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ON THE CURVATURE OF FINSLER MANIFOLD UNDER PROJECTIVE TRANSFORMATION

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Abstract

Using the Grifone formalism of the connection Γ [8] and the notations of Frölicher-Nijenhuis [4], for a Finsler manifold we establish that there exists a tensor W invariant under the projective transformation and that the curvature R of the homogenous Grifone-Ehersmann connection $\Gamma = [J, S]$ can be written as $R = \omega \wedge J + \pi \otimes C,$ if and only if the tensor W vanishes, where ω and π are semi-basic forms, J is the vertical endomorphism and C is the Liouville vector field.

0. Introduction

In this work we use the terminology and the notations of Frölicher-Nijenhuis [4] and the results given by Grifone [8]. In fact the terminology of Frölicher-Nijenhuis [1, 4] is efficient in the calculus of derivation and bracket of differential vectorial form. (M, E) is a Finsler manifold where M is a differential manifold of dimension n, $E : TM \to \mathbb{R}$ is a homogenous Lagrangian function of degree 2 [8], and the rank of E is $2n$ at every point of

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TM such that $E(0) = 0$, C^{∞} on $TM \setminus \{0\}$ and C^1 at the section 0. The canonical spray S defined by $i_S dd_J E = -dE$ is homogenous of degree 2 [8, 9]. The integral curve of S is a solution of the Euler-Lagrange equation $\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{x}^i}\right) - \frac{\partial E}{\partial x^i} = 0$. The projective transformation of partial differential equation has been studied in [10]. But in this paper, we define the projective transformation in the Grifone terminology, by the vertical translation of the spray S, how it defines the differential equation. The projective transformation is $S + \lambda C$, where λ is a homogenous function of degree 1 and C is the vertical canonical vector field. With this canonical spray, we associate a homogenous Grifone-Ehersmann connection $\Gamma = [J, S]$, where J is the almost tangent structure on M. The geodesics of Γ are the integral curves of the spray S. After introducing the notion of the derivation of the differential form and studying necessary relations between the inner product, the exterior derivation and the bracket of derivations, we provide:

(1) An extension of the notation of a trace to a semi-basic vectorial form.

(2) The relation between the curvature R of the connection $[J, S]$ and the new curvature R_* of the new connection $\Gamma_* = [J, S + \lambda C]$ obtained under a projective transformation.

(3) The existence of a tensor field W invariant under the projective transformation.

(4) The proof that the tensor W vanishes if and only if the curvature R is written as $R = \omega \wedge J + \pi \otimes C$, where ω and π are semi-basic forms, $i_{JX}\omega = 0$ and $i_{JX}\pi = 0$, and ω (resp. π) is a 1 (resp. 2) scalar tensor field on TM.

1. Derivations

Let M be a manifold of dimension n. We denote by $\Lambda(M)$ the graduate algebra of the skew-symmetric differential forms on M.

Definition 1.0.1. A mapping $D : \Lambda(M) \to \Lambda(M)$ is called *derivation of degree r* of $\Lambda(M)$, if:

(i) $D(\Lambda^p(M)) \subset \Lambda^{p+r}(M)$;

(ii) $\forall \omega_1, \omega_2 \in \Lambda^p(M)$; $D(\omega_1 + \omega_2) = D(\omega_1) + D(\omega_2)$;

(iii) $\forall k \in \mathbb{R}$ $Dk = 0$;

(iv) $\forall \alpha \in \Lambda^p(M)$, $\beta \in \Lambda^r(M)$;

$$D(\alpha \wedge \beta) = (D\alpha) \wedge \beta + (-1)^{pr} \alpha \wedge (D\beta).$$

The bracket of two derivations D_1 and D_2 of degrees r_1 and r_2 is a derivation of degree $r_1 + r_2$ on $\Lambda(M)$ given by: $[D_1, D_2] = D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1$. The Jacobi identity is

$$(-1)^{r_1 r_3}[D_1, [D_2, D_3]] + (-1)^{r_2 r_1}[D_2, [D_3, D_1]] + (-1)^{r_3 r_2}[D_3, [D_1, D_2]] = 0.$$

D is completely defined by the action on $\Lambda^0(M)$ and $\Lambda^1(M)$ only. Every derivation of degree $r \leq -2$ vanishes.

1.1. Exterior differentiation

Definition 1.1.1. The *exterior differentiation* is a derivation d of degree 1 such that for every $\omega \in \Lambda^p(M)$,

$$d\omega(X_1, ..., X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} X_i.\omega(X_1, ..., \hat{X}_i, ..., X_{p+1})$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{p+1})$$

If $p = 0$, then $df(X) = X.f$.

Note that $d \circ d = 0$ and $[d, d] = 0$.

1.2. Inner product

Definition 1.2.1. Let $\alpha \in \wedge^p T_m M$, $\beta \in \wedge^q T_m M$ and $L \in \vee^\ell(M)$. Then the exterior product of two scalar forms α and β is the $p + q$*-exterior form* $\alpha \wedge \beta$ defined by

$$\alpha \wedge \beta(X_1, \ldots, X_{p+q})$$

$$= \frac{1}{p!q!}\sum_\sigma \varepsilon(\sigma)\alpha(X_{\sigma(1)}, \ldots, X_{\sigma(p)})\beta(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}).$$

We generalize this operation to mixed product:

$$\alpha \wedge L(X_1, \ldots, X_{p+\ell})$$

$$= \frac{1}{p!\ell!}\sum_\sigma \varepsilon(\sigma)\alpha(X_{\sigma(1)}, \ldots, X_{\sigma(p)})L(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+\ell)}).$$

Definition 1.2.2. The inner product i_L of differential form with respect to the ℓ-vectorial form L is a *derivation of degree* $\ell - 1$ defined by:

(i) $\forall f \in \Lambda^0(M) \quad i_L(f) = 0,$

(ii) $\forall \omega \in \Lambda^p(M) \quad i_L(\omega) = \omega \barwedge L$ where

$$(\omega \barwedge L)(X_1, \ldots, X_{p+\ell-1})$$

$$:= \frac{1}{(p-1)!\ell!}\sum_\sigma \varepsilon(\sigma)\omega(L(X_{\sigma(1)}, \ldots, X_{\sigma(\ell)}), X_{\sigma(\ell+1)}, \ldots, X_{\sigma(p+\ell-1)}),$$

where σ belongs to the set of permutations of $p + \ell - 1$ elements.

There exists an extension of the inner product defined above to the q-vectorial form Q. In fact, $i_L(Q) = Q \barwedge L$ and $(Q \barwedge L)(X_1, \ldots, X_{q+\ell-1}) :=$ $\frac{1}{(q-1)!\ell!}\sum_\sigma \varepsilon(\sigma)Q(L(X_{\sigma(1)}, \ldots, X_{\sigma(\ell)}), X_{\sigma(p+1)}, \ldots, X_{\sigma(\ell+q-1)}).$

If $q = 0$, then $Q \barwedge L = 0$.

Note that $[i_L, i_P] = i_L i_P - (-1)^{(\ell-1)(p-1)} i_P i_L$.

Proposition 1.2.1. *The form* $\omega \wedge D$ *is a derivation of degree equal to the sum of the degree of the scalar form* ω *and the degree of the derivation* $D : i_{\omega \wedge L} = \omega \wedge i_L$ [1], $i_{\omega \wedge L}$ *and* $\omega \wedge i_L$ *are inner products*. *If* α *is a* 1-*scalar form*, *then* $i_{\omega \wedge L}\alpha = \alpha(\omega \wedge L) = \omega \wedge (\alpha \circ L) = \omega \wedge i_L \alpha$.

Proposition 1.2.2. *If L and P are*, *respectively*, ℓ *and p vectorial forms*, *then* [2, 3],

$$[i_L, i_P] = i_{P \barwedge L} - (-1)^{(\ell-1)(p-1)} i_{L \barwedge P} = i_{i_L P} - (-1)^{(\ell-1)(p-1)} i_{i_P L}.$$

If $\ell = 0$, *we have* $[i_X, i_P] = i_{P \barwedge X} = i_{i_X P}$ [1].

