alpha}}\left[1+\frac{\theta x^{\alpha}}{\theta+1}\right]^{\lambda}}{1+e^{-\theta\lambda x^{\alpha}}\left[1+\frac{\theta x^{\alpha}}{\theta+1}\right]^{\lambda}}, \quad x>0,\ \theta,\ \alpha,\ \lambda>0. \tag{5}$$

The pdf corresponding to (5) is as follows:

$$f(x)=\frac{2\lambda\alpha\theta^{2}(x^{\alpha-1}+x^{2\alpha-1})e^{-\theta\lambda x^{\alpha}}\left[1+\frac{\theta x^{\alpha}}{\theta+1}\right]^{\lambda-1}}{(\theta+1)\left[1+e^{-\theta\lambda x^{\alpha}}\left[1+\frac{\theta x^{\alpha}}{\theta+1}\right]^{\lambda}\right]^{2}}, \quad x>0,\ \theta,\ \alpha,\ \lambda>0. \tag{6}$$

Further, the survival function of X, denoted by $\overline{F}(x)$, is as follows:

$$\overline{F}(x)=\frac{2e^{-\theta\lambda x^{\alpha}}\left[1+\frac{\theta x^{\alpha}}{\theta+1}\right]^{\lambda}}{1+e^{-\theta\lambda x^{\alpha}}\left[1+\frac{\theta x^{\alpha}}{\theta+1}\right]^{\lambda}}.$$

Additionally, the hazard rate function (hrf), say $h(x)$, can be written as follows:

$$h(x)=\frac{\lambda\alpha\theta^{2}(x^{\alpha-1}+x^{2\alpha-1})}{(\theta+1)\left[1+\frac{\theta x^{\alpha}}{\theta+1}\right]\left[1+e^{-\theta\lambda x^{\alpha}}\left[1+\frac{\theta x^{\alpha}}{\theta+1}\right]^{\lambda}\right]}.$$

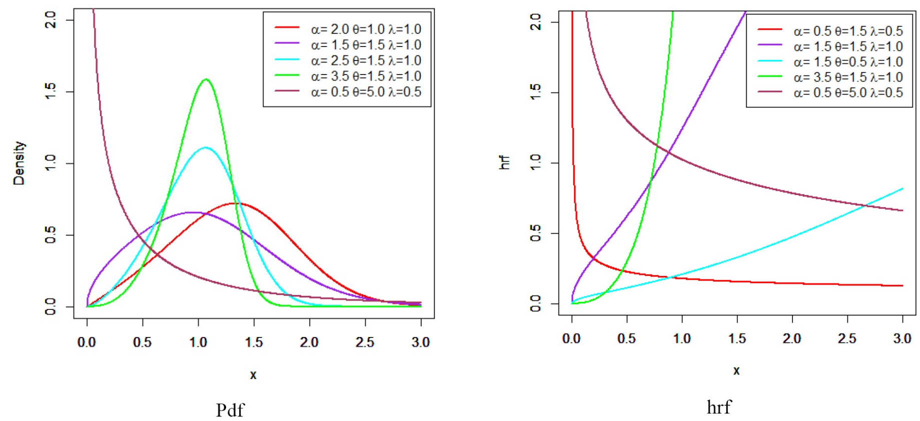


Figure 1. Plots of pdf and hrf for selected parameter values.

For $\alpha = 1,$ the pdf (6) reduces to a new model called *TIHL-Lindley distribution*. The pdf and hrf plots for the TIIHLPL are presented in Figure 1. As seen from Figure 1, densities of TIIHLPL distribution take different shapes like, symmetric, left skewed, reversed *J* shaped and unimodal. And it is clear from Figure 1 that the hrf plots take different shapes according to different values of parameters. It can be increasing, decreasing, up-side down and *J* shaped.

3. Some Statistical Properties

This section provides some statistical properties of TIHLPL distribution.

