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SOME PROPERTIES OF STRONGLY PARACOMPACT MAPPINGS

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Abstract

In this paper, we continue our study of strongly paracompact mappings, see [2]. More properties of strongly paracompact mappings are obtained. It is proved that paracompact locally compact regular mappings are strongly paracompact. And every paracompact locally compact mapping is a disjoint union of closed and open Lindelöf mappings.

1. Preliminaries

Definitions and theorems mentioned in this section can be found in [1, 3, 6, 8], for more details one can consult them.

Unless otherwise stated, Y is a fixed topological space with topology τ, collection of all neighborhoods (nbd(s)) of $y \in Y$ is denoted by $N(y)$. If $f: X \to Y$ is a mapping and $A \subseteq X$ and $B \subseteq X$, then $[A]_B$ means the closure of A in B. For continuous mappings $f: X \to Y$ and $g: Z \to Y$, a continuous mapping $\lambda: X \to Z$ such that $f = g\lambda$ is called a morphism of f into g and is denoted by $\lambda: f \to g$. λ is called surjective, closed, perfect

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surjective, closed, perfect etc., respectively. Open covers will be denoted by $\hat{U}, \hat{V}, \hat{W}, \ldots..$

Definition 1.1. A mapping $f : X \to Y$ is called a T_i*-mapping*, $i = 0$, 1, 2, if for every $x, x^* \in f^{-1}y$ such that $x \neq x^*$ the following conditions are, respectively, satisfied:

(1) $i = 0$: At least one of the points x and x^* has a nbd in X not containing the other point.

(2) $i = 1$: Each of the points x and x^* has a nbd in X not containing the other point.

(3) $i = 2$: The points x and x^* have disjoint nbds in X.

Definition 1.2. If A and B are subsets of X, then we say that A and B are

(1) *nbd separated* in $U \subseteq X$,

(2) *functionally separated* in $U \subseteq X$,

if, respectively, the sets $A \cap U$ and $B \cap U$,

(1) have disjoint nbds in U.

(2) there exists a continuous function $f : U \to [0, 1]$ such that $A \cap U \subseteq f^{-1}(0)$ and $B \cap U \subseteq f^{-1}(1)$.

Definition 1.3. A mapping $f : X \to Y$ is said to be *completely regular* (*regular*) if for every $x \in X$ and every closed set F in X such that $x \notin F$ there exists a neighborhood $O \in N(fx)$ such that $\{x\}$ and F are functionally separated (nbd separated) in $f^{-1}O$.

A completely regular (regular) T_0-mapping is called a *Tychonoff* or $T_{3\frac{1}{2}}$-*mapping* (*regular* or T_3*-mapping*).

Definition 1.4. A mapping $f : X \to Y$ is called *functionally prenormal* (*prenormal*) if for every $y \in Y$ and every disjoint closed (in X) sets F and H there exists a neighborhood O of y such that F and H are functionally separated (nbd separated) in $f^{-1}O$. If for every open subset O of Y the mapping $f|_{f^{-1}O} : f^{-1}O \to O$ is functionally prenormal (prenormal), then f is called *functionally normal* (*normal*). A normal T_3-mapping is called T_4*-mapping*.

A mapping $g : A \to B$ is said to be a (*closed*, *open*, *dense*, etc.) *submapping* of the mapping $f : X \to Y$ if g is the restriction of f on the (closed, open, dense, etc.) subset A of the space X and $g(A) = f(A) \subseteq B \subseteq Y$. A mapping $f : X \to Y$ is said to be *compact* if and only if f is perfect. If $f : X \to Y$ is a compact T_2-mapping and $g : A \to B$ is a submapping of f where B is a closed subset of Y, then g is compact.

Definition 1.5. For the collection of mappings $f_\alpha : X_\alpha \to Y$ such that $\alpha \in \Delta$, the subspace $P = \bigcup_{y \in Y} \prod_{\alpha \in \Delta} f_\alpha^{-1} y$ of the Tychonoff product $\prod = \prod_{\alpha \in \Delta} X_\alpha$ is called the *fan product* of the spaces $\{X_\alpha;\ \alpha \in \Delta\}$.

