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TANGENT BUNDLE OF A HYPERSURFACE IN R^4

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Abstract

In this paper, we consider a special case for the relevant work in a 2011 paper of Sharief Deshmukh and Suha B. Al-Shaikh published in Beitr. Algebra Geom. [9] wherein we study the tangent bundle of M^3 as a submanifold of R^4. Since we know that R^4 has three Hermitian complex structures, using them together with the unit normal to the hypersurface M^3, we get three orthonormal unit vectors on M^3. By means of these three vectors, we study the properties of *TM* equipped with the induced metric represented in [9].

1. Introduction

Given a Riemannian manifold (M, g), one can define several Riemannian metrics on the tangent bundle $T(M)$ of M. Maybe the best known example is the Sasaki metric introduced in [1]. Although the Sasaki metric is naturally defined, it is very rigid. For example, the Sasaki metric is

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not, generally, Einstein. The tangent bundle $T(M)$ with the Sasaki metric is never locally symmetric unless the metric g on the base manifold is flat (see [2]). Musso and Tricerri [3] have proved that the Sasaki metric has constant scalar curvature if and only if (M, g) is locally Euclidian. In the same paper, they have given an explicit expression of a positive definite Riemannian metric introduced by Cheeger and Gromoll in [4] and called this metric the Cheeger-Gromoll metric. Sekizawa (see [5]), computed geometric objects related to this metric. Later, Gudmundsson and Kappos in [6] have completed these results and have shown that the scalar curvature of the Cheeger-Gromoll metric is never constant if the metric on the base manifold has constant sectional curvature. Furthermore, Abbassi and Sarih have proved that $T(M)$ with Cheeger-Gromoll metric is never a space of constant sectional curvature (cf. [7]). Recently, efforts are made to study the geometry of the tangent bundle of a hypersurface M in the Euclidean space R^{n+1} (cf. [8]), where the authors have shown that the induced metric on its tangent bundle TM as a submanifold of the Euclidean space R^{2n+2} is not a natural metric. Deshmukh and Al-Shaikh in [9], extend the study initiated in [8] and computed geometric objects related to this metric. In this paper, we consider a special case for the relevant work in [9], because we study the tangent bundle of M^3 as a submanifold of R^4. Since we know that R^4 has three Hermitian complex structures, using them together with the unit normal to the hypersurface M^3, we get three orthonormal unit vectors on M^3, and using those three vectors we study the properties of TM equipped with the induced metric represented in [9].

2. Preliminaries

Let (M, g) be a Riemannian manifold and TM be its tangent bundle with projection map $\pi : TM \to M$. Then for each $(p, u) \in TM$, the tangent space $T_{(p,u)}TM = \mathfrak{H}_{(p,u)} \oplus \mathfrak{V}_{(p,u)}$, where $\mathfrak{V}_{(p,u)}$ is kernel of $d\pi_{(p,u)} : T_{(p,u)}TM \to T_pM$ and $\mathfrak{H}_{(p,u)}$ is the kernel of the connection map $K_{(p,u)} : T_{(p,u)}(TM) \to T_p M$ with respect to the Riemannian connection on (M, g). The subspaces $\mathfrak{H}_{(p,u)}$, $\mathfrak{V}_{(p,u)}$ are called the *horizontal* and *vertical subspaces*, respectively. Consequently, the Lie algebra of smooth vector fields $\mathfrak{X}(TM)$ on the tangent bundle *TM* admits the decomposition $\mathfrak{X}(TM) = \mathfrak{H} \oplus \mathfrak{V}$, where $\mathfrak{H}$ is called the *horizontal distribution* and $\mathfrak{V}$ is called the *vertical distribution* on the tangent bundle *TM*. For each $X_p \in T_p M$, the horizontal lift of X_p to a point $z = (p, u) \in TM$ is the unique vector $X_z^h \in \mathfrak{H}_z$ such that $d\pi(X_z^h) = X_p \circ \pi$ and the vertical lift of X_p to a point $z = (p, u) \in TM$ is the unique vector $X_z^v \in \mathfrak{V}_z$ such that $X_z^v(df) = X_p(f)$ for all functions $f \in C^\infty(M)$, where *df* is the function defined by $(df)(p, u) = u(f)$. Also, for a vector field $X \in \mathfrak{X}(M)$, the horizontal lift of *X* is a vector field $X^h \in \mathfrak{X}(TM)$ whose value at a point (p, u) is the horizontal lift of $X(p)$ to (p, u), the vertical lift X^v of *X* is defined similarly. For $X \in \mathfrak{X}(M)$ the horizontal and vertical lifts X^h, X^v of *X* are the uniquely determined vector fields on *TM* satisfying

$$d\pi(X_z^h) = X_{\pi(z)},\ K(X_z^h) = 0_{\pi(z)},\ d\pi(X_z^v) = 0_{\pi(z)},\ K(X_z^v) = X_{\pi(z)}.$$

Also for a smooth function $f \in C^\infty(M)$ and vector fields $X, Y \in \mathfrak{X}(M)$, we have $(fX)^h = (f \circ \pi)X^h$, $(fX)^v = (f \circ \pi)X^v$, $(X + Y)^h = X^h + Y^h$ and $(X + Y)^v = X^v + Y^v$. If $\dim M = m$ and (U, ϕ) is a chart on *M* with local coordinates $x^1, x^2, ..., x^m$, then $(\pi^{-1}(U), \overline{\Phi})$ is a chart on *TM* with local coordinates $x^1, ..., x^m, y^1, ..., y^m$, where $x^i = x^i \circ \pi$ and $y^i = dx^i$, $i = 1, ..., m$. Throughout this paper we use Einstein convention for summation, that is, the repeated indices are summed on their ranges. For horizontal and vertical lifts of smooth vector fields, we have the following:

