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MINIMUM-ERROR DISCRIMINATION FOR GEOMETRICALLY UNIFORM STATES

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Abstract

In this paper, we compare optimality conditions for the least square measurement and the (generalized) pretty good measurement for the minimum-error discrimination problem and obtain the optimal measurement for an ensemble of geometrically uniform states by applying the least square measurement.

1. Introduction and Problem Setting

In this paper, we study and compare the optimality conditions for the least square measurement (LSM) given in [2, 3] and the pretty good measurement (PGM) given in [4, 5] for the minimum error discrimination problem (cf. [1, 6]). We will show that the optimal PGM or generalized PGM gives the optimal LSM.

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Suppose that Alice wants to transform classical information to Bob using a quantum mechanical channel. Alice prepares a quantum state from a collection of known states and sends the state to Bob. Bob detects the information by using an appropriate measurement. A major assumption in this game is that both parties make a prior arrangement concerning the ensemble of quantum states. If the quantum states are mutually orthogonal, then Bob apply an optimal orthogonal measurement that will determine the state correctly with probability one. But if the prepared states are not orthogonal, then there is no measurement for Bob to distinguish perfectly between them. Thus the problem for Bob is to construct a measurement optimized to distinguish between nonorthogonal pure quantum states.

Formally, we may formulate the optimization problem in the following way. Let $\mathcal{H}$ be a d-dimensional Hilbert space. In preparation, we have an ensemble $P = \{p_i, |\psi_i\rangle\langle\psi_i|\}_{i=1}^{d}$, where $|\psi_1\rangle, ..., |\psi_d\rangle$ constitutes linearly independent pure states in $\mathcal{H}$ and span$\{|\psi_1\rangle, ..., |\psi_d\rangle\} = \mathcal{H}$. The probabilities $p_1, ..., p_d$ is referred as a priori probability, $p_i > 0$ and $\sum_{i=1}^{d} p_i = 1$. Alice choose a quantum state using the probability distribution $\{p_i\}$ and sends it to Bob and then Bob must figure out the state using an appropriate measurement, which minimizes the probability of a detection error. More explicitly, we seek the positive operator valued measurement (POVM) with elements $\{E_1, E_2, ..., E_d\}$ that maximizes the probability of success

$$p_s = \sum_{i=1}^{d} p_i \langle\psi_i|E_i|\psi_i\rangle$$

subject to $E_i \geq 0$ for all i and $\sum_{i=1}^{d} E_i = \text{Id}$. Equivalently, we seek the matrix Z that minimizes Tr Z subject to $Z \geq p_i|\psi_i\rangle\langle\psi_i|$ for all i. The duality problem can be summarized as follows:

$$\operatorname*{Max}_{\{E_i\}} p_s = \operatorname*{Min}_{\{Z\}} \operatorname{Tr} Z.$$

If $\{E_i\}_{i=1}^{d}$ is an element of the optimal POVM, then for some Hermitian matrix Z,

$$\sum_{i=1}^{d} p_i \langle \psi_i | E_i | \psi_i \rangle = \operatorname{Tr} Z$$

and hence it satisfies

$$(Z - p_i |\psi_i\rangle\langle\psi_i|)E_i = E_i(Z - p_i|\psi_i\rangle\langle\psi_i|) = 0.$$

Summing over i and using the relation $\sum_{i=1}^{d} E_i = \mathrm{Id},$ we have

$$Z = \sum_{i=1}^{d} p_i E_i |\psi_i\rangle\langle\psi_i|$$

$$= \sum_{i=1}^{d} p_i |\psi_i\rangle\langle\psi_i| E_i.$$

Thus we get the following relations:

$$E_j(p_j|\psi_j\rangle\langle\psi_j| - p_i|\psi_i\rangle\langle\psi_i|)E_i = 0$$

and

$$\sum_{i=1}^{d} p_i |\psi_i\rangle\langle\psi_i| E_i \geq p_j |\psi_j\rangle\langle\psi_j|.$$

From this we have

Theorem 1 [5]**.** *An optimal d-POVM* $\{E_i\}_{i=1}^{d}$ *satisfies the relations*

(1) $E_j(p_j|\psi_j\rangle\langle\psi_j| - p_i|\psi_i\rangle\langle\psi_i|)E_i = 0,$

(2) $\sum_{i=1}^{d} p_i |\psi_i\rangle\langle\psi_i| E_i \geq p_j |\psi_j\rangle\langle\psi_j|.$