3.1. Important representation

The pdf and cdf expansions of TIHLPL are provided, which are useful in studying most statistical properties of TIHLPL distribution. From a generalized binomial series, it is known that, for $|z| < 1,$ and β is a positive real non-integer,

$$(1+z)^{-\beta} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta + i - 1}{i} z^i. \tag{7}$$

Then, by applying the binomial theorem (7) in pdf (6), we have

$$f(x) = \sum_{i=0}^{\infty} \frac{(-1)^i (i+1) 2\lambda\alpha\theta^2}{\theta + 1} (x^{\alpha-1} + x^{2\alpha-1})$$

$$\cdot\, e^{-\theta\lambda(i+1)x^\alpha} \left[1 + \frac{\theta x^\alpha}{\theta + 1}\right]^{\lambda(i+1)-1}. \tag{8}$$

Now, using the generalized binomial theorem, we can write

$$\left[1 + \frac{\theta x^\alpha}{\theta + 1}\right]^{\lambda(i+1)-1} = \sum_{j=0}^{\infty} \left(\frac{\theta}{\theta + 1}\right)^j \binom{\lambda(i+1)-1}{j} x^{\alpha j}. \tag{9}$$

Inserting the expansion (9), then the pdf (8) will be converted to

$$f(x) = \sum_{j=0}^{\infty} \eta_j (x^{\alpha(j+1)-1} + x^{\alpha(j+2)-1}) e^{-\theta\lambda(i+1)x^\alpha}, \tag{10}$$

where

$$\eta_j = \sum_{i=0}^{\infty} (-1)^i (i+1) 2\lambda\alpha\theta \left(\frac{\theta}{\theta + 1}\right)^{j+1} \binom{\lambda(i+1)-1}{j}.$$

Further, an extra expansion for the $[F(x)]^s$, for s an integer, is derived, again the binomial expansion is worked out:

$$[F(x)]^s = \sum_{k=0}^{s} \sum_{u=0}^{\infty} (-1)^{k+u} \binom{s+k-1}{k} \binom{s}{u} e^{-\theta\lambda(k+l)x^\alpha} \left[1 + \frac{\theta x^\alpha}{\theta + 1}\right]^{\lambda(k+u)}.$$

Again using the binomial expansion, $[F(x)]^s$ is given by

$$[F(x)]^s = \sum_{z=0}^{\infty} S_z x^{\alpha z} e^{-\theta\lambda(k+u)x^\alpha}, \tag{11}$$

where

$$S_z = \sum_{k=0}^{z}\sum_{u=0}^{\infty}(-1)^{k+u}\binom{s+k-1}{k}\binom{s}{u}\binom{\lambda(k+u)}{z}\left[\frac{\theta}{\theta+1}\right]^{z}.$$

3.2. The probability weighted moments (PWMs)

For a random variable X, the PWMs, denoted by $\tau_{r,s}$, can be calculated according to the following relation:

$$\tau_{r,s} = E[X^r F(x)^s] = \int_{-\infty}^{\infty} x^r f(x)(F(x))^s dx. \tag{12}$$

Inserting (10), (11) in (12), the PWMs of TIHLPL will be converted to

$$\tau_{r,s} = \sum_{j,z=0}^{\infty} \eta_j S_z \int_0^{\infty} (x^{r+\alpha(j+z+1)-1} + x^{r+\alpha(j+z+2)-1})e^{-\theta\lambda(i+k+u+1)x^{\alpha}} dx,$$

then

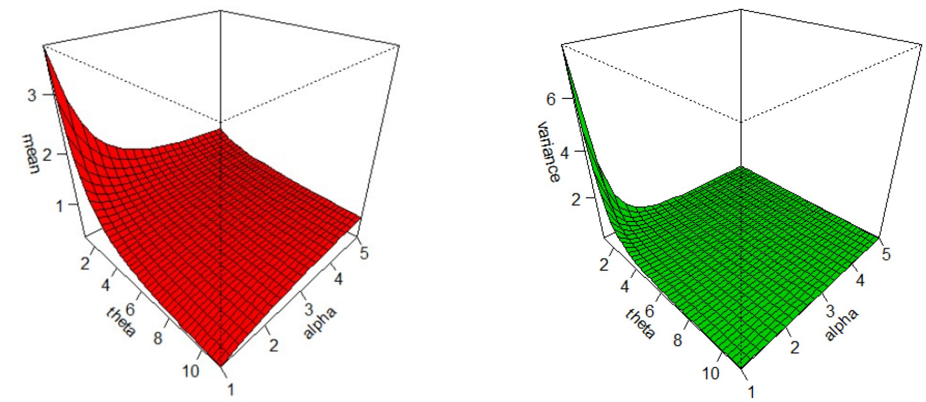
$$\tau_{r,s} = \sum_{j,z=0}^{\infty} \frac{\eta_j S_z}{\alpha}\left(\frac{\Gamma\left(\frac{r}{\alpha}+j+z+1\right)}{[\theta\lambda(i+k+u+1)]^{\frac{r}{\alpha}+j+z+1}} + \frac{\Gamma\left(\frac{r}{\alpha}+j+z+2\right)}{[\theta\lambda(i+k+u+1)]^{\frac{r}{\alpha}+j+z+2}}\right).$$