Lemma 2.1 ([6]). *Let* (M, g) *be a Riemannian manifold and*

$X, Z \in \mathfrak{X}(M)$, be locally represented by $X = \xi^i \dfrac{\partial}{\partial x^i}$ and $Z = \eta^i \dfrac{\partial}{\partial x^i}$. Then the vertical and horizontal lifts X^v and X^h of X at the point $Z \in TM$ are given by

$$(X^v)_Z = \xi^i \frac{\partial}{\partial y^i}, \quad (X^h)_Z = \xi^i \frac{\partial}{\partial x^i} - \xi^j \eta^k \Gamma^i_{jk} \frac{\partial}{\partial y^i},$$

where the coefficients Γ^i_{jk} are the Christoffel symbols of the Riemannian connection ∇ on (M, g).

3. Tangent Bundle of the Hypersurface

Let *M* be an orientable hypersurface of the Euclidean space R^{2n} with immersion $f : M \to R^{2n}$ and *TM* be its tangent bundle with immersion $F : TM \to R^{4n}$. We denote the induced metric on *M*, *TM* by g, $\overline{g}$ respectively and the Euclidean metric on R^{2n} as well as on R^{4n} by $\langle\, ,\rangle$. We also denote by ∇, $\overline{\nabla}$, D and $\overline{D}$ the Riemannian connections on *M*, *TM*, R^{2n}, and R^{4n} respectively, and recall that the connection coefficient (Christoffel symbols) Γ^k_{ij} of the Euclidean connection are zero. Let *N* and *S* be the unit normal vector field and the shape operator of the hypersurface *M*.

We know that if $x^1, \ldots, x^{2n-1}$ are the local coordinates on *M*, then the corresponding local coordinates on *TM* are $x^1, x^2, \ldots, x^{2n-1}, y^1, y^2, \ldots, y^{2n-1}$, where $x^i = x^i \circ \pi$ and $y^i = dx^i$, $i = 1, 2, \ldots, 2n - 1$.

Similarly if $u^1, \ldots, u^{2n}$ are the local coordinates on R^{2n}, then we get corresponding coordinates $u^1, u^2, \ldots, u^{2n}, v^1, v^2, \ldots, v^{2n}$ on R^{4n}, where

$$\left(\frac{\partial}{\partial u^i}\right)^v = \frac{\partial}{\partial v^i}$$

$$\left(\frac{\partial}{\partial u^i}\right)^h = \frac{\partial}{\partial u^i}, \quad i = 1, \ldots, 2n.$$

We have the following theorem which is a consequence of the fact that an immersion of M in N induces an immersion of TM in TN.

Theorem 3.1 [8]. *The map* $F : TM \to R^{4n}$ *is an immersion with the matrix for* $dF_P : T_P(TM) \to T_{F(P)}(R^{4n})$ *at* $P = (p, X_p) \in TM$, *as the* $(4n) \times (4n-2)$ *matrix given by*

$$dF_P = \begin{bmatrix} df_{p(2n)\times(2n-1)} & 0_{(2n)\times(2n-1)} \\ \left(\dfrac{\partial^2 f^i}{\partial x^j \partial x^k}(p)\, y^k(P)\right)_{(2n)\times(2n-1)} & df_{p(2n)\times(2n-1)} \end{bmatrix},$$

where

$$F = \left(f^1 \circ \pi,\ f^2 \circ \pi,\ \ldots,\ f^{2n} \circ \pi,\ \left(\frac{\partial f^1}{\partial x^i} \circ \pi\right) y^i,\ \ldots,\ \left(\frac{\partial f^{2n}}{\partial x^i} \circ \pi\right) y^i\right).$$

Thus the tangent bundle TM of the hypersurface M of the Euclidean space R^{2n} is a submanifold of R^{4n}. For the hypersurface M of the Euclidean space R^{2n}, we have the following Gauss and Weingarten formulae

$$D_X Y = \nabla_X Y + \langle S(X), Y \rangle N$$

$$D_X N = -S(X),$$

where $X, Y \in \mathfrak{X}(M)$ and S denotes the shape operator (Weingarten map) $S : \mathfrak{X}(M) \to \mathfrak{X}(M)$. Similarly for the submanifold TM of the Euclidean space R^{4n}, we have the Gauss and Weingarten formulae:

$$\overline{D}_X Y = \overline{\nabla}_X Y + h(X, Y)$$

$$\overline{D}_X \hat{N} = -\overline{S}_{\hat{N}}(X) + \nabla^{\perp}_X \hat{N},$$

where $X, Y \in \mathfrak{X}(TM)$ and $\overline{S}_{\hat{N}}$ denotes the Weingarten map in the direction of the normal $\hat{N}$ which is $\overline{S}_{\hat{N}} : \mathfrak{X}(TM) \to \mathfrak{X}(TM)$, and is related to the second fundamental form *h* by

$$\langle h(X, Y), \hat{N} \rangle = \overline{g}(\overline{S}_{\hat{N}}(X), Y).$$

Also we observe that for $X \in \mathfrak{X}(M)$ the vertical lift X^v of *X* to *TM*, as $X^v \in \ker d\pi$ we have $d\pi(X^v) = 0$ that is $df(d\pi(X^v)) = 0$ or equivalently we get $d(f \circ \pi)(X^v) = 0$, that is, $d(\widetilde{\pi} \circ F)(X^v) = 0$ which gives $dF(X^v) \in \ker d\widetilde{\pi} = \overline{\mathfrak{V}}$.