2. The Least Square Measurements

In this section, we study the least square measurements for minimum-error discrimination problem discussed in Section 1, following [2] and [3]. For a preparation, let $P = \{p_i, |\psi_i\rangle\langle\psi_i|\}_{i=1}^{d}$ be an ensemble of linearly independent pure states in d-dimensional Hilbert space $\mathcal{H}_d$ such that $\mathcal{H}_d =$ span$\{\psi_i\}_{i=1}^{d}$ and let $\rho = \sum_{i=1}^{d} p_i|\psi_i\rangle\langle\psi_i|$. Then the least square measurement consists of the measurement operators $\{E_i\}_{i=1}^{d}$ such that $E_i = |\mu_i\rangle\langle\mu_i|$, where

$$|\mu_i\rangle = \sqrt{p_i}\rho^{-\frac{1}{2}}|\psi_i\rangle.$$

Note that

$$\sum_{i=1}^{d}|\mu_i\rangle\langle\mu_i| = \sum_{i=1}^{d} p_i\rho^{-\frac{1}{2}}|\psi_i\rangle\langle\psi_i|\rho^{-\frac{1}{2}} = \text{Id}.$$

Theorem 2. *Let* $P = \{p_i, |\psi_i\rangle\langle\psi_i|\}_{i=1}^{d}$ *be an ensemble of quantum states with prior probabilities* $\{p_i\}_{i=1}^{d}$ *and let* $\rho = \sum_{i=1}^{d} p_i|\psi_i\rangle\langle\psi_i|$. *The least square measurement* $\{E_i = |\mu_i\rangle\langle\mu_i|\}_{i=1}^{d}$ *with* $|\mu_i\rangle = \sqrt{p_i}\rho^{-\frac{1}{2}}|\psi_i\rangle$ *maximizes the probability of success if, for each i,*

$$p_i\langle\mu_i|\psi_i\rangle = p_i\langle\psi_i|\rho^{-\frac{1}{2}}|\psi_i\rangle = C,$$

where C is a constant independent of i.

Proof. Suppose that $p_i\langle\mu_i|\psi_i\rangle = C$ and we will find a Hermitian matrix Z such that

(1) $Z \geq p_i|\psi_i\rangle\langle\psi_i|$ for all i,

(2) $(Z - |\psi_i\rangle\langle\psi_i|)|\mu_i\rangle\langle\mu_i| = 0.$

Define $Z = C\sqrt{\rho}$. Then since $p_i \langle \mu_i | \psi_i \rangle = C$, each eigenvalue of the matrix $p_i \rho^{-\frac{1}{4}} | \psi_i \rangle \langle \psi_i | \rho^{-\frac{1}{4}}$ takes the value 0 or C and hence

$$p_i \rho^{-\frac{1}{4}} | \psi_i \rangle \langle \psi_i | \rho^{-\frac{1}{4}} \leq C \cdot \mathrm{Id}.$$

Now we get

$$p_i | \psi_i \rangle \langle \psi_i | = p_i \rho^{\frac{1}{4}} (\rho^{-\frac{1}{4}} | \psi_i \rangle \langle \psi_i | \rho^{-\frac{1}{4}}) \rho^{\frac{1}{4}} \leq C\sqrt{\rho} = Z.$$

This proves $Z \geq p_i | \psi_i \rangle \langle \psi_i |$ for all i. Since

$$(Z - | \psi_i \rangle \langle \psi_i |) \mu_i = (\alpha\sqrt{\rho} - | \psi_i \rangle \langle \psi_i |) \rho^{-\frac{1}{2}} | \psi_i \rangle$$

$$= \alpha | \psi_i \rangle - \alpha | \psi_i \rangle = 0,$$

we have $(Z - | \psi_i \rangle \langle \psi_i |) \mu_i \mu_i^* = 0$, which proves the theorem. □

3. The Generalized Pretty Good Measurements

In this section, we study the generalized pretty good measurements which is defined in [5], see also [6]. We will find the optimality condition for the problem of minimum-error discrimination and we discuss a special case, which is called *pretty good measurement* studied in [4], and we show that the optimality condition for this case is equivalent to the least square measurements discussed in Section 2.