Definition 1.6. For the collection of mappings $f_\alpha : X_\alpha \to Y$ such that $\alpha \in \Delta$, if $p_\alpha : \prod \to X_\alpha$ is the projection of $\prod$ onto X_α, then the restriction π_α of p_α on P, where P is the fan product of the spaces $\{X_\alpha;\ \alpha \in \Delta\}$, is called the *projection* of the fan product onto the factor X_α.

Definition 1.7. Let P be the fan product of the spaces $\{X_\alpha;\ \alpha \in \Delta\}$. If $p : P \to Y$ is defined such that $p = f_\alpha \pi_\alpha;\ \alpha \in \Delta$, then p is called the *projection* of the fan product. If $\{f_\alpha;\ \alpha \in \Delta\} = \{f, g\}$ such that $f : X \to Y$ and $g : Z \to Y$, then we write $X_f \times_g Z$ to denote the fan product of the spaces X and Z, and π_α will be denoted by π_X or π_Z.

Theorem 1.8. *For any continuous mapping* $f : X \to Y$ *the following conditions are equivalent*:

(1) *f is compact.*

(2) *For every continuous mapping* $g : Z \to Y$ *the projection* π_Z *is closed.*

(3) *If* $g : Z \to Y$ *is functionally normal, then the projection* π_Z *is closed.*

Definition 1.9. A mapping $f : X \to Y$ is called *paracompact* if for every $y \in X$ and every open (in X) cover $\hat{U} = \{U_\alpha;\ \alpha \in \Delta\}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}y$ is covered by $\hat{U}$ and $(f^{-1}O_y \wedge \hat{U})$ has an open (in X) y-locally finite refinement in $f^{-1}O_y$.

Theorem 1.10. (1) *Every compact mapping is paracompact.*

(2) *Every paracompact Hausdorff mapping is* T_4.

(3) *Every closed submapping of a paracompact mapping is paracompact.*

Theorem 1.11. *If* $f : X \to Y$ *is a regular mapping, then the following conditions are equivalent*:

(1) *f is paracompact and* T_2.

(2) *For every* $y \in Y$ *and every open (in* X*) cover* $\hat{U}$ *of* $f^{-1}y$ *there exists* $O_y \in N(y)$ *such that* $f^{-1}y$ *is covered by* $\hat{U}$ *and* $(f^{-1}O_y \wedge \hat{U})$ *has an open (in* X*)* σ*-locally finite refinement* $\hat{V}$ *in* $f^{-1}O_y$*; that is* $\hat{V} = \bigcup_{i<\omega} \hat{V}_i$, *where* $\hat{V}_i$ *is locally finite in* $f^{-1}O_y$ *for every* $i < \omega$.

Definition 1.12. A mapping $f : X \to Y$ is called *Lindelöf* if *f* is closed and for every $y \in Y$ we have $f^{-1}y$ is Lindelöf.

In [6] Lindelöf mappings are assumed regular.

Definition 1.13. A mapping $f : X \to Y$ is called *countably paracompact* if for every $y \in Y$ and every countable open (in X) cover $\hat{U}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}y$ is covered by $\hat{U}$ and $f^{-1}O_y \wedge \hat{U}$ has an open (in X) y-locally finite refinement in $f^{-1}O_y$.

Theorem 1.14. *Let $f : X \to Y$ be a mapping. Then the following two conditions are equivalent*:

(1) *f is countably paracompact.*

(2) *For every $y \in Y$ and every countable open* (*in X*) *cover $\hat{U}$ of $f^{-1}y$ there exists a neighborhood $O_y \in N(y)$ such that $f^{-1}O_y$ is covered by $\hat{U}$ and $f^{-1}O_y \wedge \hat{U}$ has an open* (*in X*) *locally finite refinement in $f^{-1}O_y$.*

Theorem 1.15. *Let $f : X \to Y$ be a mapping. Then the following two conditions are equivalent*:

(1) *f is normal and countably paracompact.*

(2) *For every $y \in Y$ and every countable open* (*in X*) *cover $\hat{U} = \{U_i;\ i \in N\}$ of $f^{-1}y$ there exists a neighborhood $O_y \in N(y)$ such that $f^{-1}O_y$ is covered by $\hat{U}$; furthermore for every $i = 1, 2, \ldots$ there exists $O_{i(y)} \in N(y)$, where $O_{i(y)} \subseteq O_y$, and there exists a closed* (*in $f^{-1}O_{i(y)}$*) *subset $F_i \subseteq f^{-1}O_{i(y)} \cap U_i$ such that $f^{-1}O_y = \bigcup_{i=1}^{\infty} F_i$.*

A fibrewise version of Dowker's Theorem is introduced in [2].

Theorem 1.16 [2]**.** *If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are any two mappings such that $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is closed and $f_2 : X_2 \to Y_2$ is a compact MT-mapping with at least one infinite fiber, then*

$f_1 : X_1 \to Y_1$ is normal countably paracompact if and only if $f_1 \times f_2$ is normal.

2. Properties of Strongly Paracompact Mappings

Definition 2.1. Let $f : X \to Y$ be a mapping. For every $y \in Y$ the open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha\in\Delta}$ of $f^{-1}y$ is said to be *y-starfinite* if for every $x \in f^{-1}y$ there exists $U_\alpha \in \hat{U}$ such that $x \in U_\alpha$ and the family $S(U_\alpha, \hat{U}) = \{U_\beta \in \hat{U}; U_\beta \bigcap U_\alpha \neq \Phi\}$ is finite.

Definition 2.2 [3]. A mapping $f : X \to Y$ is called *strongly paracompact* if for every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha\in\Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a y-starfinite open (in X) refinement in $f^{-1}O_y$.

Theorem 2.3 [3]. (1) *Every strongly paracompact mapping is paracompact.*

(2) *Every strongly paracomapct Hausdorf mapping is normal.*

Theorem 2.4 [3]. *If $f : X \to Y$ is a functionally normal mapping, then the following conditions are equivalent.*

(1) *f is strongly paracompact.*

(2) *For every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha\in\Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a y-closed starfinite locally finite refinement in $f^{-1}O_y$.*

(3) *For every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha\in\Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a starcountable y-closed locally finite refinement in $f^{-1}O_y$.*

(4) *For every* $y \in Y$ *and every open* (*in* X) *cover* $\hat{U} = \{U_\alpha\}_{\alpha\in\Delta}$ *of* $f^{-1}y$ *there exists* $O_y \in N(y)$ *such that* $f^{-1}O_y \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$ *and* $\hat{U} \wedge f^{-1}O_y$ *has a starcountable open* (*in* X) *refinement in* $f^{-1}O_y$.

Corollary 2.5. *Every Lindelöf functionally normal mapping is strongly paracomapct*.

If the fibres of a mapping f are strongly paracompact spaces, then f is not necessarily a strongly paracompact mapping, even if f is closed and Tychonoff.

Example 2.6. Let L be the Niemytzki plane and let L_1 be the line $y = 0$. Then L_1 is closed in L and so the quotient mapping $q : L \to L/L_1$ is closed. Since L is Tychonof space, q is Tychonof. And since each fibre of q is discrete, the fibres of q are strongly paracompact space. Since there exists two closed (in L) subsets $A \subset L_1$ and $B \subset L_1$ which are not nbd separated, we have q is not a prenormal mapping, and so cannot be strongly paracompact.

In the category *Top* arbitrary disjoint sum of strongly paracompact spaces is strongly paracompact. This is true for finite sum in the category Top_Y; more precisely.

Theorem 2.7. *Let* $f_i : X_i \to Y$ *be a strongly paracompact mapping for every* $i = 1, 2, ..., n$. *Then* $f : \coprod_{i=1}^n X_i \to Y$, *where* $f = \coprod_{i=1}^n f_i$, *is strongly paracompact*.