In fact if α *is a* 1-*scalar form*, *then*

$$i_L \circ i_P \alpha = i_L(\alpha \circ P) = (\alpha \circ P) \barwedge L = \alpha \circ (P \barwedge L) = i_{P \barwedge L}\alpha.$$

1.3. Lie derivation

The Lie derivation with respect to a ℓ-vectorial form L is a derivation d_L of $\Lambda(M)$ defined by $d_L = [i_L, d]$. Remark that $[d_L, d] = 0$, and if $d_L = 0$, then $L = 0$. In fact if $\forall f \in C^{\infty}(M)$ $d_L f = 0$, then $\forall f$ $df \circ L = 0 \Rightarrow L = 0$.

Proposition 1.3.1. *If* ω *is a p*-*scalar form and L is a* ℓ-*vectorial form*, *then*

$$d_{\omega \wedge L} = \omega \wedge d_L + (-1)^{(p+\ell)} d\omega \wedge i_L.$$

Proof. $d_{\omega \wedge L}$ and $\omega \wedge d_L$ are derivations of the same degree.

$$\begin{aligned} d_{\omega \wedge L} f &= i_{\omega \wedge L} df = df \circ (\omega \wedge L) = \omega \wedge (i_L df) \\ &= \omega \wedge d_L f - (-1)^{(p+\ell-1)} d\omega \wedge i_L f = \omega \wedge d_L f + (-1)^{(p+\ell)} d\omega \wedge i_L f. \end{aligned}$$

$$\begin{aligned} d_{\omega\wedge L}\alpha &= i_{\omega\wedge L}d\alpha - (-1)^{(p+\ell-1)}di_{\omega\wedge L}\alpha \\ &= \omega\wedge i_L d\alpha - (-1)^{(p+\ell-1)}d((\omega\wedge i_L)\alpha) \\ &= \omega\wedge i_L d\alpha - (-1)^{(p+\ell-1)}(d\omega\wedge i_L\alpha + (-1)^p\omega\wedge di_L\alpha) \\ &= \omega\wedge d_L\alpha + (-1)^{(p+\ell)}(d\omega\wedge i_L\alpha). \end{aligned}$$

1.4. Derivations of type i_*

Definition 1.4.1. A derivation D of $\wedge(M)$ is called of *type* i_* if $\forall f \in C^\infty(M)$, $Df = 0$.

Theorem 1.4.1. *If D is a derivation of type i_* of degree $(\ell - 1)$, then there exists a unique ℓ-vectorial form L such that $i_L = D$,* ([4] *and* [1]).

Proof. In fact, $\forall\omega \in \Lambda^1(M)$, we define L by

$$L(X_1, \ldots, X_\ell, \omega) = \omega L(X_1, \ldots, X_\ell) = (D\omega)(X_1, \ldots, X_\ell).$$

Proposition 1.4.1. *The bracket of two derivations of type i_* is a derivation of type i_** [4, 1].

In fact, if D_1 and D_2 are two derivations of type i_*, then for a function f on M, we have $[D_1, D_2]\cdot f = 0$.

1.5. Derivation of type d_*

Definition 1.5.1. A derivation of type d_* is a derivation D of $\wedge(M)$ such that $[D, d] = 0$.

Proposition 1.5.1. *A derivation $d_L := [i_L, d]$ is of type d_* where L is a ℓ-vectorial form* ([4]).

In fact, $d_L d = [i_L, d]d = i_L dd - (-1)^{\ell-1}di_L d = (-1)^\ell di_L d$ and $dd_L = di_L d$, imply $d_L d = (-1)^\ell dd_L$ and hence $[d_L, d] = 0$.

Proposition 1.5.2. *The bracket of two derivations of type* d_* *is a derivation of type* d_* ([4]).

In fact, if D_1 and D_2 are two derivations of type d_*, then

$(-1)^{r_2}[d, [D_1, D_2]] = -(-1)^{r_1}[D_1, [D_2, d]] - (-1)^{r_1 r_2}[D_2, [d, D_1]] = 0.$

Theorem 1.5.1. *If D is a derivation of degree* ℓ *of type* d_*, *then there exists a unique* ℓ*-vectorial form L such that* $D = d_L$ [4, 1].

In fact, L is defined by $Df = df \circ L$ for any $f \in \Lambda^0(M)$.

Theorem 1.5.2. *Every derivation D of* $\wedge(M)$ *of degree* ℓ *is a unique sum of two derivations where the first is of type* i_* *and the second is of type* d_* [4, 1].

There exists ℓ vectorial form L such that for any function f on M, we have $Df = d_L f$. Since $D - d_L$ is a derivation of type i_*, there exists a $\ell + 1$ vectorial form P such that $D - d_L = i_P$. Therefore, $D = d_L + i_P$.

2. Frölicher-Nijenhuis and Nijenhuis-Richardson Bracket [1]

2.1. Frölicher-Nijenhuis bracket

Definition 2.1.1. The *Frölicher-Nijenhuis bracket* $[L, P]$ of two vectorial forms $L \in \vee^{\ell}(M)$ and $P \in \vee^{p}(M)$ is defined by: $[d_L, d_P] = d_{[L, P]}$.

Proposition 2.1.1 [2]. *If L, P and Q are vectorial forms of degree* ℓ, p *and q, respectively, then the following hold*:

(1) *If* $\ell = p = 0$, *then the bracket* $[L, P]$ *is the bracket of vector fields.*

(2) $[L, P] = (-1)^{(\ell p+1)}[P, L]$ *and* $[I, L] = 0$.

(3) $(-1)^{\ell q}[L, [P, Q]] + (-1)^{p\ell}[P, [Q, L]] + (-1)^{qp}[Q, [L, P]] = 0.$

Proof. (1) If X and Y are two vector fields and $f \in C^{\infty}(M)$, then

$$[d_X, d_Y] \cdot f = d_X \circ d_Y \cdot f - d_Y \circ d_X \cdot f = X \cdot Y \cdot f - Y \cdot X \cdot f$$

$$= [X, Y] \cdot f = d_{[X, Y]} f.$$

The bracket of vector fields is a particular case of bracket of vectorial forms.

(2) $d_{[L, P]} = [d_L, d_P] = -(-1)^{\ell p}[d_P, d_L] = (-1)^{\ell p+1} d_{[P, L]}$, then $[L, P] = (-1)^{(\ell p+1)}[P, L]$ and $[I, L] = -[L, I]$. For $f \in C^{\infty}(M)$ we have $d_{[I, L]}(f) = 0$. Since $[d_I, d_L](f) = d_{[I, L]} f = [d, d_L] f = 0$, $[I, L] = 0$.

(3) This is the Jacobi identity for derivations.

Remark 2.1.1. The explicit form of bracket [2, 3] in the case $\ell = p = 1$ is:

$$[L, P](X, Y) = LP[X, Y] + PL[X, Y] + [LX, PY] + [PX, LY]$$

$$- L[PX, Y] - P[LX, Y] - L[X, PY] - P[X, LY].$$

We deduce that for a vector field X:

$$i_X[L, P] = [L(X), P] + [P(X), L] - P[X, L] - L[X, P].$$

If $\ell = p = 1$ and $L = P$, then we obtain

$$[L, L](X, Y) = 2L^2[X, Y] + 2[LX, LY] - 2L[LX, Y] - 2L[X, LY],$$

and $[L, X]Y = [LY, X] - L[Y, X]$. In fact, $i_X[L, L] = 2[LX, L] - L[X, L]$.

Proposition 2.1.2 [1-4]**.** *The decomposition of the derivation* $[i_L, d_P]$ *in the unique sum of two derivations one of type* d_* *and second of type* i_* *proves that*

$$[i_L, d_P] = d_{P \overline{\wedge} L} + (-1)^p i_{[L, P]}.$$

In fact the two members of the equality are derivations of degree $\ell + p - 1$. We prove that they have the same action on the functions and on the 1-scalar forms. Let f be a function on M. We have:

$$[i_L, d_P]f = i_L \circ d_P f = i_L \circ i_P df = i_{P \overline{\wedge} L} df = d_{P \overline{\wedge} L} f + (-1)^p i_{[L, P]} f.$$

Similarly

$$(-1)^{(\ell-1)p} i_{[L, P]} df = (-1)^{(\ell-1)p} d_{[L, P]} f = (-1)^{(\ell-1)p} [d_L, d_P] f$$

$$= (-1)^{(\ell-1)p} [[i_L, d], d_P] f = -(-1)^p [[d_P, i_L], d] f$$

$$= (-1)^p (-1)^{p(\ell-1)} [[i_L, d_P], d] f$$

$$= (-1)^{p\ell} [i_L, d_P] df - (-1)^{p\ell} (-1)^{(\ell+p-1)} d[i_L, d_P] f$$

$$= (-1)^{p\ell} [i_L, d_P] df - (-1)^{p\ell} (-1)^{(\ell+p-1)} dd_{P \overline{\wedge} L} f$$

$$= (-1)^{p\ell} [i_L, d_P] df - (-1)^{p\ell} (-1)^{(\ell+p-1)} (-1)^{(\ell+p-1)} d_{P \overline{\wedge} L} df$$

$$= (-1)^{p\ell} [i_L, d_P] df - (-1)^{p\ell} d_{P \overline{\wedge} L} df,$$

then $(-1)^p i_{[L, P]} df = [i_L, d_P] df - d_{P \overline{\wedge} L} df$ and $[i_L, d_P] = d_{P \overline{\wedge} L} + (-1)^p i_{[L, P]}$. We deduce that for $P = X$, and have $[i_L, d_X] = i_{[L, X]}$.