3.3. Moments

In this subsection, we derive the rth moment for the TIHLPL distribution. If X has the pdf (10), then rth moment is obtained as follows:

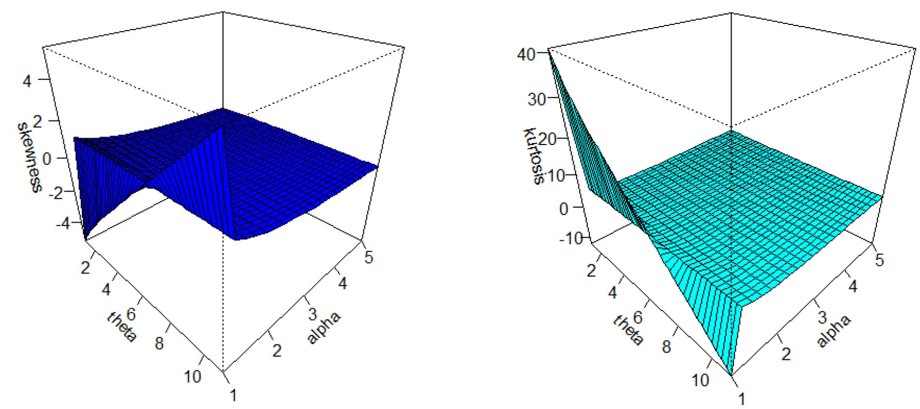
$$\mu'_r = \sum_{j=0}^{\infty}\eta_j \int_0^{\infty} (x^{r+\alpha(j+1)-1} + x^{r+\alpha(j+2)-1})e^{-\theta\lambda(i+1)x^{\alpha}} dx$$

$$= \sum_{j=0}^{\infty}\frac{\eta_j}{\alpha}\left(\frac{\Gamma\left(\frac{r}{\alpha}+j+1\right)}{[\theta\lambda(i+1)]^{\frac{r}{\alpha}+j+1}} + \frac{\Gamma\left(\frac{r}{\alpha}+j+2\right)}{[\theta\lambda(i+1)]^{\frac{r}{\alpha}+j+2}}\right).$$



mean variance

Figure 2. Plots of mean and variance.



skewness kurtosis

Figure 3. Plots of skewness and Kurtosis.

Figures 2 and 3 illustrate the mean, variance, skewness and kurtosis whose forms depend basically on the parameters α and θ.

Furthermore, for a random variable *X*, the moment generating function of TIHLPL distribution is given by

$$M_X(t) = \sum_{s=0}^{\infty} \frac{t^s}{s!} \mu'_r = \sum_{s,\, j=0}^{\infty} \frac{t^s}{s!} \frac{\eta_j}{\alpha} \left(\frac{\Gamma\left(\frac{r}{\alpha} + j + 1\right)}{[\theta\lambda(i+1)]^{\frac{r}{\alpha}+j+1}} + \frac{\Gamma\left(\frac{r}{\alpha} + j + 2\right)}{[\theta\lambda(i+1)]^{\frac{r}{\alpha}+j+2}} \right).$$

3.4. Order statistics

Order statistics have been extensively applied in many fields of statistics, such as reliability and life testing. Let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables with their corresponding continuous distribution function $F(x)$. Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the corresponding ordered random sample from a population of size n. According to David [9], the pdf of the rth order statistic is defined as

$$f_{X_{r:n}}(x) = \frac{f(x)}{B(r,\, n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} F(x)^{v+r-1}, \tag{13}$$