We have the following lemma.

Lemma 3.1 [9]. *If* (M, g) *is an orientable hypersurface of* R^{2n}, *and* $(TM, \overline{g})$ *is its tangent bundle as submanifold of* R^{4n}, *then the metric* $\overline{g}$ *on TM for* $P = (p, u) \in TM$, *satisfies*

(1) $\overline{g}_P(X_P^h, Y_P^h) = g_P(X_p, Y_p) + g_p(S_p(X_p), u)g_p(S_p(Y_p), u)$

(2) $\overline{g}_P(X_P^h, Y_P^v) = 0$

(3) $\overline{g}(X^v, Y^v) = g_p(X_p Y_p)$.

Remark. We observe that the metrics defined on *TM* using the Riemannian metrics of *M* (such as Sasaki metric, Cheeger-Gromoll metric, and Oproiu metric) are natural metrics in the sense that the submersion $\pi : TM \to M$ becomes a Riemannian submersion with respect to these metrics. However, the induced metric on the tangent bundle *TM* of a hypersurface *M* of the Euclidean space R^{2n}, as a submanifold of R^{4n} is not a natural metric because of the presence of the term $g_p(S_p(X_p), u)g_p(S_p(Y_p), u)$ in the inner product of horizontal vectors in *TM*.

In what follows, we drop the suffixes and it is understood from the context of the entities appearing in the equation.

Theorem 3.2 [9]**.** *Let* (M, g) *be an orientable hypersurface of* R^{2n}, *and* $(TM, \bar{g})$ *be its tangent bundle as a submanifold of* R^{4n}, *and* (M, g) *be equipped with the connection* ∇ *and TM be equipped with the connection* $\overline{\nabla}$. *Then the connection on TM is given as follows*:

(i) $\overline{\nabla}_{X^h} Y^h = (\nabla_X Y)^h - \frac{1}{2}(R(X, Y)u)^v$

(ii) $\overline{\nabla}_{X^h} Y^v = (\nabla_X Y)^v + g(S(X), Y) \circ \pi N^v$

(iii) $\overline{\nabla}_{X^v} Y^h = g(S(X), Y) \circ \pi N^v$

(iv) $\overline{\nabla}_{X^v} Y^v = 0.$

Moreover we have the following lemmas:

Lemma 3.2 [9]**.** *Let M be an orientable hypersurface of* R^{2n}. *Then for* $X, Y \in \mathfrak{X}(M)$,

(1) $h(X^v, Y^v) = 0$

(2) $h(X^v, Y^h) = 0$

(4) $h(X^h, Y^h) = g(S(X), Y) \circ \pi N^h.$

Lemma 3.3 [9]**.** *For an orientable hypersurface M of* R^{2n} *and* $X \in \mathfrak{X}(M)$,

(1) $\overline{D}_{X^v} N^v = 0$

(2) $\overline{D}_{X^v} N^h = 0$

(3) $\overline{D}_{X^h} N^v = -(S(X))^v$

(4) $\overline{D}_{X^h} N^h = -(S(X))^h.$

Lemma 3.4 [9]**.** *For the tangent bundle TM of an orientable hypersurface M of the Euclidean space* R^{2n},

(i) $h(X^v, N^v) = 0$, (ii) $\overline{\nabla}_{X^v} N^v = 0$,

(iii) $h(X^h, N^v) = 0$, (iv) $\overline{\nabla}_{X^h} N^v = -(S(X))^v$, $X \in \mathfrak{X}(M)$.

4. Structure Induced on *M*

Let M^3 be an orientable hypersurface of the Euclidean space R^4 with immersion $f : M^3 \to R^4$ and *TM* be its tangent bundle with immersion $F : TM \to R^8$. We denote the induced metrics on *M*, *TM* by g, $\overline{g}$ respectively and the Euclidean metric on R^4 as well as on R^8 by $\langle\, ,\rangle$. We also denote by ∇, $\overline{\nabla}$, D and $\overline{D}$ the Riemannian connections on *M*, *TM*, R^4, and R^8 respectively. Let *N* and *S* be the unit normal vector field and the shape operator of the hypersurface *M*. We know that R^4 has 3 Hermitian complex structures J_1, J_2, J_3, Hermitian in the sense that $J_i^2 = -I$ and $\langle J_i X, J_i Y\rangle = \langle X, Y\rangle, \forall i = 1, 2, 3,$ which give that $\langle J_i X, X\rangle = 0$. For those three complex structures we have the relations

$$J_1 \circ J_2 = -J_2 \circ J_1 = J_3$$

$$J_2 \circ J_3 = -J_3 \circ J_2 = J_1$$

$$J_3 \circ J_1 = -J_1 \circ J_3 = J_2.$$

Suppose that *N* is a unit normal to the hypersurface *M*. Then this unit normal will induce 3 unit vector fields on *M*, defined by $\xi_i = -J_i N$, $\forall i = 1, 2, 3,$ with dual 1-form $\eta_i(X) = g(X, \xi_i)$. The set $\{\xi_1, \xi_2, \xi_3\}$ is a set of normal orthonormal vectors of M^3, because

$$
\begin{aligned}
g(\xi_1, \xi_2) &= \langle J_1\xi_1, J_1\xi_2\rangle \\
&= \langle N, J_1\xi_2\rangle \\
&= -\langle J_1 N, \xi_2\rangle \\
&= \langle J_3 J_2 N, \xi_2\rangle \\
&= -\langle J_3\xi_2, \xi_2\rangle \\
&= 0
\end{aligned}
$$

and $g(\xi_i, \xi_i) = \langle J_i\xi_i, J_i\xi_i\rangle = \langle N, N\rangle = 1.$