Proof. Let $y \in Y$ and $\hat{U} = \{U_\alpha\}_{\alpha\in\Delta}$ be an open $\left(\text{in } \coprod_{i=1}^n X_i\right)$ cover of $f^{-1}y$. For every $i \in \{1, 2, ..., n\}$ the family $\hat{U}_i = \{U_\alpha \cap X_i\}_{\alpha\in\Delta}$ is an open (in X_i) cover of $f_i^{-1}y$. Since for every $i = 1, 2, ..., n$, f_i is strongly paracompact, there exists $O_{yi} \in N(y)$ such that $f^{-1}O_{yi} \subseteq \bigcup \hat{U}_i$ and

$\hat{U}_i \wedge f_i^{-1} O_{yi}$ has a starfinite open (in X_i) refinement $\hat{V}_i$ in $f_i^{-1} O_{yi}$. Set $O_y = \bigcap_{i=1}^{n} O_{yi}$ and $\hat{V} = \bigcup_{i=1}^{n} \hat{V}_i$. Then $\hat{V} \cap f^{-1} O_y$ is a starfinite open $\left(\text{in} \coprod_{i=1}^{n} X_i\right)$ refinement of $\hat{U} \wedge f^{-1} O_y$ in $f^{-1} O_y$; that is f is strongly paracompact. □

In the category *Top* strongly paracompactness is an inverse invariant of perfect mappings and it is an invariant of open perfect mappings but it is not an invariant of perfect mappings. We will prove the same in the category Top_Y.

Firstly, consider any perfect mapping $p : X \to Y$ such that X is strongly paracompact and Y is not strongly paracompact. Let $f : X \to \{a\}$ and $g : Y \to \{a\}$ such that $fx = gy = a$ for every $x \in X$ and $y \in Y$. It is clear that f is strongly paracompact and g is not, and p is a perfect morphism of f onto g; hence being strongly paracompact mapping is not an invariant of perfect morphisms.

Theorem 2.8. *If* $\lambda : X \to Z$ *is an open perfect morphism of a strongly paracompact mapping f onto g, where* $f : X \to Y$ *and* $g : Z \to Y$, *then g is strongly paracompact.*

Proof. It is clear that g is closed. Let $y \in Y$ and let $\hat{W} = \{V_\alpha\}_{\alpha \in \Delta}$ be an open (in Z) cover of $g^{-1} y$. Then $\hat{U} = \{\lambda^{-1} V_\alpha\}_{\alpha \in \Delta}$ is an open (in X) cover of $f^{-1} y$, so there exists $O_y \in N(y)$ such that $f^{-1} O_y \subseteq \bigcup_{\alpha \in \Delta} \lambda^{-1} V_\alpha$ and $\hat{U} \wedge f^{-1} O_y$ has a starfinite open (in X) refinement in $f^{-1} O_y$, say $\hat{W}^* = \{V_\beta ; \beta \in B\}$. For every $n \in N$ and every $\beta_1, \beta_2, \ldots, \beta_n \in B$ set

$$U(\beta_1, \beta_2, \ldots, \beta_n) = \left(\bigcap_{i=1}^{n} \lambda V_{\beta i}\right) \bigcap \left(g^{-1} O_y - \lambda\left(X - \bigcup_{i=1}^{n} V_{\beta i}\right)\right).$$