$B(\cdot, \cdot)$ stands for beta function. The pdf of the rth order statistic for TIHLPL distribution is derived by substituting (11) and (12) in (13), replacing s with $v + r - 1$,

$$f_{X_{r:n}}(x) = \frac{1}{B(r,\, n-r+1)} \sum_{v=0}^{n-r} \sum_{j,\, z=0}^{\infty} \eta^*$$

$$\cdot \int_0^{\infty} (x^{r+\alpha(j+z+1)-1} + x^{r+\alpha(j+z+2)-1}) e^{-\theta\lambda(i+k+u+1)x^{\alpha}} dx, \tag{14}$$

where

$$\eta^* = (-1)^v \binom{n-r}{v} \eta_j S_z.$$

The distribution of the smallest and largest order statistics can be obtained individually from (14) by setting $r = 1$ and $r = n.$ Further, the kth moment of rth order statistics for TIHLPL distribution is defined by:

$$E(X_{r:n}^{k}) = \int_{-\infty}^{\infty} x^{k} f_{x_{r:n}}(x) dx. \tag{15}$$

By substituting (14) in (15), it leads to

$$E(X_{r:n}^{k}) = \frac{1}{B(r,\, n-r+1)} \sum_{v=0}^{n-r} \sum_{j,\, z=0}^{\infty} \frac{\eta^{*}}{\alpha}$$

$$\cdot \left(\frac{\Gamma\left(\frac{r}{\alpha} + j + z + 1\right)}{[\theta\lambda(i + k + u + 1)]^{\frac{r}{\alpha}+j+z+1}} + \frac{\Gamma\left(\frac{r}{\alpha} + j + z + 2\right)}{[\theta\lambda(i + k + u + 1)]^{\frac{r}{\alpha}+j+z+2}} \right).$$

4. Maximum Likelihood Method

This section deals with the maximum likelihood estimators of the unknown parameters for the TIHLPL distribution on the basis of complete samples. Let $X_1, ..., X_n$ be the observed values from the TIHLPL distribution with set of parameters $\Phi = (\theta, \alpha, \lambda)^{T}$. The log-likelihood function for parameter vector $\Phi = (\theta, \alpha, \lambda)^{T}$ is obtained as follows:

$$\ln L(\Phi) = n \ln 2\lambda + n \ln \alpha + 2n \ln \theta - n \ln(\theta + 1)$$

$$+ \sum_{i=1}^{n} \ln(x_i^{\alpha-1} + x_i^{2\alpha-1}) - \theta\lambda \sum_{i=1}^{n} x_i^{\alpha}$$

$$+ (\lambda - 1) \sum_{i=1}^{n} \ln\left[1 + \frac{\theta x_i^{\alpha}}{\theta + 1}\right] - 2 \sum_{i=1}^{n} \ln\left[1 + e^{-\lambda\theta x_i^{\alpha}} \left[1 + \frac{\theta x_i^{\alpha}}{\theta + 1}\right]^{\lambda}\right].$$

The elements of the score function $U(\Phi) = (U_\theta, U_\alpha, U_\lambda)$ are given by

$$U_\theta = \frac{2n}{\theta} - \frac{n}{\theta+1} - \lambda\sum_{i=1}^{n} x_i^\alpha + \left(\frac{\lambda-1}{\theta+1}\right)\sum_{i=1}^{n}\left[\frac{x_i^\alpha}{1+\theta(1+x_i^\alpha)}\right]$$

$$-2\sum_{i=1}^{n}\frac{\lambda x_i^\alpha e^{-\lambda\theta x_i^\alpha}\left[1+\frac{\theta x_i^\alpha}{\theta+1}\right]^{\lambda-1}\left[\frac{1}{(\theta+1)^2}-\frac{\theta x_i^\alpha}{\theta+1}-1\right]}{1+e^{-\lambda\theta x_i^\alpha}\left[1+\frac{\theta x_i^\alpha}{\theta+1}\right]^{\lambda}},$$

$$U_\alpha = \frac{n}{\alpha} - \lambda\theta\sum_{i=1}^{n} x_i^\alpha \ln(x_i) + \theta(\lambda-1)\sum_{i=1}^{n}\left[\frac{x_i^\alpha \ln(x_i)}{1+\theta(1+x_i^\alpha)}\right]$$

$$+\sum_{i=1}^{n}\left[\frac{x_i^{\alpha-1}\ln(x_i)(1+x_i^\alpha)}{1+\theta(1+x_i^\alpha)}\right]$$

$$-2\sum_{i=1}^{n}\frac{\lambda\theta x_i^\alpha \ln(x_i) e^{-\lambda\theta x_i^\alpha}\left[1+\frac{\theta x_i^\alpha}{\theta+1}\right]^{\lambda-1}\left[\frac{1}{\theta+1}-\frac{\theta x_i^\alpha}{\theta+1}-1\right]}{1+e^{-\lambda\theta x_i^\alpha}\left[1+\frac{\theta x_i^\alpha}{\theta+1}\right]^{\lambda}}$$