Let $P = \{p_i, | \psi_i \rangle \langle \psi_i |\}_{i=1}^{d}$ be an ensemble of quantum states and let $Q = \{q_i, | \psi_i \rangle \langle \psi_i |\}_{i=1}^{d}$ be another ensemble of quantum states with prior probability $\{q_i\}_{i=1}^{d}$. For the ensemble Q, let $\rho_q = \sum_{i=1}^{d} q_i | \psi_i \rangle \langle \psi_i |$ and define

$$E_i = q_i \rho_q^{-\frac{1}{2}} | \psi_i \rangle \langle \psi_i | \rho_q^{-\frac{1}{2}}.$$

Then it is easy to see that E_i are well-defined and $E_i \geq 0$ for each *i*. Also, we have

$$\sum_{i=1}^{d} E_i = \sum_{i=1}^{d} q_i \rho_q^{-\frac{1}{2}} |\psi_i\rangle\langle\psi_i| \rho_q^{-\frac{1}{2}}$$

$$= \rho_q^{-\frac{1}{2}} \sum_{i=1}^{d} q_i |\psi_i\rangle\langle\psi_i| \rho_q^{-\frac{1}{2}}$$

$$= \rho_q^{-\frac{1}{2}} \rho_q \rho_q^{-\frac{1}{2}} = \mathrm{Id}.$$

Theorem 3. *Let* $P = \{p_i, |\psi_i\rangle\langle\psi_i|\}_{i=1}^{d}$ *and* $Q = \{q_i, |\psi_i\rangle\langle\psi_i|\}_{i=1}^{d}$ *be two ensembles of linearly independent pure states and let* $\rho_p = \sum_{i=1}^{d} p_i |\psi_i\rangle\langle\psi_i|$ *and* $\rho_q = \sum_{i=1}^{d} q_i |\psi_i\rangle\langle\psi_i|$. *Define* $E_i = q_i \rho_q^{-\frac{1}{2}} |\psi_i\rangle\langle\psi_i| \rho_q^{-\frac{1}{2}}$. *Then the POVM* $\{E_i\}_{i=1}^{d}$ *is optimal for the minimum-error discriminant problem if* $p_i \langle\psi_i| \rho_q^{-\frac{1}{2}} |\psi_i\rangle = C$ *for all i.*

Proof. We will show that the elements E_i of measurement satisfy the conditions given in Theorem 1.

Let $Z = \sum_{i=1}^{d} p_i |\psi_i\rangle\langle\psi_i| E_i$. Then

$$Z = \sum_{i=1}^{d} p_i |\psi_i\rangle\langle\psi_i| q_i \rho_q^{-\frac{1}{2}} |\psi_i\rangle\langle\psi_i| \rho_q^{-\frac{1}{2}}$$

$$= \sum_{i=1}^{d} p_i q_i |\psi_i\rangle\langle\psi_i| \rho_q^{-\frac{1}{2}} |\psi_i\rangle\langle\psi_i| \rho_q^{-\frac{1}{2}}$$

$$= C\sum_{i=1}^{d} q_i|\psi_i\rangle\langle\psi_i|\rho_q^{-\frac{1}{2}}$$

$$= C\rho_q\rho^{-\frac{1}{2}} = C\sqrt{\rho_q}.$$

Thus Z is Hermitian if $p_i\langle\psi_i|\rho_q^{-\frac{1}{2}}|\psi_i\rangle$ is constant.

Since $Z = C\sqrt{\rho_q}$,

$$p_i\langle\psi_i|Z^{-1}|\psi_i\rangle = p_i\langle\psi_i|C^{-1}\rho_q^{-\frac{1}{2}}|\psi_i\rangle$$

$$= \frac{1}{C}\, p_i\langle\psi_i|\rho_q^{-\frac{1}{2}}|\psi_i\rangle$$

$$= \frac{1}{C}\cdot C = 1.$$

This is equivalent to $Z = p_i|\psi_i\rangle\langle\psi_i|$. □

In Theorem 3, if $P = Q$, then the condition for optimality in the pretty good measurement is simply the same as the one in the least square measurement given in Theorem 2. In other words, the generalized pretty good measurement implies the least square measurement.

Let $P = \{p_i, |\psi_i\rangle\langle\psi_i|\}_{i=1}^{d}$ be an ensemble of linearly independent pure states in $\mathcal{H}$ as above and let $\rho = \sum_{i=1}^{d} p_i|\psi_i\rangle\langle\psi_i|$. For an orthonormal basis $\{i\}_{i=1}^{d}$ for the Hilbert space $\mathcal{H}$, the Gram matrix associated with the ensemble is given by

$$S := \sum_{i,\, j=1}^{d}\sqrt{p_i p_j}\langle\psi_i|\psi_j\rangle|i\rangle\langle j|$$

or

$$\langle i|S|j\rangle = S_{ij} = \sqrt{p_i p_j}\langle \psi_i | \psi_j \rangle.$$