Since λ is open and closed, $U(\beta_1, \beta_2, ..., \beta_n)$ is open in Z for every $n \in N$. Let $\hat{W}^{**}$ be the family of all such sets. It is clear that $\hat{W}^{**}$ is an open refinement of $\hat{W}$. We shall show that $\hat{W}^{**}$ is a cover of $g^{-1}y$. Let $z \in g^{-1}y = \lambda f^{-1}y$. Then $\lambda^{-1}z$ is a compact subset of $f^{-1}y$, so that $\hat{W}^*$ is an open (in X) cover of $\lambda^{-1}z$; hence there exists a natural number n such that $\lambda^{-1}z$ is covered by $V_{\beta 1}, V_{\beta 2}, ..., V_{\beta n}$, we can assume that $V_{\beta i}$ intersects $\lambda^{-1}z$, for every i, so that $z \in \lambda V_{\beta i}$ for every i. Since $\lambda^{-1}z \subseteq \bigcup_{i=1}^{n} V_{\beta i}$, we have $z \in \left(\bigcap_{i=1}^{n} \lambda V_{\beta i}\right) \bigcap \left(g^{-1}O_y - \lambda\left(X - \bigcup_{i=1}^{n} V_{\beta i}\right)\right) = U(\beta_1, \beta_2, ..., \beta_n)$; which means $\hat{W}^{**}$ covers $g^{-1}z$. Now, we will show that $\hat{W}^{**}$ is starfinite. Let $U(\beta_1, \beta_2, ..., \beta_n) \in \hat{W}^{**}$. For each $i = 1, 2, ..., n$ let Δ_i be a finite subset of B such that $V_\beta \cap V_{\beta i} \neq \Phi$ if and only if $\beta \in \Delta_i$. Set $\Delta^* = \bigcup_{i=1}^{n} V_i$; it is clear that Δ^* is finite: Now, we shall show that $U(\beta_1, \beta_2, ..., \beta_n) \bigcap U(\alpha_1, \alpha_2, ..., \alpha_m) = \Phi$ for every $U(\alpha_1, \alpha_2, ..., \alpha_m) \in \hat{W}^{**}$ such that $\alpha_s \notin \Delta^*$ for some $s \in \{1, 2, ..., m\}$. It is clear that if $x \in V_{\alpha_s}$, then $\lambda(x) \notin g^{-1}O_y - \lambda\left(X - \bigcup_{i=1}^{n} V_{\beta i}\right)$; which implies that $\lambda V_{\alpha_s} \cap U(\beta_1, \beta_2, ..., \beta_n) = \Phi$, so that $U(\alpha_1, \alpha_2, ..., \alpha_m) \subseteq \lambda V_{\alpha_s}$ and $U(\beta_1, \beta_2, ..., \beta_n) \cap U(\alpha_1, \alpha_2, ..., \alpha_m) = \Phi$, which means; if $U(\beta_1, \beta_2, ..., \beta_n) \bigcap U(\alpha_1, \alpha_2, ..., \alpha_m) \neq \Phi$, then $\alpha_1, \alpha_2, ..., \alpha_m \in \Delta^*$, and $\hat{W}^{**}$ is a starfinite open (in Z) cover of $g^{-1}y$. □

Now, we will show that strongly paracompact mappings are inverse invariants of perfect mappings.

Theorem 2.9. *If $\lambda : X \to Z$ is a perfect morphism of f onto the strongly* *paracompact mapping g such that* $f : X \to Y$ *and* $g : Z \to Y$, *then f is strongly paracompct*.

Proof. It is clear that f is closed. Let $y \in Y$ and $\hat{V} = \{V_\alpha\}_{\alpha \in \Delta}$ be an open (in X) cover of $f^{-1}y$. Then for each $z \in g^{-1}y$ we have $\lambda^{-1}z$ is a compact subset of $f^{-1}y$ covered by $\hat{V}$. Since λ is perfect, there exists U_z (an open subset of Z) such that $\lambda^{-1}U_z$ is covered by a finite subfamily of $\hat{V}$. Set $\hat{U} = \{U_z; z \in g^{-1}y\}$. Since $\hat{U}$ is an open (in Z) cover of $g^{-1}y$ and g is strongly paracompact, there exists $O_y \in N(y)$ such that $g^{-1}O_y$ is covered by $\hat{U}$ and $\hat{U} \wedge g^{-1}O_y$ has a starfinite open (in Z) refinement $\hat{V} = \{V_\beta;$ $\beta \in B\}$ in $g^{-1}O_y$. The family $\{\lambda^{-1}V_\beta; \beta \in B\}$ is a starfinite open (in X) cover of $f^{-1}y$. But for every $\beta \in B$ there exists U_z such that $z \in g^{-1}y$, V_β is contained in U_z and there exists a finite subset Δ_z of Δ such that U_z is contained in $\bigcup\{V_\alpha; \alpha \in \Delta_z\}$. Let $\hat{V}_\beta = \{V_\alpha \cap \lambda^{-1}V_\beta; \alpha \in \Delta_z\}$. Then $\hat{V}^{**} = \bigcup\{\hat{V}_\beta, \beta \in B\}$ is a starfinite open (in X) refinement of $\hat{V} \wedge f^{-1}O_y$ in $f^{-1}O_y$. And this completes the proof. □

Theorem 2.10. *If* $f : X \to Y$ *is a perfect mapping and* $g : Z \to Y$ *is a strongly paracompact mapping*, *then* $p(= f \times g) : P(= X_f \times_g Z) \to Y$ *is strongly paracompact*.