and

$$U_\lambda = \frac{n}{\lambda} - \theta\sum_{i=1}^{n} x_i^\alpha + \sum_{i=1}^{n}\ln\left[1+\frac{\theta x_i^\alpha}{\theta+1}\right]$$

$$-2\sum_{i=1}^{n}\frac{\left(e^{-\lambda\theta x_i^\alpha}\left[1+\frac{\theta x_i^\alpha}{\theta+1}\right]\right)^{\lambda}\ln\left(e^{-\lambda\theta x_i^\alpha}\left[1+\frac{\theta x_i^\alpha}{\theta+1}\right]\right)}{1+e^{-\lambda\theta x_i^\alpha}\left[1+\frac{\theta x_i^\alpha}{\theta+1}\right]^{\lambda}}.$$

Setting U_θ, U_α and U_λ equal to zero and solving these equations simultaneously yield the maximum likelihood estimate (MLE) $\hat{\Phi} = (\hat{\theta}, \hat{\alpha}, \hat{\lambda})$

of $\Phi = (\theta, \alpha, \lambda)^T$. These equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods.

5. Data Analysis

In this section, we use two real data sets to illustrate the importance and flexibility of the TIHLPL distribution. We compare the fits of the TIHLPL model with some models namely: the Weibull Weibull (WW) (Abouelmagd et al. [1]), the beta Weibull (BW) (Lee et al. [20]), Mcdonald Weibull (McW) (Cordeiro et al. [7]) and exponentiated Weibull (EW) (Mudholkar and Srivastava [22]) distributions.

The maximized log-likelihood (-2ℓ), Akaike information criterion (AIC), the corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (A^*) and Cramér-von Mises (W^*) statistics are used for model selection.

Example 1. The data have been obtained from Nicholas and Padgett [24]. The data represent tensile strength of 100 observations of carbon fibers and they are:

3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

For the data in Example 1, Table 1 gives the MLEs of the fitted models and their standard errors (SEs) in parenthesis. The values of goodness-of-fit statistics are listed in Table 2.

Table 1. The MLEs and SEs of the model parameters for first data set

Model	Estimates (SEs)				
TIHLPL $(\lambda, \theta, \alpha)$	9.256×10^{-3} (0.0037)	12.878 (3.698)	2.356 (0.157)		
WW $(\alpha, \beta, \lambda, \gamma)$	18.394 (1.582)	13.273 (0.236)	0.493 (0.073)	0.159 (0.086)	
BW (a, b, λ, γ)	34.051 (0.961)	14.541 (0.19)	0.833 (0.11)	0.427 (0.077)	
McW $(a, b, \lambda, \gamma, c)$	35.28 (0.916)	18.125 (0.254)	0.813 (0.13)	0.399 (0.085)	1.548 (6.993)
EW (λ, γ, a)	5.77 (0.103)	0.295 (0.057)	1135 (0.662)		

Table 2. Goodness-of-fit statistics for first data set

Model	-2ℓ	AIC	CAIC	BIC	HQIC	A^*	W^*
TIHLPL	286.062	292.062	292.312	292.062	295.225	0.44532	0.05693
WW	299.747	307.747	309.347	305.656	309.54	0.45081	0.06256
BW	317.214	325.214	326.814	325.214	329.431	1.22496	0.23356
McW	308.116	318.116	319.716	318.116	323.388	1.22090	0.23286
EW	373.861	377.861	378.305	376.815	378.757	2.81959	0.51324

Example 2. The second data set is obtained from Tahir et al. [25] and represents failure times of 84 aircraft windshield. The data are:

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82, 3.0, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