Define

$$M|i\rangle := \sqrt{p_i}|\psi_i\rangle \quad \text{or} \quad M = \sum_{i=1}^{d} \sqrt{p_i}|\psi_i\rangle\langle i|.$$

Then we may write ρ and S in terms of the matrix M as follows:

$$\begin{aligned}\rho &= \sum_{i=1}^{d} p_i|\psi_i\rangle\langle\psi_i| \\ &= \left(\sum_{i=1}^{d} \sqrt{p_i}|\psi_i\rangle\langle i|\right)\left(\sum_{j=1}^{d} \sqrt{p_j}|j\rangle\langle\psi_j|\right) \\ &= MM^*\end{aligned}$$

and

$$\begin{aligned}S &= \sum_{i,j} \sqrt{p_i p_j}\langle\psi_i|\psi_j\rangle|i\rangle\langle j| \\ &= \left(\sum_{i=1}^{d} \sqrt{p_i}|i\rangle\langle\psi_i|\right)\left(\sum_{j=1}^{d} \sqrt{p_j}|\psi_j\rangle\langle j|\right) \\ &= M^*M.\end{aligned}$$

By the singular value decomposition, there are unitaries U and V such that

$$M = UDV^*,$$

where $D = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_d)$ is the diagonal matrix with diagonal entries λ_i are non-negative real numbers. We have then

$$\rho = MM^* = UD^2U^* \quad \text{and} \quad S = M^*M = VD^2V^*$$

or equivalently

$$\sqrt{\rho} = UDU^* \quad \text{and} \quad \sqrt{S} = VDV^*.$$

Now we may reformulate Theorem 3 using the Gram matrix in the following way:

Theorem 4. *Let* $P = \{p_i, |\psi_i\rangle\langle\psi_i|\}_{i=1}^{d}$ *and* $Q = \{q_i, |\psi_i\rangle\langle\psi_i|\}_{i=1}^{d}$ *be two ensembles of linearly independent pure states and let* $\rho_p = \sum_{i=1}^{d} p_i|\psi_i\rangle\langle\psi_i|$ *and* $\rho_q = \sum_{i=1}^{d} q_i|\psi_i\rangle\langle\psi_i|$. *Define* $E_i = q_i\rho_q^{-\frac{1}{2}}|\psi_i\rangle\langle\psi_i|\rho_q^{-\frac{1}{2}}$. *Then the POVM* $\{E_i\}_{i=1}^{d}$ *is optimal for the minimum-error discriminant problem if* $p_i = \dfrac{q_i C}{\langle i|\sqrt{S}|i\rangle}$ *for all* i.

Proof. Note that

$$\begin{aligned} q_i\langle\psi_i|\rho^{-\frac{1}{2}}|\psi\rangle &= \langle i|M^*\rho^{-\frac{1}{2}}M|i\rangle \\ &= \langle i|VDU^*UD^{-1}U^*UDV|i\rangle \\ &= \langle i|VDV^*|i\rangle \\ &= \langle i|\sqrt{S}|i\rangle. \end{aligned}$$

Thus if $p_i\langle\psi_i|\rho^{-\frac{1}{2}}|\psi_i\rangle = C$, then $\langle\psi_i|\rho^{-\frac{1}{2}}|\psi_i\rangle = \dfrac{C}{p_i}$ and hence

$$q_i\langle\psi_i|\rho^{-\frac{1}{2}}|\psi_i\rangle = \frac{q_i C}{p_i} = \langle i|\sqrt{S}|i\rangle$$

and the result follows. □

4. Geometrically Uniform States

Let $\mathcal{G} = \{U_i\}_{i=0}^{d-1}$ be a finite abelian group of unitary matrices. For simplicity, we let $U_0 = \mathrm{Id}$ and since $\mathcal{G}$ is a group, for all i, $U_i^* = U_i^{-1} = U_j$

for some $j \in \{0, 1, ..., d-1\}$. Also, for all i, j, $U_i U_j = U_k \in \mathcal{G}$ for some $k \in \{0, 1, ..., d-1\}$. The set $\mathcal{S} = \{\rho_i = U_i \rho U_i^* \,|\, U_i \in \mathcal{G}\}$ is called a *geometrically uniform* (*GU*) *states set* generated by the group $\mathcal{G}$ with a single generator ρ. Also, one may define a GU states set in the following way. A quantum state $\{\rho_i\}_{i=1}^{m}$ is a GU set if there is a symmetry represented by a unitary transformation U such that

$$\rho_{i+1} = U\rho_i U^*, \quad U^m = \mathrm{I}, \ \text{ for } i = 1, 2, ..., m.$$