Note. For $X \in \mathfrak{X}(M)$, $J_i X = \phi_i(X) + \eta_i(X)N$, where $\phi_i(X)$ is the tangential component of $J_i X$. Then it follows that ϕ_i is a (1, 1) tensor field on M^3. Using $J_i^2 = -I$, it is easy to see that $(\phi_i, \xi_i, \eta_i, g)$ defines an almost contact metric structure on M^3, that is,

$$\phi_i^2 X = -X + \eta_i(X)\xi_i,\ \eta_i(\xi_i) = 1,\ \eta_i \circ \phi_i = 0,\ \phi_i(\xi_i) = 0$$

and $g(\phi_i X, \phi_i Y) = g(X, Y) - \eta_i(X)\eta_i(Y),\ X, Y \in \mathfrak{X}(M).$

We immediately obtain the following

Lemma 4.1. *Let M^3 be an orientable hypersurface of R^4. Then the structure $(\phi_i, \xi_i, \eta_i, g), \forall i = 1, 2, 3$ on M^3 satisfies*

(i) $(\nabla_X \phi_i)(X) = \eta_i(X)SX - g(SX, X)\xi_i$

(ii) $D_X \xi_i = J_i S(X)$

(iii) $\nabla_X \xi_i = \phi_i SX.$

Note. We have the following relations

(1)

$$
\begin{aligned}
\Phi_2\xi_1 &= -\xi_3 \\
\Phi_2\xi_3 &= \xi_1 \\
\Phi_2\xi_2 &= 0
\end{aligned}
$$

(2)

$$\Phi_3\xi_1 = \xi_2$$

$$\Phi_3\xi_2 = -\xi_1$$

$$\Phi_3\xi_3 = 0$$

(3)

$$\Phi_1\xi_1 = 0$$

$$\Phi_1\xi_2 = \xi_3$$

$$\Phi_1\xi_3 = -\xi_2.$$

Note. Let $\Phi_1 S = S\Phi_1$. Then use $\Phi_1 SX = \nabla_X \xi_1$, and take $X = \xi_1$ $\Rightarrow \Phi_1 S\xi_1 = \nabla_{\xi_1}\xi_1 \Rightarrow \nabla_{\xi_1}\xi_1 = S\Phi_1\xi_1 = S(0) = 0.$

Theorem 4.1. *Let M be an orientable real hypersurface of R^4. If $\Phi_1 S = S\Phi_1$, and $\Phi_2 S = S\Phi_2$, then*

$$g(S\xi_i, \xi_j) = 0 \textit{ when}, \ i \neq j$$

and

$$g(S\xi_1, \xi_1) = g(S\xi_2, \xi_2) = g(S\xi_3, \xi_3).$$

Proof. We have

1.

$$\begin{aligned} g(S\xi_1, \xi_2) &= -g(S\xi_1, \Phi_1\xi_3) \\ &= g(\Phi_1 S\xi_1, \xi_3) \\ &= g(S\Phi_1\xi_1, \xi_3) \\ &= 0 \end{aligned}$$

2.

$$\begin{aligned} g(S\xi_1, \xi_3) &= g(S\xi_1, \Phi_1\xi_2) \\ &= -g(\Phi_1 S\xi_1, \xi_2) \\ &= g(S\Phi_1\xi_1, \xi_2) \\ &= 0 \end{aligned}$$

3.

$$\begin{aligned} g(S\xi_2, \xi_3) &= g(S\xi_2, \Phi_1\xi_2) \\ &= -g(\Phi_1 S\xi_2, \xi_2) \\ &= -g(S\Phi_1\xi_2, \xi_2) \\ &= -g(S\xi_3, \xi_2) \\ &= -g(\xi_3, S\xi_2) \\ &\Rightarrow g(S\xi_2, \xi_3) = 0 \end{aligned}$$

4.

$$\begin{aligned} g(S\xi_3, \xi_3) &= g(S\Phi_1\xi_2, \Phi_1\xi_2) \\ &= g(S\xi_2, \xi_2) \end{aligned}$$

and similarly

$$\begin{aligned} g(S\xi_1, \xi_1) &= g(S\Phi_2\xi_3, \Phi_2\xi_3) \\ &= g(S\xi_3, \xi_3). \end{aligned}$$