In this paper, we will use the brackets of the vectorial forms and the derivations with complicated formulas. That is why we prove the general case which will allow us to use all needed particular cases in all of the following.

Proposition 2.1.3. *Let* α *be a p-scalar form*, β *be a q-scalar form*, L *be a* ℓ *-vectorial form and* N *be a m-vectorial form*. *Then*

$$[\alpha \wedge L, \beta \wedge N]$$

$$= \alpha \wedge d_L \beta \wedge N - (-1)^{(m+q)(\ell+p)} \beta \wedge d_N \alpha \wedge L + (-1)^{q\ell} (\alpha \wedge \beta \wedge [L, N]$$

$$- (-1)^{(m+q)(\ell+1)} \alpha \wedge d\beta \wedge L \overline{\wedge} N + (-1)^{(\ell+p)(q+1)} \beta \wedge d\alpha \wedge (N \overline{\wedge} L)$$

$$+ (-1)^{\ell+p} d\alpha \wedge i_L \beta \wedge N - (-1)^{(p+\ell+1)(q+m)} d\beta \wedge i_N \alpha \wedge L.$$

Proof. See the Appendix.

We deduce from the above result all the equalities proved by [2, 4, 3] given in the following proposition.

Proposition 2.1.4. *X and Y are vector fields on M. Let L, N be respectively ℓ and m-vectorial forms on M. Let ϕ be a p-scalar form, ω and ψ be q-scalar form on the manifold M and α be a 1-differential form. Finally, we design by f and g functions on M. Then the following equalities* hold [2, 3]:

(1) $[fL, gN] = fd_L g \wedge N - (-1)^{m\ell} gd_N f \wedge L + fg[L, N] - (-1)^{m\ell+m} fdg$

$$\wedge (L \barwedge N) + (-1)^{\ell} gdf \wedge (N \barwedge L).$$

(2) $[X, \omega \wedge L] = \theta_X \omega \wedge L + \omega \wedge [X, L]$.

(3) $[\phi \otimes X, \psi \otimes Y] = \phi \wedge \psi \otimes [X, Y] + \phi \wedge d_X \psi \otimes Y - d_Y \phi \wedge \psi \otimes X +$

$$(-1)^p (d\phi \wedge i_X \psi \otimes Y + i_Y \phi \wedge d\psi \otimes X).$$

(4) $[X, \alpha \otimes Y] = d_X \alpha \otimes Y + \alpha \otimes [X, Y]$.

(5) *If* $\ell = q = 1$ *we have* $[L, \psi \otimes Y] = (d_L \psi) \otimes Y - \psi \wedge [L, Y] - d\psi \otimes LY$.

2.2. Nijenhuis-Richardson bracket

Definition 2.2.1. The *Nijenhuis-Richardson bracket* of the ℓ-vectorial form L and the p-vectorial form P is defined by $[L, P]^{\circ} := P \barwedge L - (-1)^{(\ell-1)(p-1)} L \barwedge P$.

We deduce from the above part that:

$$[L, P]^{\circ} = -(-1)^{(\ell-1)(p-1)} [P, L]^{\circ} \text{ and } [i_L, i_P] = i_{[L, P]^{\circ}} \text{ [2, 3].}$$

Proposition 2.2.1. *If* $a = \ell - 1$, $b = p - 1$ *and* $c = q - 1$, *then*

$$(-1)^{ac} [[L, P]^{\circ}, Q]^{\circ} + (-1)^{ba} [[P, Q]^{\circ}, L]^{\circ} + (-1)^{cb} [[Q, L]^{\circ}, P]^{\circ} = 0.$$

Proposition 2.2.2. *The following holds*:

$$[i_L, d_P] - (-1)^{(\ell-1)(p-1)}[i_P, d_L] = d_{[L, P]^\circ} \text{ [4, 3].}$$

In fact:

$$[i_L, d_P] = d_{P \overline{\wedge} L} + (-1)^p i_{[L, P]}$$

and we have

$$-(-1)^{(\ell-1)(p-1)}[i_P, d_L] = -(-1)^{(\ell-1)(p-1)} d_{L \overline{\wedge} P} - (-1)^{(\ell-1)(p-1)} (-1)^\ell i_{[P, L]}.$$

Thus $[i_L, d_P] - (-1)^{(\ell-1)(p-1)}[i_P, d_L] = d_{[L, P]^\circ}$.

3. Grifone-Ehresmann Connection

Let $P : TM \to M$ be the tangent bundle of *M* and $\Pi : TTM \to TM$ be the tangent bundle of *TM*. We have the exact sequence:

$$0 \to TM \times_M TM \xrightarrow{i} TTM \xrightarrow{j} TM \times_M TM \to 0,$$

where $i(v, w) := \frac{d}{dt}(v + tw)|_t = 0$ is the natural injection in the vertical bundle and $j := (\Pi, P^T)$ [8].

Definition 3.0.2. The *almost tangent structure* or *vertical endomorphism* on *M* is the 1-vectorial form $J := i \circ j$ on *TM*.

Locally if the chart (U, x^i) is a coordinates system on *M*, then the associated canonical system on *TM* is $(\overline{U}, (x^\alpha, y^\alpha))$ and $J = dx^\alpha \otimes \frac{\partial}{\partial y^\alpha}$.

We verify easily that $J^2 = 0$, $[J, J] = 0$ and $\ker J = \operatorname{Im} J = T^v$, where $T^v = Ker(P^T)$ is the vertical space.

Definition 3.0.3. A *canonical field* or *Liouville field* is the vertical vector field C on TM defined, at $z \in TM$, by $C_z := i(z, z)$. Locally, $C = y^{\alpha} \dfrac{\partial}{\partial y^{\alpha}}$.

Definition 3.0.4. A vectorial l-form L (resp. scalar p-form ω) on TM is called *homogeneous of degree* r if $[C, L] = (r-1)L$ (resp. $d_C \omega = r\omega$) where $d_C = [i_C, d]$ is the Lie derivation.

Proposition 3.0.3. *J is homogeneous of degree* 0 *and* $[J, J] = 0$. (*It means that* $[C, J] = -J$ *and* $d_{[J,J]} = [d_J, d_J] = 2d_J \circ d_J = 0$ [8]).

Definition 3.0.5. A *spray* is a vector field S on TM such that $JS = C$ [8, 7].

In classical system (x^{α}, y^{α}), $S = y^{\alpha} \dfrac{\partial}{\partial x^{\alpha}} - 2G^{\alpha}(x, y) \dfrac{\partial}{\partial y^{\alpha}}$. Consequently, S is homogeneous of degree 2 $([C, S] = S)$ if and only if the functions $G^{\alpha}(x, y)$ are homogeneous of degree 2 with respect to y^{α}. The vector field $S^* := [C, S] - S$ is vertical and is called the *deviation* of S.

Proposition 3.0.4. *For any* $X \in \chi(TM)$, *and two arbitrary sprays S and* S', *we have the following equalities* [8]:

(1) $J[JX, S] = JX$,

(2) $X - [JX, S]$ *is vertical vector field*,

(3) $X = J[X, S] + [JX, S] - J[[JX, S], S]$,

(4) $J[S, S'] = S - S'$.

Definition 3.0.6. A p-scalar form $\omega \in \bigotimes^p T^*TM$ is called *semi-basic* if $i_{JX}\omega = 0$. Similarly, a ℓ-vectorial form L is called *semi-basic* if $\operatorname{Im} L \subset VTM$ and $i_{JX} L = 0$.