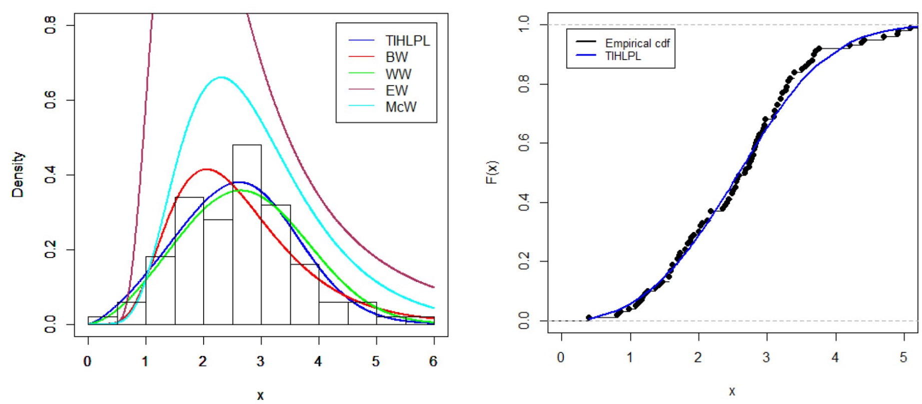
Table 3 lists the MLEs of the fitted models and their SEs in parenthesis. The values of goodness-of-fit statistics are presented in Table 4.

Table 3. The MLEs and SEs for second data set

Model	Estimates (SEs)				
TIHLPL $(\lambda, \theta, \alpha)$	9.138×10^{-3} (0.00486)	19.605 (4.816)	2.008 (0.176)		
WW $(\alpha, \beta, \lambda, \gamma)$	20.862 (1.44)	3.752 (0.298)	0.199 (0.069)	0.545 (0.113)	
BW (a, b, λ, γ)	53.874 (2.717)	20.528 (0.278)	1.076 (0.278)	0.231 (0.184)	
McW $(a, b, \lambda, \gamma, c)$	51.321 (5.329)	19.762 (0.605)	1.119 (0.48)	0.23 (0.424)	1.525 (38.539)
EW (λ, γ, a)	7.017 (0.134)	0.144 (0.063)	1773 (0.827)		

Table 4. Goodness-of-fit statistics for second data set

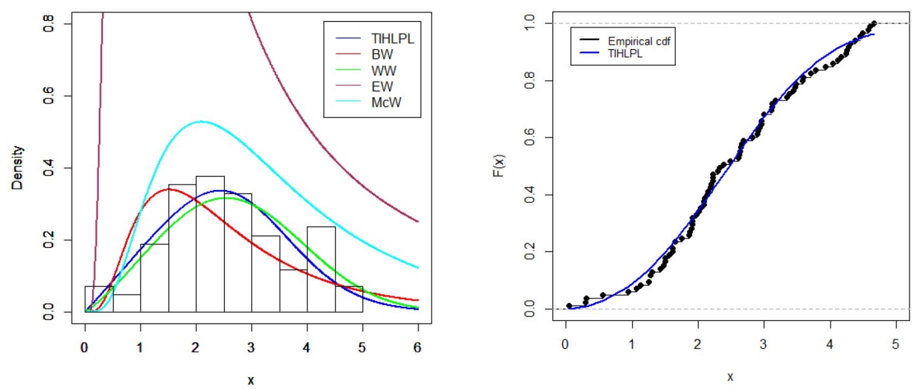
Model	-2ℓ	AIC	CAIC	BIC	HQIC	A^*	W^*
TIHLPL	257.777	263.777	264.077	263.55	266.708	0.5966	0.06727
WW	261.389	269.389	269.895	269.086	273.298	0.65619	0.07529
BW	289.948	297.948	298.455	297.645	301.857	3.34711	0.48715
McW	283.983	293.983	294.752	293.604	298.869	3.33313	0.4847
EW	320.347	326.347	326.647	324.196	326.302	32.74879	7.04167



(a) (b)

Figure 4. Estimated pdf and cdf plots for first data set.

It is observed from Table 4 that the WW distribution gives a better fit than other fitted models. Plots of the histogram, fitted densities and estimated cdfs are displayed in Figure 5.



(a) (b)

Figure 5. Estimated pdf and cdf plots for second data set.

It is noted from Tables 2 and 4 that the TIHLPL distribution provides a better fit than other competitive fitted models. It has the smallest values for goodness-of-fit statistics among all fitted models. Plots of the histogram, fitted densities and estimated cdfs are shown in Figures 4 and 5, respectively. These figures supported the conclusion drawn from the numerical values in Tables 2 and 4.

6. Conclusion

In this paper, we propose a three-parameter model extended from the power Lindley model, named the TIHLPL distribution. The TIHLPL model is motivated by the wide use of the PL distribution in practice and also for the fact that the generalization provides more flexibility to analyze positive real-life data. We derive explicit expressions for the moments and order statistics. The maximum likelihood estimation of the model parameters is investigated. The practical importance of the TIHLPL distribution is demonstrated by means of two data sets.

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