From this theorem we get the following

Corollary 4.1. *Let M be an orientable real hypersurface of $\mathfrak{R}^4$. Then under the assumption that the shape operator S of M in $\mathfrak{R}^4$ has the conditions*

$$S\Phi_1 = \Phi_1 S, \textit{ and } \Phi_2 S = S\Phi_2$$

we have

$$g(SX, Y) = g(S\xi_1, \xi_1)g(X, Y), \forall X, Y \in \mathfrak{X}(M).$$

Proof. For $X \in \mathfrak{X}(M)$, we can write $X = a\xi_1 + b\xi_2 + c\xi_3$. This implies that

$$\begin{aligned} g(SX, X) &= g(aS\xi_1 + bS\xi_2 + cS\xi_3, a\xi_1 + b\xi_2 + c\xi_3) \\ &= a^2 g(S\xi_1, \xi_1) + b^2 g(S\xi_2, \xi_2) + c^2 g(S\xi_3, \xi_3) \\ &= g(S\xi_1, \xi_1)[a^2 + b^2 + c^2] \\ &= g(S\xi_1, \xi_1)g(X, X) \\ &= g(g(S\xi_1, \xi_1)X, X) \\ &= g(S\xi_1, \xi_1)g(X, X). \end{aligned}$$

This gives

$$g(S(X+Y), (X+Y)) = g(S\xi_1, \xi_1)g(X+Y, X+Y)$$

which implies that $g(SX, Y) = g(S\xi_1, \xi_1)g(X, Y), \forall X, Y \in \mathfrak{X}(M)$.

Note. Since $S = fI$, where $f = g(S\xi_1, \xi_1)$, $(\nabla_X S)(Y) = X(f)Y$, and $(\nabla_X S)(Y) = (\nabla_Y S)(X)$, for a hypersurface of the Euclidean space. This implies that $X(f)Y = Y(f)(X)$. Taking Y as a unit vector field, and taking inner product with Y for both sides of the equation, it follows that

$$X(f)g(Y, Y) = Y(f)g(X, Y)$$

$$X(f) = Y(f)g(X, Y).$$

Assume $X = \xi_1$ and $Y = \xi_2 \Rightarrow \xi_1(f) = 0$. Then we get $\xi_2(f) = 0$ and $\xi_3(f) = 0$, which imply that $X(f) = 0$, $\forall X \Rightarrow f = g(S\xi_1, \xi_1)$ is a constant.

Theorem 4.2. *Let M be an orientable real hypersurface of* R^4, *and TM be its tangent bundle. Then* $\{\xi_1^v, \xi_2^v, \xi_3^v\}$ *is a set of linearly independent vectors*.

Proof. Assume that

$$\lambda_1\xi_1^v + \lambda_2\xi_2^v + \lambda_3\xi_3^v = 0.$$

Taking inner product with ξ_1^v, we have

$$\lambda_1\bar{g}(\xi_1^v, \xi_1^v) + \lambda_2\bar{g}(\xi_2^v, \xi_1^v) + \lambda_3\bar{g}(\xi_3^v, \xi_1^v) = 0$$

$$\Rightarrow$$

$$\lambda_1 g(\xi_1, \xi_1) + \lambda_2 g(\xi_2, \xi_1) + \lambda_3 g(\xi_3, \xi_1) = 0$$

$$\Rightarrow$$

$$\lambda_1 = 0.$$

Similarly we can show that $\lambda_2 = \lambda_3 = 0$.

Theorem 4.3. *Let M be an orientable real hypersurface of* $\mathfrak{R}^4$. *If* $S\Phi_1 = \Phi_1 S$, *and* $\Phi_2 S = S\Phi_2$, *then the sectional curvature of the horizontal plane sections is a constant*.

Proof. Since

$$\widetilde{R}(X^h, Y^h; Z^h, W^h) = \{g(S(Y), Z)g(S(X), W) - g(S(X), Z)g(S(Y), W)\} \circ \pi$$

using that $g(SX, Y) = g(S\xi_1, \xi_1)g(X, Y)$, and that $\alpha = g(S\xi_1, \xi_1) =$ constant *t*, we get

$$\widetilde{R}(X^h, Y^h; Z^h, W^h) = \alpha^2\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \circ \pi,$$

which proves the result.

Theorem 4.4. *The tangent bundle* TM^3 *of the hypersurface* M^3 *of* R^4 *under the conditions* $S\Phi_1 = \Phi_1 S,$ *and* $\Phi_2 S = S\Phi_2$ *is flat if and only if M is flat*.

Proof. Because of the existence of the Hopf condition on the hypersurface *M*, we get that *M* has a constant sectional curvature, say *c*. Thus

$$R(X, Y; Z, W) = c\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}.$$

Substitute this in the relation

$$\widetilde{R}(X^h, Y^h; Z^h, W^h) = \alpha^2\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \circ \pi$$

to get

$$\widetilde{R}(X^h, Y^h; Z^h, W^h) = \frac{\alpha^2}{c} R(X, Y; Z, W).$$

Since $\widetilde{R} = 0$ except for the horizontal vectors, it follows that $\widetilde{R} = 0$ $\Leftrightarrow R = 0,$ which proves the result.

Corollary 4.2. *The tangent bundle* TM^3 *of the hypersurface* M^3 *of* R^4 *under the conditions* $S\Phi_1 = \Phi_1 S,$ *and* $\Phi_2 S = S\Phi_2$ *is locally symmetric if and only if M is locally symmetric*.