Proposition 3.0.5. *For any* 1-*vectorial semi-basic form* T, *we have* $T[J, S] = J[T, S] = T$ [8].

Proposition 3.0.6. *Let L be a semi-basic l-form* $(l \geq 1)$ *on TM and S be a spray. Then the potential* L° *of L is defined by* $L^0 := i_S L$ *and it is independent of the choice of S* [8].

Proposition 3.0.7. *If L is a homogenous* ℓ*-vectorial form of degree r*, *then* [8]

(1) $L = \dfrac{1}{r + \ell}([J, L]^{\circ} + [J, L^{\circ}]),$

(2) *If L is closed* (*i.e.* $[J, L] = 0$), *then* $L = \dfrac{1}{r + 1}\ell([J, L^{\circ}]).$

3.1. Grifone-Ehresmann connection

Definition 3.1.1. A *Grifone-Ehresmann connection* on M is a $(1 - 1)$ tensor field Γ on *TM* such that $J\Gamma = J$ and $\Gamma J = -J$ [8].

The local form of the connection Γ in the classical system (x^i, y^i) is

$$\Gamma = dx^i \otimes \frac{\partial}{\partial y^i} - 2\Gamma_i^j dx^i \otimes \frac{\partial}{\partial y^j} - dy^i \otimes \frac{\partial}{\partial y^i},$$

where Γ_i^j are the Christoffel coefficients of the connection Γ which satisfies $\Gamma \circ \Gamma = I$. If S is a spray, then $[J, S]$ is a connection. If $\Gamma = [J, S]$ and $S = y^i \dfrac{\partial}{\partial x^i} - 2G^i(x, y)\dfrac{\partial}{\partial y^i}$, then $G^j = \dfrac{1}{2} y^i \Gamma_i^j$.

Definition 3.1.2. The *torsion H*, *weak torsion t* and the *strong torsion T* of Γ are defined respectively by $H := \dfrac{1}{2}[C, \Gamma]$, $t := \dfrac{1}{2}[J, \Gamma]$ and $T := t^0 - H$.

Proposition 3.1.1. *If* Γ *is homogeneous of degree* 1, *then* $T^0 = H = 0$ [8].

Remark 3.1.1. The two mappings $h := \frac{1}{2}(I + \Gamma)$ and $v = \frac{1}{2}(I - \Gamma)$, satisfy $h \circ h = h$ and $v \circ v = v$ in TTM. We deduce that the horizontal distribution associated to Γ is $H(TM) := \operatorname{Im} h$ and $V(TM) = \operatorname{Im} v$, such that $TTM = H(TM) \oplus V(TM)$ [8]. The local form is given by $h = dx^i \otimes \frac{\partial}{\partial x^i} - \Gamma_i^j dx^i \otimes \frac{\partial}{\partial y^j}$ and $v = \Gamma_i^j dx^i \otimes \frac{\partial}{\partial y^j} + dy^i \otimes \frac{\partial}{\partial y^i}$.

Definition 3.1.3. The *curvature of the connection* Γ is $R = -\frac{1}{2}[h, h]$.

Proposition 3.1.2. *If* $X \in \chi(TM)$ *and S is an arbitrary spray*, *then we have the following equalities* [8]:

$$h[JX, S] = hX,\ Jh = J,\ hJ = 0,\ Jv = 0 \text{ and } vJ = J.$$

Definition 3.1.4. The *associated spray S* of a connection Γ is a spray defined by $S := hS'$, where S' is an arbitrary spray.

This definition is verified since the difference between two sprays is a vertical vector field.

Proposition 3.1.3. *We have* $T^0 + S^* = 0$.

Proposition 3.1.4. *Let* λ *be a homogenous function of degree* 1 *on M*. *Then*:

(1) $[h, d_J\lambda \otimes C] = d_h d_J \lambda \otimes C$,

(2) $[h, \lambda J] = d_h\lambda \wedge J$,

(3) $[\lambda J, C] = 0$,

(4) $[J, \lambda C] = \lambda J + d_J\lambda \otimes C$,

(5) $d_C d_J \lambda = 0$,

(6) $[i_C, d_J] = i_J$ *and* $i_C d_J \lambda = 0$.

Proof. (1) $[h, d_J\lambda \otimes C] = d_h d_J \lambda \otimes C + d_J\lambda[h, C] - dd_J\lambda \otimes h(C) = d_h d_J\lambda \otimes C$,

(2) $[h, \lambda J] = d_h\lambda \wedge J - d\lambda \wedge hJ + \lambda[h, J] = d_h\lambda \wedge J$,

(3) $[\lambda J, C] = \lambda[J, C] - C.\lambda J = \lambda J - \lambda J = 0$,

(4) $[J, \lambda C] = \lambda[J, C] + d_J\lambda \otimes C = \lambda J + d_J\lambda \otimes C$,

(5) $d_C d_J \lambda = d_J d_C \lambda + d_{[C, J]}\lambda = d_J\lambda - d_J\lambda = 0$,

(6) $[i_C, d_J] = -i_{[C, J]} = i_J$ and $i_C d_J \lambda = -d_J i_C \lambda + i_J\lambda = 0$.

Proposition 3.1.5. *Let* Γ *be a connection of curvature R. Then* [8]:

(1) $[J, R] = [h, t]$,

(2) $[h, R] = 0$,

(3) $[C, R] = -[h, H]$,

(4) *If* Γ *is homogenous of degree* 1, *then* $H = 0$ *and* $[C, R] = 0$.

3.2. Almost complex structure associated to Γ

Definition 3.2.1. An *almost complex structure* on *TM* is a 1-vectorial form *F* on *TM* such that $F^2 = -I$.

Proposition 3.2.1. *Let* Γ *be a Grifone connection on M. Then there exists a unique almost complex structure F on TM,* C^{∞} *on* $TM - \{0\}$, *such that* $FJ = h$ *and* $Fh = -J$. *Consequently,* $JF = v$ [8].

The local form of *F* is given by:

$$F = \Gamma_i^j dx^i \otimes \frac{\partial}{\partial x^j} + dy^i \otimes \frac{\partial}{\partial x^i} - \Gamma_i^s \Gamma_s^j dx^i \otimes \frac{\partial}{\partial y^j} - dx^i \otimes \frac{\partial}{\partial y^i}$$

$$- \Gamma_i^j dy^i \otimes \frac{\partial}{\partial y^j}.$$

Proposition 3.2.2. *If the almost complex structure* F *satisfies* $F = h[S, h] - J$, $(h[S, h] - J) \circ J = h[S, h]J = h[S, hJ] - h \circ h[S, J] = h \circ [J, S] = h(h[S, h] - J) \circ h = h[S, h]h - J = h[S, h] - h \circ h[S, h] - J = -J$, *then* $F = h[S, h] - J$.

Proposition 3.2.3. *If* (M, E) *is a Finsler manifold, then the connection* $\Gamma = [J, S_E]$ *is homogenous of degree* 1 *and we have* $[C, F] = F + 2J$.

Proof. $[C, F]J = [C, FJ] - F[C, J] = [C, h] + FJ = FJ = FJ + 2JJ = (F + 2J)J$ and $[C, F]h = [C, Fh] - F[C, h] = -[C, J] = J = Fh = (F + 2J)h$, and hence $[C, F] = F + 2J$.

3.3. Finsler manifold in Grifone formalism

Definition 3.3.1. A *Finsler manifold* (M, E) is a differentiable manifold M equipped with Lagrangian mapping $E : TM \to \mathbb{R}_+$, where $E(0) = 0$, C^∞ on $TM - \{0\}$, C^1 on the zero section and such that:

(1) $d_C E = 2E$,

(2) The 2-differential form $\Omega = dd_J E$ is of maximal rank (i.e. symplectic form on *TM*) [8].

The local form is

$$\Omega = \frac{1}{2}\left(\frac{\partial^2 E}{\partial x^i \partial y^j} - \frac{\partial^2 E}{\partial y^i \partial x^j}\right) dx^i \wedge dx^j - \frac{\partial^2 E}{\partial y^i \partial y^j} dx^i \wedge dy^j.$$

Proposition 3.3.1. *The symplectic form* $\Omega = dd_J E$ *defines a metric* $\overline{g}$ *on the vertical fibre bundle VTM, and we have* $i_J \Omega = 0$ *where* $\overline{g}(JX, JY) = \Omega(JX, Y)$ [8].

Proposition 3.3.2. *There exists on the Finsler manifold* (M, E) *a spray called canonical spray* S_E *defined by* $i_{S_E}\Omega = -dE$ [8, 9, 7].