Proof. The projection mapping $\Pi_Z : X_f \times_g Z \to Z$ is a perfect morphism of p onto g. If g is a strongly paracompact mapping, then, by the previous theorem, p is a strongly paracompact mapping. □

Next, we will show that strongly paracompact mappings and paracompact mappings are equivalent within the realm of locally compact mappings.

Theorem 2.11. *If* $f : X \to Y$ *is a regular locally compact mapping, then the following two conditions are equivalent*:

(1) *f is strongly paracomapct*.

(2) *f is paracompact*.

Proof. (1) $\to$ (2) is obvious!

(2) $\to$ (1) Let $\hat{U} = \{U_\alpha\}_{\alpha \in \Delta}$ be an open (in X) cover of $f^{-1}y$. For every $x \in f^{-1}y$ there exists an open (in X) neighborhood W_x of x and $O_{y(x)} \in N(y)$ such that $[W_x]_{f^{-1}O_{y(x)}} \subseteq U_{\alpha(x)} \cap f^{-1}O_y$ for some $\alpha(x) \in \Delta$ and the restriction $f_x : [W_x]_{f^{-1}(O_{y(x)})} \to Y$ of f is a compact mapping. Let $\hat{W} = \{W_x;\, x \in f^{-1}y\}$. Then $\hat{W}$ is an open cover of $f^{-1}y$ and since f is paracompact, there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{W}$ and $\hat{W} \wedge f^{-1}O_y$ has a locally finite open (in X) refinement $\hat{V}$ in $f^{-1}O_y$. For every $x \in f^{-1}y$ let $O^*_{y(x)} = O_y \cap O_{y(x)}$. Now, for every $V \in \hat{V}$ there exists $x \in f^{-1}y$ such that $V \subseteq W_x \cap f^{-1}O_y$, so that $[V]_{f^{-1}O^*_{y(x)}} \subseteq U_{\alpha(x)} \cap f^{-1}O_y$. Since $f_x : [W_x]_{f^{-1}O_{y(x)}} \to Y$ is compact and $O^*_{y(x)} \subseteq O_{y(x)}$, we have the restriction $f_V : [V]_{f^{-1}O^*_{y(x)}} \to Y$ of f is compact. For every $t \in f_V^{-1}y$ there exists an open (in X) neighborhood H_t of t such that H_t intersects finitely many elements of $\hat{V}$, so that the family $\hat{H}_V = \{H_t \cap [V]_{f^{-1}O^*_{y(x)}};\, t \in f_V^{-1}y\}$ is an open (in $[V]_{f^{-1}O^*_{y(x)}}$) cover of $f_V^{-1}y$. Since f_V is compact, there exists $O_V \in N(y)$ such that $f^{-1}O_V$ is contained in finitely many elements of $\hat{H}_V$. Let $V^* = V \cap f^{-1}O_V$ and $\hat{V}^* = \{V^*;\, V \in \hat{V}\}$. Then $\hat{V}^*$ is a refinement of $\hat{U}$. But every element of

$\hat{V}^*$ is contained in finitely many elements of $\hat{H}$ and every element of $\hat{H}$ intersects finitely many elements of $\hat{V}$, so that every element of $\hat{V}^*$ intersects finitely many elements of $\hat{V}^*$ i.e. $\hat{V}^*$ is a starfinite open (in X) cover of $f^{-1}y$. Since f is closed, there exists $O_y^* \in N(y)$ such that $f^{-1}O_y^* \subseteq \bigcup \hat{V}^*$; hence $f^{-1}O_y^* \wedge \hat{V}^*$ is a starfinite open (in X) refinement of $f^{-1}O_y^* \wedge \hat{U}$ in $f^{-1}O_y^*$. This completes the proof. □

We know that every paracompact locally compact space is a disjoint union of Lindelöf closed and open subsets. The following theorem shows the corresponding statement in the category Top_Y is true.