Proof. This follows immediately from the relations

$$\widetilde{R}(X^h, Y^h; Z^h, W^h) = \frac{\alpha^2}{c} R(X, Y; Z, W)$$

$$\Rightarrow$$

$$(\overline{\nabla}_{X^h}\widetilde{R})(Y^h, Z^h)W^h = \frac{\alpha^2}{c}(\nabla_X R)(Y, Z)W$$

and $(\overline{\nabla}_{X^i}\widetilde{R})(Y^j, Z^k)W^l = 0,$ for $i, j, k, l \in \{h, v\}.$

5. Curvatures of *TM*

Assuming the conditions $S\Phi_1 = \Phi_1 S$, and $\Phi_2 S = S\Phi_2$, we obtain that $g(SX, Y) = g(S\xi_1, \xi_1) g(X, Y), \forall X, Y \in \mathfrak{X}(M)$, where $\alpha = g(S\xi_1, \xi_1)$ is a constant. Then Lemma 3.1 takes the form:

(i) $\bar{g}_P(X_P^h, Y_P^h) = g_p(X_p, Y_p) + \alpha^2 g_p(X_p, u) g_p(Y_p, u)$

(ii) $\bar{g}_P(X_P^h, Y_P^v) = 0$

(iii) $\bar{g}_P(X_P^v, Y_P^v) = g_p(X_p, Y_p)$

and Lemma 3.2 takes the form:

(i) $h(X^v, Y^v) = 0$

(ii) $h(X^v, Y^h) = 0$

(iii) $h(X^h, Y^h) = \alpha g(X, Y) \circ \pi\ N^h$.

Note. We can choose an orthonormal frame for *TM* as follows.

For a given point $(p, u) \in TM$, with $u \neq 0$, let $\{e_1, e_2, e_3\}$ be an orthonormal basis for M at p, such that $e_1 = \dfrac{u}{|u|}$. Then from this basis, we define the horizontal and vertical lifts by $f_1 = \dfrac{1}{\sqrt{\lambda}} e_1^h$, where $\lambda = 1 + (\alpha \cdot |u|)^2$, $f_i = e_i^h$, for $j = \{2, 3\}$ and $f_{j+3} = e_j^v$, for $j = \{1, 2, 3\}$. Now, we can easily prove that the set $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ forms an orthonormal basis for *TM*.

Proof. We have

(1)

$$\bar{g}(f_1, f_1) = \bar{g}\left(\frac{1}{\sqrt{\lambda}} e_1^h, \frac{1}{\sqrt{\lambda}} e_1^h\right)$$

$$= \frac{1}{\lambda}\{g(e_1, e_1) + \alpha^2 g(e_1, u)g(e_1, u)\}$$

$$= \frac{1}{\lambda}\left\{1 + \frac{\alpha^2}{|u|^2} g(u, u)g(u, u)\right\}$$

$$= \frac{1}{\lambda}\{1 + \alpha^2 |u|^2\}$$

$$= 1$$

(2)

$$\bar{g}(f_2, f_2) = \bar{g}(e_2^h, e_2^h)$$

$$= g(e_2, e_2) + \alpha^2 g(e_2, u)g(e_2, u),$$

$$= 1, \left(\text{since } e_1 = \frac{u}{|u|} \perp e_2\right).$$

Similarly, we obtain that $\bar{g}(f_3, f_3) = 1.$

(3) We can easily show that

$$\bar{g}(f_4, f_4) = \bar{g}(f_5, f_5) = \bar{g}(f_6, f_6) = 1,$$

and that the inner product of any pair of the set $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ is zero.

Lemma 5.1. *For M a Hopf hypersurface of R^4, the mean curvature* $\alpha = g(S\xi_1, \xi_2).$

Proof. We have

$$\alpha = \frac{1}{n}\sum_{i=1}^{n} g(Se_i, e_i)$$

for the orthonormal basis $\{e_1, e_2, e_3\}$ of *M*, and hence

$$\alpha = \frac{1}{3}\sum_{i=1}^{3} g(Se_i, e_i)$$

$$= \frac{1}{3}[g(Se_1, e_1) + g(Se_2, e_2) + g(Se_3, e_2)]$$

$$= \frac{1}{3} g(S\xi_1, \xi_1)[g(e_1, e_1) + g(e_2, e_2) + g(e_3, e_3)]$$

$$= \frac{1}{3} g(S\xi_1, \xi_1)[3]$$

$$= g(S\xi_1, \xi_1).$$

Theorem 5.1. *For the tangent bundle* TM^3 *of the hypersurface* M^3 *of* R^4 *under the conditions* $S\Phi_1 = \Phi_1 S$, *and* $\Phi_2 S = S\Phi_2$,

$$H = \alpha\left[\frac{1+2\lambda}{6\lambda}\right]N^h.$$

Proof. Since for the orthonormal basis $\{e_1, e_2, e_3\}$ of M, we have that $\{f_1, f_2, f_3, f_4, f_5, f_6\} = \left\{\frac{1}{\sqrt{\lambda}} e_1^h, e_2^h, e_3^h, e_1^v, e_2^v, e_3^v\right\}$ is an orthonormal basis for TM. Hence

$$H = \frac{1}{2 \times 3}\left[\sum_{i=1}^{6} h(f_i, f_i)\right]$$

$$= \frac{1}{6}\left[\frac{1}{\lambda} h(e_1^h, e_1^h) + h(e_2^h, e_2^h) + h(e_3^h, e_3^h) + h(e_1^v, e_1^v) + h(e_1^v, e_1^v) + h(e_1^v, e_1^v)\right]$$

$$= \frac{1}{6}\left[\frac{1}{\lambda}\alpha g(e_1, e_1)N^h + \alpha g(e_1, e_1)N^h + \alpha g(e_1, e_1)N^h\right]$$

$$= \frac{\alpha}{6}\left[\frac{1}{\lambda} + 2\right]N^h$$

$$= \alpha\left[\frac{1+2\lambda}{6\lambda}\right]N^h,$$

where $\alpha = g(S\xi_1\xi_1)$ is the mean curvature.