The local form of this spray is $S_E = y^i \frac{\partial}{\partial x^i} - 2G^j(x, y)\frac{\partial}{\partial y^i}$ where $G^i = \frac{1}{2}\bar{g}^{ij}\left(y^s \frac{\partial^2 E}{\partial x^s \partial y^j} - \frac{\partial E}{\partial x^j}\right)$.

Remark 3.3.1. If E is homogenous of degree 2, then G^i is homogenous of degree 2 and $[C, S_E] = S_E$. In this case the spray associated to $[J, S_E]$ is S_E. The Christoffel coefficient are homogenous of degree 1 and we have $[C, \Gamma] = 0$. But $G^j = \frac{1}{2} y^i \Gamma_i^j$, then $\Gamma = [J, S]$ is homogenous of degree 1, hence $[C, \Gamma] = 0$.

Every connection Γ on (M, E) defines a metric g, extension of $\bar{g}$ to TM, defined by $g(X, Y) = \bar{g}(JX, JY) + \bar{g}(vX, vY)$ and $g(hX, vY) = 0$.

4. Trace of Semi-basic Forms

The trace of a semi-basic form is the extension of the trace of a linear form.

Definition 4.0.2. The trace of a p-vectorial semi-basic form L is the $(p-1)$-vectorial semi-basic form $tr(L)$ defined by: $\forall X \in TTM$; $i_X tr(L) := tr(i_X L)$ and that if $p = 1$ we have $tr(L) := trace(FL)$ where F is the almost complex structure associated to an arbitrary Grifone connection Γ.

Proposition 4.0.3. *The definition of a trace of semi-basic form is independent of the choice of the Grifone connection* Γ.

Proof. Let F (resp. F') be the complex structure associated to Grifone connection Γ (resp. Γ'). It is sufficient to prove the property for $p = 1$. In fact, the form $(F - F')L$ is semi-basic because $J(F - F')L = (v - v')L = L - L = 0$, then $trFL = trF'L$.

Proposition 4.0.4. *Let* ω *be a* 1-*scalar semi-basic on TM and X be a vertical vector field on TM. Then* $tr(\omega \otimes X) = \omega(FX)$.

Proof. In fact, if $\{e_i\}_i$, $i = 1, ..., 2n,$ is a local orthonormal basis of *TTM* associated to the metric *g* deduced from a Grifone connection on *M*, and θ^i is its dual basis, then $tr(\omega \otimes X) = trace(\omega \otimes FX) = \theta^i(\omega(e_i)FX) = \omega(FX).$

Proposition 4.0.5. *If* ω *is a q-scalar semi-basic on TM, L is a* 1-*vectorial semi-basic form on TM and X is a vector field on TM, then* $tr(\omega \otimes LX) = (-1)^{q+1} i_{FLX}\omega.$

If $q = 1,$ then we have $tr(\omega \otimes LX) = trace(\omega \otimes FLX) = i_{FLX}\omega.$

Suppose that the property is true to the order $q - 1,$ then for $Y \in TTM$ we have

$$i_Y(tr(\omega \otimes LX)) = tr(i_Y(\omega \otimes LX)) = tr((i_Y\omega) \otimes LX) = (-1)^q i_{FLX} i_Y \omega$$

$$= (-1)^q ([i_{FLX}, i_Y] - i_Y i_{FLX}\omega) = i_Y((-1)^{q+1} i_{FLX}\omega).$$

Thus $tr(\omega \otimes LX) = (-1)^{q+1} i_{FLX}\omega.$

Proposition 4.0.6. *Let* ω *be a q-scalar semi-basic on TM, L be a* ℓ-*vectorial semi-basic form on TM and X be a vector field on TM. Then* $tr(\omega \wedge L) = \omega \wedge tr(L) - (-1)^{q(\ell+1)} i_{FL}\omega.$

Proof. We proceed by induction on ℓ and q. If $\ell = q = 1,$ then we have:

$$i_X tr(\omega \wedge L) = tr(i_X(\omega \wedge L)) = tr((\omega(X))L) - tr(\omega \otimes LX)$$

$$= \omega(X)tr(L) - i_{FLX}\omega = i_X(\omega tr(L)) - i_X i_{FL}\omega + i_{FL} i_X \omega$$

$$= i_X(\omega \wedge tr(L) - i_{FL}\omega).$$

In fact $i_X\omega$ is a function, and hence $tr(\omega \wedge L) = \omega \wedge tr(L) - i_{FL}\omega.$

We suppose that it is true to the order $q - 1,$ with $\ell = 1.$ Let ω be a q-scalar semi-basic form and L be a 1-vectorial semi-basic form. Then

$$i_X(tr(\omega \wedge L)) = tr(i_X(\omega \wedge L)) = tr((i_X\omega) \wedge L) + (-1)^q tr(\omega \otimes LX)$$

$$= i_X \omega tr(L) - (-1)^{2(q-1)} i_{FL} i_X \omega + (-1)^{2q+1} i_{FLX} \omega$$

$$= (i_X \omega) tr(L) - i_{FL} i_X \omega - i_{FLX} \omega = (i_X \omega) \wedge tr(L) - i_X i_{FL} \omega$$

$$= i_X (\omega \wedge tr(L) - i_{FL} \omega).$$

Finally, we suppose that the property is true for $q = 1$ and $\ell - 1$. Then

$$i_X (tr(\omega \wedge L)) = tr(\omega(X) L) - tr(\omega \wedge i_X L)$$

$$= \omega(X) tr(L) - \omega \wedge tr(i_X L) + (-1)^{\ell} i_{(F i_X L)} \omega.$$

But using the formula $[i_L, i_P] = i_{P \overline{\wedge} L} - (-1)^{(\ell-1)(p-1)} i_{L \overline{\wedge} P}$ and that ω is semi-basic, we deduce that

$$i_{(F i_X L)} \omega = i_{i_X L \overline{\wedge} F} \omega - i_F i_{i_X L} \omega + i_{i_X L} i_F \omega = \omega \circ F \circ i_X L = i_X (\omega \circ F \circ L)$$

$$= i_X (i_{F \overline{\wedge} L} \omega) = i_X (i_L i_F \omega - i_F i_L \omega + i_{L \overline{\wedge} F} \omega).$$

But $i_L \omega = i_{L \overline{\wedge} F} \omega = 0$, since L and ω are semi-basic forms. Thus $i_X (tr(\omega \wedge L)) = \omega(X) tr(L) - \omega \wedge tr(i_X L) + (-1)^p i_X i_{FL} \omega = i_X (\omega \wedge tr(L) - (-1)^{p+1} i_{FL} \omega)$. Therefore ℓ and q, we have $tr(\omega \wedge L) = \omega \wedge tr(L) - (-1)^{q(\ell+1)} i_{FL}(\omega)$.

Corollary 4.0.1. *Let ω be a q-scalar semi-basic form on TM and L be a ℓ-vectorial semi-basic form on TM. Then*

(1) $tr(J) = n$ *and* $tr(\omega \wedge J) = (n - q)\omega$.

(2) *If* $q = 1$, $tr(\omega \wedge L) = \omega \wedge tr(L) + (-1)^{\ell} i_{FL} \omega$,

(3) $tr(\omega \otimes C) = (-1)^{q+1} \omega^{\circ}$.

Proof. $tr(J) = Trace(FJ) = Trace(h) = n$. Since ω is a semi-basic form and $FJ = h$, $i_h \omega = q\omega$ and $tr(\omega \wedge J) = \omega tr(J) - i_{FJ} \omega = (n - q)\omega$. Finally,

$$tr(\omega \otimes C) = tr(\omega \otimes JS) = (-1)^{q+1} i_{FJS}\omega = (-1)^{q+1} i_{hS}\omega$$
$$= (-1)^{q+1} i_S \omega = (-1)^{q+1} \omega^\circ.$$

Proposition 4.0.7. *If L is a 1-vectorial semi-basic form on TM and X is a vector field on TM, then* $X \cdot tr(L) = Trace([X, FL])$.