Theorem 2.12. *If* $f : X \to Y$ *is a paracomapct locally compact mapping, then there exists* $O_y \in N(y)$ *such that* $f^{-1}O_y = \bigcup_{t \in T} C_i$, *where* $\{C_t\}_{t \in T}$ *is a family of pairwise disjoint closed and open subsets of* $f^{-1}O_y$, *and for every* $t \in T$ *the restriction* $f_t : C_t \to Y$ *of f on* C_t *is Lindelöf.*

Proof. Since f is locally compact and paracompact, f is strongly paracompact (by the previous theorem). For every $x \in X$ there exists an open (in X) neighborhood W_x of x and $O_{y(x)} \in N(y)$ such that the restriction $f_x : [W_x]_{f^{-1}O_{y(x)}} \to Y$ of f is compact. Let $\hat{W} = \{W_x;\ x \in X\}$. Then $\hat{W}$ is an open cover of $f^{-1}y$, so that there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{W}$ and $\hat{W} \wedge f^{-1}O_y$ has a starfinite open (in X) refinement $\hat{V}$ in $f^{-1}O_y$. We can write $\hat{V} = \bigcup_{t \in T} \hat{V}_t$ such that $\{\hat{V}_t;\ t \in T\}$ is the set of all components of $\hat{V}$ and for every $t \in T$ we have $\hat{V}_t = \{V_{t,i};\ i \in N\}$ such that $V_{t,i}$ is an open subset of X for every $i \in N$ and $t \in T$. Let $C_t = \bigcup_{i \in N} V_{t,i}$. Then C_t is a closed and open subset of $f^{-1}O_y$ and the family $\{C_t\}_{t \in T}$ is a cover of $f^{-1}O_y$ consisting of closed and open pairwise disjoint subsets of X.

To complete the proof it suffices to show that for every $t \in T$ the restriction $f_t : C_t \to Y$ of f on C_t is Lindelöf. It is clear that f_t is paracompact. Since C_t is a closed subset of $f^{-1}O_y$, we have $C_t = [C_t]_{f^{-1}O_y} = \left[\bigcup_{i \in N} V_{t,i}\right]_{f^{-1}O_y} = \bigcup_{i \in N} [V_{t,i}]_{f^{-1}O_y}$. For every $i \in N$ let $f_{t,i} : [V_{t,i}]_{f^{-1}O_{y(x)}} \to Y$ be the restriction of f on $[V_{t,i}]_{f^{-1}O_{y(x)}}$. For every $i \in N$ there exists $x \in X$ such that $V_{t,i} \subseteq W_x$; which implies that $[V_{t,i}]_{f^{-1}O_{y(x)}} \subseteq [W_x]_{f^{-1}O_{y(x)}}$, and since $f_x : [W_x]_{f^{-1}(O_{y(x)})} \to Y$ is a compact mapping for every $x \in X$, we have $f_{t,i}$ is compact. Let $\hat{U}$ be an open cover of $f_t^{-1}y$. Then $\hat{U}$ is an open cover of $f_{t,i}^{-1}y$ for every $i \in N$. Since $f_{t,i}$ is compact, there exists $O_i \in N(y)$ such that $f_{t,i}^{-1}O_i \subseteq \bigcup \hat{U}_i$, where $\hat{U}_i$ is a finite subfamily of $\hat{U}$. It is clear that $\hat{V} = \bigcup\{\hat{U}_i;\, i \in N\}$ is a countable subfamily of $\hat{U}$ and covers $f_t^{-1}y$. Finally, since f_t is closed, there exists $O_y^* \in N(y)$ such that $f_t^{-1}O_y^* \subseteq \bigcup \hat{V}$; which means f_t is Lindelöf. And this completes the proof.

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