Lemma 5.2. *The tangent bundle* TM^3 *of the hypersurface* M^3 *of* R^4 *under the conditions* $S\Phi_1 = \Phi_1 S$, *and* $\Phi_2 S = S\Phi_2$ *has a non-negative sectional curvature* $\widetilde{K}$ *which satisfies the following*

$$\widetilde{K}(f_i, f_j) = \alpha^2, \text{ for } i, j = \{1, 2, 3\}, i \neq j$$

and

$$\widetilde{K}(f_i, f_k) = 0, \text{ for } i = \{1, 2, 3\}, k = \{4, 5, 6\}.$$

Proof. At a point (p, u) on *TM*, using $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ as an orthonormal basis for the tangent space $T_{(p,u)}TM$, the sectional curvature $\widetilde{K}$ is given by

$$\widetilde{K}(f_1, f_2) = \frac{\widetilde{R}(f_1, f_2, f_2, f_1)}{\| f_1 \wedge f_2 \|^2}$$

$$= \frac{\widetilde{R}\left(\frac{1}{\sqrt{\lambda}} e_1^h, e_2^h, e_2^h, \frac{1}{\sqrt{\lambda}} e_1^h\right)}{\left\| \frac{1}{\sqrt{\lambda}} e_1^h \wedge e_2^h \right\|^2}$$

$$= \frac{\frac{1}{\lambda}\alpha^2\{g(e_2, e_2)g(e_1, e_1) - g(e_1, e_2)^2\}}{\frac{1}{\lambda}\{g(e_2, e_2)g(e_1, e_1) - g(e_1, e_2)^2\}}$$

$$= \alpha^2.$$

Similarly, we get that $\widetilde{K}(f_1, f_3) = \alpha^2$, and

$$\widetilde{K}(f_2, f_2) = \frac{\widetilde{R}(f_2, f_3, f_3, f_2)}{\| f_1 \wedge f_2 \|^2}$$

$$= \frac{\widetilde{R}(e_2^h, e_3^h, e_3^h, e_2^h)}{\| e_2^h \wedge e_3^h \|^2}$$

$$= \frac{\alpha^2\{g(e_2, e_2)g(e_3, e_3) - g(e_3, e_2)^2\}}{\{g(e_2, e_2)g(e_3, e_3) - g(e_3, e_2)^2\}}$$

$$= \alpha^2.$$

Similarly,

$$\widetilde{K}(f_1, f_k) = \frac{\widetilde{R}(f_1, f_k, f_k, f_1)}{\| f_1 \wedge f_k \|^2}$$

$$= \frac{\widetilde{R}\left(\frac{1}{\sqrt{\lambda}} e_1^h, e_k^v, e_k^v, \frac{1}{\sqrt{\lambda}} e_1^h\right)}{\left\| \frac{1}{\sqrt{\lambda}} e_1^h \wedge e_k^v \right\|^2}$$

$$= 0,$$

and $\widetilde{K}(f_2, f_k) = \widetilde{K}(f_3, f_k) = 0.$

Theorem 5.2. *If* TM^3 *is the tangent bundle of the hypersurface* M^3 *of* R^4. *Then under the conditions* $S\Phi_1 = \Phi_1 S$, *and* $\Phi_2 S = S\Phi_2$, *the Ricci curvature* $\widetilde{Ric}$ *of* $(TM, \overline{g})$ *satisfies the following*:

$$\widetilde{Ric}(X^h, Y^h) = \frac{\alpha^2}{\lambda}\{g(X^h, Y^h) - 2\alpha^2 g(X, u)g(Y, u)\} + 3\alpha^2 g(X, Y)$$

and

$$\widetilde{Ric}(X^h, Y^v) = \widetilde{Ric}(X^v, Y^v) = 0.$$

Proof. We have

$$\widetilde{Ric}(X, Y) = \sum_{i=1}^{6} \tilde{R}(f_i, X, Y, f_i)$$

for X^h, Y^h, and hence

$\widetilde{Ric}(X^h, Y^h)$

$$= \sum_{i=1}^{6} \tilde{R}(f_i, X^h, Y^h, f_i)$$

$$= \sum_{i=1}^{3} \widetilde{R}(f_i, X^h, Y^h, f_i)$$

$$= \widetilde{R}\left(\frac{1}{\sqrt{\lambda}} e_1^h, X^h, Y^h, \frac{1}{\sqrt{\lambda}} e_1^h\right) + \widetilde{R}(e_2^h, X^h, Y^h, e_2^h)$$

$$+ \widetilde{R}(e_3^h, X^h, Y^h, e_3^h)$$

$$= \alpha^2 \left\{\frac{1}{\lambda} g(X, Y) g(e_1, e_1) + \frac{1}{\lambda} g(e_1, X) g(Y, e_1)\right\}$$

$$+ \alpha^2 \{g(X, Y) g(e_2, e_2)\} + \alpha^2 \{g(X, Y) g(e_3, e_3) + g(e_3, X) g(Y, e_3)\}$$

$$= \frac{\alpha^2}{\lambda} \{g(X, Y) + g(e_1, X) g(Y, e_1)\} + \alpha^2 \{g(X, Y) + g(e_2, X) g(Y, e_2)\}$$