Proof. Let $e = \{e_i\}_i$ be a local orthonormal basis on *TM* and $\theta = \{\theta^i\}_i$ be its dual basis. In this system $F = F_i^j \theta^i \otimes e_j$, $L = L_i^j \theta^i \otimes e_j$ and $FL = F_j^p L_i^j \theta^i \otimes e_p$. Then $Trace\, FL = F_j^p L_p^j$ and $X \cdot tr(FL) = (X \cdot F_j^s) L_s^j + F_j^s (X \cdot L_s^j)$. Now,

$$\theta^i([X, FL])(e_i) = -\theta^i([FL, X]e_i) = -\theta^i([FL, X]e_i)$$
$$= \theta^i(FL[e_i, X]) - \theta^i([FLe_i, X]) = \theta^i(FL(e_i \cdot X^s e_s)) + X^s \theta^i(FL[e_i, e_s])$$
$$- \theta^i([F_j^p L_i^j e_p, X])$$
$$= e_i \cdot X^s \theta^i(FL(e_s)) + X^s \theta^i(FL[e_i, e_s]) - \theta^i(F_j^p L_i^j (e_p \cdot X^s) e_s)$$
$$+ \theta^i(X^s(e_s \cdot F_j^p L_i^j) e_p) - X^s F_j^p L_i^j \theta^i([e_p, e_s])$$
$$= F_j^p L_s^j (e_p \cdot X^s) + X^s \theta^i(FL[e_i, e_s]) - F_j^p L_i^j (e_p \cdot X^i)$$
$$+ X^s(e_s \cdot F_j^p L_i^j) - X^s F_j^p L_i^j \theta^i([e_p, e_s])$$
$$= X^s e_s \cdot (F_j^i L_i^j) = X \cdot tr(L).$$

Corollary 4.0.2. *If the 1-vectorial semi-basic form L is homogenous of degree r, then $tr(L)$ is a homogenous function of degree r.*

Proof. The connection $\Gamma = [J, S]$ is homogenous of degree 1. If $[C, L] = (r-1)L$, then

$$C \cdot tr(L) = trace([C,\ FL])$$
$$= trace([C,\ F]L) + trace(F[C,\ L])$$
$$= trace((F + 2J)L) + trace(F(r-1)L)$$
$$= rtrace(FL) = rtr(L).$$

5. Projective Transformation

Definition 5.0.3. *The projective transformation of a spray S, homogenous of degree 2, is the vector field $S_* = S + \lambda C$, where λ is a homogenous function of degree 1, C^1 on TM and C^∞ on $TM \setminus \{0\}$.*

Proposition 5.0.8. *S_* is a homogenous spray of degree 2.*

Proof. $JS_* = JS + J(\lambda C) = C$ and $[C,\ S_*] = [C,\ S] + [C,\ \lambda C] = S + (C \cdot \lambda)C = S + \lambda C = S_*$.

Proposition 5.0.9. *If $f \in C^\infty(TM)$ is homogenous function of degree 1, then $d_h f$ and $d_J f^2$ are homogenous 1-scalar semi-basic forms of degree 1.*

Proof. $d_h f(JX) = (df)(hJX) = 0$ and $d_J f^2(JX) = (df^2)(JJX) = 0$. They are homogenous, in fact $d_C d_h f = d_h d_C f + d_{[C,\,h]} f = d_h f$ since the connection is homogenous $[C,\ \Gamma] = 0$ and $[C,\ h] = 0$. On other hand, $[C,\ J] = -J$ and f^2 are homogenous of degree 2. Thus, we have $d_C d_J f^2 = d_J d_C f^2 + d_{[C,\,J]} f^2 = 2d_J f^2 - d_J f^2 = d_J f^2$.

6. Curvature of Homogenous Connection under Projective Transformation

We consider in this section the connection $\Gamma = [J,\ S]$ associated to the homogenous spray of degree 2. In particular case where the manifold M is a Finsler manifold equipped with the homogenous lagrangian E of degree 2, we deduce that the associated spray is homogenous of degree 2. We consider in this section the projective transformation $S_* = S + \lambda C$. The curvature (resp. horizontal projector) of $[J, S]$ is R (resp. h) and the curvature of the connection $[J, S_*]$ is designed by R_*.

6.1. Invariant tensor under projective transformation

Using Proposition 2.1.3 we deduce the following particular cases, which will be used in later parts:

Proposition 6.1.1. *Using the equalities of Proposition* 3.1.2, *Proposition* 4.1.4, *and Proposition* 3.1.4 *we easily establish the following equalities*:

(1) $[d_J\lambda \otimes C, d_J\lambda \otimes C] = 2d_J\lambda \wedge d_C d_J\lambda \otimes C - 2i_C d_J\lambda \wedge dd_J\lambda \otimes C = 0,$

(2) $[\lambda J, \lambda J] = 2\lambda d_J\lambda \wedge J,$

(3) $$[d_J\lambda \otimes C, \lambda J] = d_{\lambda J} d_J\lambda \otimes C + d_J\lambda \wedge [\lambda J, C] - dd_J\lambda \otimes \lambda JC$$
$$= d_{\lambda J} d_J\lambda \otimes C,$$

(4) $$d_{fJ} d_J g = i_{fJ} dd_J g - di_{fJ} d_J g = f i_J dd_J g - df \wedge i_J d_J g - f di_J d_J g$$
$$= f d_J d_J g - df \wedge df \circ J \circ J = 0.$$

Proposition 6.1.2.

$$R_* = R - \omega \wedge J + d_J\omega \otimes C \ \textit{ where } \ \omega := \frac{1}{2}\left(d_h\lambda + \frac{1}{4} d_J\lambda^2\right),$$

$$\Gamma_* = [J, S + \lambda C] = [J, S] + (d_J\lambda) \otimes C + \lambda[J, C] = \Gamma + (d_J\lambda) \otimes C + \lambda J.$$

Proof. We have

$$R_* = -\frac{1}{8}[\Gamma_*, \Gamma_*] = -\frac{1}{8}[\Gamma + (d_J\lambda) \otimes C + \lambda J, \Gamma + (d_J\lambda) \otimes C + \lambda J]$$

$$= R - \frac{1}{8}[(d_J\lambda) \otimes C, (d_J\lambda) \otimes C] - \frac{1}{8}[\lambda J, \lambda J] - \frac{1}{4}[\Gamma, d_J\lambda \otimes C]$$

$$-\frac{1}{4}[\Gamma, \lambda J] - \frac{1}{4}[(d_J\lambda) \otimes C, \lambda J]$$

$$= R - \frac{1}{4}(d_J\lambda \wedge \theta_C d_J\lambda - i_C d_J\lambda dd_J\lambda) \otimes C - \frac{1}{4}\lambda d_J\lambda \Lambda J$$

$$-\frac{1}{2}[h, d_J\lambda \otimes C] - \frac{1}{2}[h, \lambda J]$$

$$-\frac{1}{4}(d_{\lambda J} d_J\lambda \otimes C - dd_J\lambda \otimes \lambda JC - d_J\lambda \wedge [\lambda J, C]),$$

$$R_* = R - \frac{1}{4}\lambda d_J\lambda\Lambda J - \frac{1}{2}d_h d_J\lambda \otimes C - \frac{1}{2}d_h\lambda \wedge J - \frac{1}{4}d_{\lambda J} d_J\lambda \otimes C.$$

But $\lambda d_J\lambda = \lambda d\lambda \circ J = \frac{1}{2}d(\lambda)^2 \circ J = \frac{1}{2}d_J\lambda^2$. By using Propositions 3.1.2, 4.1.4 and 7.1.1, we deduce:

$$R_* = R - \frac{1}{2}\left(\frac{1}{4}d_J(\lambda^2) + d_h\lambda\right) \wedge J - \frac{1}{2}(d_h d_J\lambda) \otimes C.$$

Let $\omega := \frac{1}{2}\left(\frac{1}{4}d_J(\lambda^2) + d_h\lambda\right)$. Then, we deduce that $d_J\omega = \frac{1}{2}d_J d_h\lambda = -\frac{1}{2}d_h d_J\lambda$, and hence $R_* = R - \omega \wedge J + d_J\omega \otimes C$.