For example, let $\mathcal{H}_d$ be a d-dimensional Hilbert space and let $\{|x_n\rangle\}_{n=0}^{d-1}$ be an orthonormal basis for $\mathcal{H}_d$. Define

$$U = \sum_{m=0}^{d-1} \exp\left[\frac{2\pi i}{d} m\right] |x_m\rangle\langle x_m| \tag{1}$$

then U is a unitary transformation on $\mathcal{H}_d$ and

$$U^n = \sum_{m=0}^{d-1} \exp\left[\frac{2\pi i}{d} mn\right] |x_m\rangle\langle x_m|.$$

This shows that the set $\mathcal{G} = \{U^m = U_m\}_{m=0}^{d-1}$ is a group of unitary transformations. Let

$$|\psi_0\rangle = \sum_{n=0}^{d-1} c_n |x_n\rangle, \quad c_n \in \mathbb{R}.$$

Then by definition of U, we have

$$U|\psi_0\rangle = \sum_{n=0}^{d-1} c_n \exp\left[\frac{2\pi i}{d} n\right] |x_n\rangle.$$

Using the relation we define the state $|\psi_k\rangle$ as follows: for all $k \in \{0, 1, ..., d-1\}$,

$$U^k|\psi_0\rangle = \sum_{n=0}^{d-1} c_n \exp\left[\frac{2\pi i}{d} nk\right]|x_m\rangle := |x_k\rangle. \tag{2}$$

We now consider an ensemble $P = \left\{\frac{1}{d}, |\psi_i\rangle\langle\psi_i|\right\}_{i=0}^{d-1}$ of linearly independent pure states in $\mathcal{H}_d$ and the ensemble will be referred as GU ensemble. Let $\rho = \sum_{i=0}^{d-1}\frac{1}{d}|\psi_i\rangle\langle\psi_i|$. Then we have

$$\rho = \sum_{i=0}^{d-1}\frac{1}{d}|\psi_i\rangle\langle\psi_i| = \frac{1}{d}\sum_{i=0}^{d-1}U_i|\psi_0\rangle\langle\psi_0|U_i^*$$

and

$$\begin{aligned}\rho U_k &= \sum_{i=0}^{d-1}\frac{1}{d}U_i|\psi_0\rangle\langle\psi_0|U_i^*U_k \\ &= U_k\sum_{i=0}^{d-1}\frac{1}{d}U_k^*U_i|\psi_0\rangle\langle\psi_0|U_i^*U_k \\ &= U_k\sum_{j=0}^{d-1}\frac{1}{d}U_j|\psi_0\rangle\langle\psi_0|U_j^* \\ &= U_k\rho\end{aligned}$$

thus U_k and ρ commute and hence $\rho^{-\frac{1}{2}}$ commutes with U_j for all j. The least square measurement operators are $E_i = |\mu_i\rangle\langle\mu_i|$ with

$$|\mu_i\rangle = \frac{1}{\sqrt{d}}\rho^{-\frac{1}{2}}|\psi_i\rangle = \frac{1}{\sqrt{d}}\rho^{-\frac{1}{2}}U_i|\psi_0\rangle. \tag{3}$$

Now we apply these measurements to the GU ensemble associated with the group $\mathcal{G} = \{U^m = U_m\}_{m=0}^{d-1}$ defined in (1). For the GU states, by (3) we define

$$|\mu_j\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} \frac{1}{c_n} \exp\left[\frac{2\pi i}{d} nj\right] |x_n\rangle. \tag{4}$$

In order to show that the measurement defined by $|\mu_j\rangle$ given in (4) is optimal, we only have to show that $\frac{1}{d}\langle\mu_i|\psi_i\rangle$ is constant by Theorem 2 or Theorem 3. From the definition of μ_i in (3) and ψ_i in (2), we have

$$\begin{aligned}\langle\mu_i|\psi_i\rangle &= \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} \frac{1}{c_n} \exp\left[-\frac{2\pi i}{d} nj\right] c_n \exp\left[\frac{2\pi i}{d} nj\right] \langle x_n|x_n\rangle \\ &= \frac{1}{\sqrt{d}}.\end{aligned}$$

Thus, $\frac{1}{d}\langle\mu_i|\psi_i\rangle$ is constant and hence $E_i = |\mu_i\rangle\langle\mu_i|$ are optimal.

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