$$+ \alpha^2 \{g(X, Y) g(e_3, X) g(Y, e_3)\}$$

$$= g(X, Y)\left\{\frac{\alpha^2}{\lambda} + 2\alpha^2\right\} + \frac{\alpha^2}{\lambda} g(e_1, X) g(Y, e_1)$$

$$+ \alpha^2 g(e_2, X) g(Y, e_2) + \alpha^2 g$$

$$= g(X, Y) \frac{\alpha^2}{\lambda} + 3\alpha^2 g(X, Y) + \left(\frac{\alpha^2}{\lambda} - \alpha^2\right) g(e_1, X) g(Y, e_1)$$

$$= \frac{\alpha^2}{\lambda} g(X, Y) + 3\alpha^2 g(X, Y) + \frac{\alpha^2}{\lambda}\left(\frac{1-\lambda}{\| u \|^2}\right) g(X, u) g(Y, u)$$

$$= \frac{\alpha^2}{\lambda} g(X, Y) + 3\alpha^2 g(X, Y) + \frac{\alpha^2}{\lambda}\left(\frac{-\alpha^2 \| u \|^2}{\| u \|^2}\right) g(X, u) g(Y, u)$$

$$= \frac{\alpha^2}{\lambda} \{g(X, Y) - \alpha^2 g(X, u) g(Y, u)\} + 3\alpha^2 g(X, Y)$$

$$= \frac{\alpha^2}{\lambda}\{g(X^h, Y^h) - 2\alpha^2 g(X, u)g(Y, u)\} + 3\alpha^2 g(X, Y).$$

Similarly we can calculate that

$$\widetilde{Ric}(X^h, Y^v) = \sum_{i=1}^{6} \tilde{R}(f_i, X^h, Y^v, f_i) = 0$$

and

$$\widetilde{Ric}(X^v, Y^v) = \sum_{i=1}^{6} \tilde{R}(f_i, X^v, Y^v, f_i) = 0.$$

Theorem 5.3. *Let* TM^3 *be the tangent bundle of the hypersurface* M^3 *of* R^4, *Then under the conditions* $S\Phi_1 = \Phi_1 S$, *and* $\Phi_2 S = S\Phi_2$, *the scalar curvature* $\widetilde{S}$ *of* $(TM, \overline{g})$ *satisfies the following*:

$$\widetilde{S} = \frac{\alpha^2}{\lambda^2}\left\{-2 + 3\lambda\left(\frac{5}{3} + 3\alpha^2\lambda\right)\right\}.$$

Thus, *TM has a constant scalar curvature*.

Proof. We have

(1) $\widetilde{Ric}(f_i, f_i) = 0,$ for $i = \{4, 5, 6\}$.

(2) $\widetilde{Ric}(f_1, f_1) = \widetilde{Ric}\left(\frac{1}{\sqrt{\lambda}}e_1^h, \frac{1}{\sqrt{\lambda}}e_1^h\right) = \frac{\alpha^2}{\lambda}\left\{1 - \frac{2\alpha^2}{\lambda} g\left(u, \frac{u}{\| u \|}\right)^2\right\} + 3\alpha^2$

$$= \frac{\alpha^2}{\lambda^2}\{\lambda - 2\alpha^2 \| u \|^2 + 3\alpha^2\lambda^2\} = \frac{\alpha^2}{\lambda^2}\{\lambda - 2 + 2\lambda + 3\alpha^2\lambda^2\}$$

$$= \frac{\alpha^2}{\lambda^2}\{-2 + 3\lambda(1 + \alpha^2\lambda)\}.$$

(3) $\widetilde{Ric}(f_2, f_2) = \widetilde{Ric}(e_2^h, e_2^h) = \frac{\alpha^2}{\lambda}\{g(e_2^h, e_2^h) - 2\alpha^2 g(e_2, u)g(e_2, u)\}$

$$+3\alpha^2 g(e_2, e_2) = \frac{\alpha^2}{\lambda} + 3\alpha^2.$$

(4) $$\widetilde{Ric}(f_3, f_3) = \widetilde{Ric}(e_3^h, e_3^h) = \frac{\alpha^2}{\lambda}\{g(e_3^h, e_3^h) - 2\alpha^2 g(e_3, u)g(e_3, u)\}$$

$$+3\alpha^2 g(e_3, e_3) = \frac{\alpha^2}{\lambda} + 3\alpha^2$$

$$\tilde{S} = \sum_{i=1}^{6} \widetilde{Ric}(f_i, f_j)$$

$$= \widetilde{Ric}(f_1, f_1) + \widetilde{Ric}(f_2, f_2) + \widetilde{Ric}(f_3, f_3)$$

$$= \frac{\alpha^2}{\lambda^2}\{-2 + 3\lambda(1 + \alpha^2\lambda)\} + 2\left\{\frac{\alpha^2}{\lambda} + 3\alpha^2\right\}$$

$$= \frac{\alpha^2}{\lambda^2}\{-2 + 3\lambda(1 + \alpha^2\lambda)\} + \frac{\alpha^2}{\lambda^2}\{2\lambda + 6\lambda\alpha^2\}$$

$$= \frac{\alpha^2}{\lambda^2}\left\{-2 + 3\lambda\left(\frac{5}{3} + 3\alpha^2\lambda\right)\right\}.$$

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