Remark 6.1.1. The form ω satisfies the following properties:

(1) $i_S\omega = i_S\frac{1}{2}\left(\frac{1}{4}d_J(\lambda^2) + d_h\lambda\right) = \frac{1}{2}\left(\frac{1}{4}i_S d_J(\lambda^2) + i_S d_h\lambda\right)$

$$= \frac{1}{2}\left(\frac{1}{4}i_S i_J d(\lambda^2) + i_S i_h d\lambda\right) = \frac{1}{2}\left(\frac{1}{4}d(\lambda^2)(C) + d\lambda\right)(hS)$$

$$= \frac{1}{2}\left(\frac{1}{4}C \cdot \lambda^2 + d\lambda\right)(S) = \frac{1}{2}\left(\frac{1}{2}\lambda^2 + d_S\lambda\right),$$

(2) $i_{[J,S]}\omega(X) = i_\Gamma\omega(hX + vX) = \omega(\Gamma hX + \Gamma vX) = \omega(hX - vY) = \omega(hX)$ $= \omega(X)$, then $i_{[J,S]}\omega = \omega$,

(3) $(d_J\omega)^\circ = i_S(d_J\omega) = [i_S, d_J]\omega - d_J i_S\omega = d_{JS}\omega - i_{[S,J]}\omega - d_J i_S\omega =$ $d_{JS}\omega - i_{[S,J]}\omega - d_J i_S\omega = d_C\omega + i_\Gamma\omega - d_J i_S\omega = 2\omega - d_J i_S\omega.$

Proposition 6.1.3. *The relation between* R *and* R_* *is*

$$R_* = R - \left(\frac{tr(R) - tr(R_*)}{n+1} + \frac{d_J(r' - r)}{n+1}\right) \wedge J + \frac{d_J tr(R) - d_J tr(R_*)}{n+1} \otimes C,$$

where $r := \dfrac{i_S(tr(R))}{1-n}$, $\omega := \dfrac{1}{2}\left(d_h\lambda + \dfrac{1}{4} d_J\lambda^2\right)$ *and* $r' = r + \omega^\circ$.

Proof. In Proposition 6.1.3, we have $R_* = R - \omega \wedge J + d_J\omega \otimes C$.

Passing to the semi-basic trace we obtain:

$$tr(R_*) = tr(R) - tr(\omega \wedge J) + tr((d_J\omega \otimes C)) = tr(R) - (n-1)\omega - (d_J\omega)^\circ.$$

Then $i_S tr(R_*) = i_S tr(R) - (n-1) i_S\omega$.

We suppose that $r := \dfrac{i_S tr(R)}{1-n}$ and $r' := r + i_S\omega = r + \dfrac{1}{2} d_S\lambda + \dfrac{1}{4}\lambda^2$. Therefore $tr(R_*) = tr(R) - (n-1)\omega - 2\omega + d_J i_S\omega = tr(R) - (n+1)\omega +$ $d_J i_S\omega$. Then $tr(R_*) = tr(R) - (n+1)\omega + d_J(r' - r)$. We deduce that $\omega =$ $\dfrac{tr(R) - tr(R_*)}{n+1} + \dfrac{d_J(r' - r)}{n+1}$ and $d_J\omega = \dfrac{d_J tr(R) - d_J tr(R_*)}{n+1}$. Finally

$$R_* = R - \left(\frac{tr(R) - tr(R_*) + d_J(r' - r)}{n+1}\right) \wedge J + \frac{d_J tr(R) - d_J tr(R_*)}{n+1} \otimes C.$$

Theorem 6.1.1. *If the manifold* M *is equipped with a Grifone connection* Γ *of curvature* R, *then the tensor*

$$W := R - \left(\frac{tr(R) - d_J r}{n+1}\right) \wedge J + \frac{d_J tr(R)}{n+1} \otimes C$$

is invariant under the projective transformation and it satisfies $tr(W) = 0$.

Proof. Using Proposition 6.1.3, we get:

$$R_* = R - \left(\frac{tr(R) - tr(R_*) + d_J(r' - r)}{n+1}\right) \wedge J + \frac{d_J tr(R) - d_J tr(R_*)}{n+1} \otimes C.$$

Then

$$R_* - \left(\frac{tr(R_*) - d_J r'}{n+1}\right) \wedge J + \frac{d_J tr(R_*)}{n+1} \otimes C$$

$$= R - \left(\frac{tr(R) - d_J r}{n+1}\right) \wedge J + \frac{d_J tr(R)}{n+1} \otimes C.$$

But R and $tr(R)$ are homogenous of degree 1. Therefore,

$$tr(W) = tr(R) - (n-1)\left(\frac{tr(R) - d_J r}{n+1}\right) - i_S \frac{d_J tr(R)}{n+1}$$

$$= tr(R) - (n-1)\left(\frac{tr(R) - d_J r}{n+1}\right) - \frac{d_C tr(R) - d_J i_S tr(R) + i_{[J,S]} tr(R)}{n+1}$$

$$= tr(R) - (n-1)\left(\frac{tr(R) - d_J r}{n+1}\right) - \frac{tr(R) - d_J i_S tr(R) + tr(R)}{n+1}$$

$$= tr(R) - (n-1)\left(\frac{tr(R) - d_J r}{n+1}\right) - \frac{2tr(R) - d_J i_S tr(R)}{n+1}$$

$$= tr(R) - tr(R) + \frac{d_J((n-1)r + i_S tr(R))}{n+1} = 0.$$

Proposition 6.1.4. *If the curvature of the homogenous Grifone connection* $\Gamma = [J, S]$ *is given by* $R = \omega \wedge J + \pi \otimes C$ *where* ω *is a* 1-*scalar semi-basic form and* π *is a* 2-*scalar semi-basic form*, *then* $\theta_C \omega = \omega$ *and* $\theta_C \pi = 0$.

Proof. Γ is homogenous of degree 1 (i.e. $[C, \Gamma] = [C, h] = 0$) and $[C, R] = 0$. If $R = \omega \wedge J + \pi \otimes C$, then $[C, \omega \wedge J + \pi \otimes C] = (\theta_C \omega - \omega) \wedge J + \theta_C \pi \otimes C = 0$. We deduce that $\theta_C \omega = \omega$ and $\theta_C \pi = 0$.

Theorem 6.1.2. *Let* (M, E) *be a Finsler manifold of dimension* $n > 2$ *equipped with a Grifone connection* $\Gamma = [J, S]$. *Then there exists a* 1-*scalar* *form* ω *and* 2-*scalar form* π *such that the curvature of* Γ *is of the form* $R = \omega \wedge J + \pi \otimes C$ *if and only if* $W = 0$.

Proof. The semi-basic trace of $R = \omega \wedge J + \pi \otimes C$ is $tr(R) = (n-1)\omega - i_S\pi$ and then $i_S tr(R) = (n-1)i_S\omega = (1-n)r$. Thus $i_S\omega = -r$. On other hand

$$0 = [J, R] = [J, \omega \wedge J + \pi \otimes C] = [J, \omega \wedge J] + [J, \pi \otimes C]$$

$$= d_J\omega \wedge J + d_J\pi \otimes C + \pi \wedge J.$$

Passing to semi-basic trace we deduce $(n-2)(d_J\omega + \pi) + i_S(d_J\pi) = 0$ and by applying d_J we obtain $(n-2)(d_J\pi) + d_J i_S(d_J\pi) = 0$. But

$$d_J i_S d_J\pi = d_J(-d_J i_S\pi + d_{JS}\pi - i_{[S,J]}\pi) = d_J(-d_J i_S\pi + \theta_C\pi + i_{I-2v}\pi)$$

$$= d_J(-d_J i_S\pi + 2\pi) = 2d_J\pi.$$

We deduce that $d_J\pi = 0$, in addition to $(n-2)(d_J\omega + \pi) + i_S(d_J\pi) = 0$. Then $\pi = -d_J\omega$ and $R = \omega \wedge J - d_J\omega \otimes C$. Finally, we prove that $W = 0$. In fact, $R = \omega \wedge J - d_J\omega \otimes C$ induces

$$tr(R) = (n-1)\omega + i_S d_J\omega = (n-1)\omega - d_J i_S\omega + d_{JS}\omega + i_\Gamma\omega$$

$$= (n-1)\omega - d_J i_S\omega + \omega + \omega = (n+1)\omega + d_J r.$$

Then the expression of ω is of the form $\omega = \dfrac{tr(R) - d_J r}{n+1}$ and the curvature R is given by $R = \dfrac{tr(R) - d_J r}{n+1} \wedge J - \dfrac{d_J tr(R) \otimes C}{n+1}$. Finally, the tensor $W := R - \left(\dfrac{tr(R) - d_J r}{n+1}\right) \wedge J + \dfrac{d_J tr(R)}{n+1} \otimes C$ vanishes and the inverse is also true.

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7. Appendix

Proposition 7.0.5. *Let f be a p-scalar form, g be a q-scalar form, L be a ℓ-vectorial form and N be an m-vectorial form. Then*

$$
\begin{aligned}
&[f \wedge L,\ g \wedge N] \\
&= f \wedge d_L g \wedge N - (-1)^{(m+q)(\ell+p)} g \wedge d_N f \wedge L \\
&\quad + (-1)^{q\ell}(f \wedge g \wedge [L,\ N] - (-1)^{(m+q)(\ell+1)} f \wedge dg \wedge L \barwedge N
\end{aligned}
$$

$$+ (-1)^{(\ell+p)(q+1)} g \wedge df \wedge (N \barwedge L) + (-1)^{\ell+p} df \wedge i_L g \wedge N$$

$$- (-1)^{(p+\ell+1)(q+m)} dg \wedge i_N f \wedge L.$$

Proof. We have

$$d_{[f \wedge L,\, g \wedge N]}$$

$$= [d_{f \wedge L}, d_{g \wedge N}]$$

$$= [f \wedge d_L + (-1)^{(\ell+p)} df \wedge i_L,\ g \wedge d_N + (-1)^{m+q} dg \wedge i_N]$$

$$= [f \wedge d_L, g \wedge d_N] + [f \wedge d_L, (-1)^{(m+q)} dg \wedge i_N]$$

$$+ [(-1)^{(\ell+p)} df \wedge i_L, g \wedge d_N] + (-1)^{\ell+m+p+q} [df \wedge i_L, dg \wedge i_N]$$

$$= f \wedge d_L g \wedge d_N + (-1)^{q\ell} f \wedge g \wedge d_L d_N - (-1)^{(m+q)(\ell+p)} g \wedge d_N f$$

$$\wedge d_L - (-1)^{(m+q)(\ell+p)} (-1)^{mp} g \wedge f \wedge d_N d_L$$

$$+ (-1)^{m+q} f \wedge d_L dg \wedge i_N + (-1)^{m+q} (-1)^{\ell(q+1)} f \wedge dg$$

$$\wedge d_L i_N - (-1)^{m+q} (-1)^{(p+\ell)(q+m)} dg \wedge i_N f \wedge (d_L)$$

$$- (-1)^{m+q} (-1)^{(p+\ell)(q+m)} (-1)^{p(m-1)} dg \wedge f \wedge (i_N d_L)$$

$$+ (-1)^{\ell+p} df \wedge i_L g \wedge d_N + (-1)^{\ell+p} (-1)^{(\ell-1)q} df \wedge g \wedge i_L d_N$$

$$- (-1)^{\ell+p} ((-1)^{(m+q)(\ell+p)}) g \wedge d_N df \wedge i_L$$

$$- (-1)^{\ell+p} ((-1)^{(m+q)(\ell+p)}) (-1)^{m(p+1)} g \wedge df \wedge d_N i_L$$

$$+ (-1)^{\ell+m+p+q} df \wedge i_L dg \wedge i_N + (-1)^{\ell+m+p+q} (-1)^{(\ell-1)(q+1)} df \wedge dg$$

$$\wedge i_L i_N - (-1)^{\ell+m+p+q} (-1)^{(m+q)(\ell+p)} dg \wedge i_N df \wedge i_L$$

$$- (-1)^{\ell+m+p+q} (-1)^{(m+q)(\ell+p)} (-1)^{(m-1)(p+1)} dg \wedge df \wedge i_N i_L$$

$$= f \wedge d_L g \wedge d_N + (-1)^{\ell+m+p+q} df \wedge i_L dg \wedge i_N$$

$$+ (-1)^{m+q} f \wedge d_L dg \wedge i_N - (-1)^{(m+q)(\ell+p)} g \wedge d_N f \wedge d_L$$

$$- (-1)^{\ell+p} ((-1)^{(m+q)(\ell+p)}) g \wedge d_N df \wedge i_L$$

$$- (-1)^{\ell+m+p+q} (-1)^{(m+q)(\ell+p)} dg \wedge i_N df \wedge i_L + (-1)^{q\ell} f \wedge g \wedge d_L d_N$$

$$- (-1)^{(m+q)(\ell+p)} (-1)^{mp} g \wedge f \wedge d_N d_L + (-1)^{m+q} (-1)^{\ell(q+1)} f \wedge dg$$

$$\wedge d_L i_N - (-1)^{m+q} (-1)^{(p+\ell)(q+m)} (-1)^{p(m-1)} dg \wedge f \wedge (i_N d_L)$$

$$+ (-1)^{\ell+p} (-1)^{(\ell-1)q} df \wedge g \wedge i_L d_N$$

$$- (-1)^{\ell+p} ((-1)^{(m+q)(\ell+p)}) (-1)^{m(p+1)} g \wedge df \wedge d_N i_L$$

$$+ (-1)^{\ell+m+p+q} (-1)^{(\ell-1)(q+1)} df \wedge dg \wedge i_L i_N$$

$$- (-1)^{\ell+m+p+q} (-1)^{(m+q)(\ell+p)} (-1)^{(m-1)(p+1)} dg \wedge df \wedge i_N i_L$$

$$+ (-1)^{\ell+p} df \wedge i_L g \wedge d_N - (-1)^{m+q} (-1)^{(p+\ell)(q+m)} dg \wedge i_N f \wedge (d_L)$$

$$= f \wedge d_L g \wedge d_N + (-1)^{\ell+m+p+q} df \wedge d_L g \wedge i_N$$

$$+ (-1)^{m+q} (-1)^{\ell} f \wedge d d_L g \wedge i_N - (-1)^{\ell+m+p+q} (-1)^{\ell} df \wedge d i_L g \wedge i_N$$

$$- (-1)^{(m+q)(\ell+p)} g \wedge d_N f \wedge d_L - (-1)^{\ell+m+p+q} (-1)^{(m+q)(\ell+p)} dg$$

$$\wedge d_N f \wedge i_L - (-1)^{\ell+p} ((-1)^{(m+q)(\ell+p)}) g \wedge d_N df \wedge i_L$$

$$+ (-1)^{\ell+m+p+q} (-1)^{(m+q)(\ell+p)} (-1)^{m} dg \wedge i_N df \wedge i_L$$

$$+ (-1)^{q\ell} f \wedge g \wedge ([d_L, d_N]) - (-1)^{(m+q)(\ell+1)} f \wedge dg \wedge (i_N d_L)$$

$$+ (-1)^{m+q} (-1)^{\ell(q+1)} f \wedge dg \wedge d_L i_N$$

$$+ (-1)^{\ell+p} (-1)^{(\ell-1)q} df \wedge g \wedge (i_L d_N - (-1)^{m(\ell-1)} d_N i_L)$$

$$+ (-1)^{\ell+m+p+q} (-1)^{(\ell-1)(q+1)} df \wedge dg \wedge (i_L i_N - (-1)^{(\ell-1)(m-1)} i_N i_L)$$

$$+ (-1)^{\ell+p} df \wedge i_L g \wedge d_N - (-1)^{m+q} (-1)^{(p+\ell)(q+m)} dg \wedge i_N f \wedge (d_L)$$

$$= d_{f \wedge d_L g \wedge N} - (-1)^{(m+q)(\ell+p)} d_{g \wedge d_N f \wedge L}$$
$$+ (-1)^{q\ell} (d_{f \wedge g \wedge [L, N]} - (-1)^{m+q} (-1)^{\ell(q+1)} (-1)^{(m-1)\ell} d_{f \wedge dg \wedge L \overline{\wedge} N}$$
$$+ (-1)^{(\ell+p)(q+1)} d_{g \wedge df \wedge (N \overline{\wedge} L)}$$
$$+ (-1)^{\ell+p} d_{df \wedge i_L g \wedge N} - (-1)^{(p+\ell+1)(q+m)} d_{dg \wedge i_N f \wedge L}.$$

Thus

$$[f \wedge L, g \wedge N]$$
$$= f \wedge d_L g \wedge N - (-1)^{(m+q)(\ell+p)} g \wedge d_N f \wedge L$$
$$+ (-1)^{q\ell} (f \wedge g \wedge [L, N] - (-1)^{(m+q)(\ell+1)} f \wedge dg \wedge (L \overline{\wedge} N)$$
$$+ (-1)^{(\ell+p)(q+1)} g \wedge df \wedge (N \overline{\wedge} L)$$
$$+ (-1)^{\ell+p} df \wedge i_L g \wedge N - (-1)^{(p+\ell+1)(q+m)} dg \wedge i_N f \wedge L.$$