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LOCAL FORM OF SYSTEMS SATISFYING SOME CONDITIONS ON THE SMALL AND THE GREAT GROWTH VECTOR

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Abstract

In this article, we give the normal form, in $\mathbb{R}^n$, of a family of distributions, fulfilling everywhere Goursat's condition [7] but satisfying singularities in the small growth vector at the origin:

$$\dim(E_{n-2}|_0) = \dim(E_{n-3}|_0) = n - 2$$

and

$$\dim(E_{n-1}|_0) = \dim(E_n|_0) = n - 1.$$

1. Main Theorem

Theorem 1. *Let E be a 2-distribution of $\mathbb{R}^n$ satisfying the condition of Goursat at any point and the small growth vector $[2, 3, \ldots, n-3, (n-2)_k, (n-1)_k, n]$, at the point $x_0 \in \mathbb{R}^n$. Then there exists a local coordinate system (x, U) about x_0, such that $E^{\perp}$ is generated by*

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$$\begin{cases} \omega_1 & = & dx^2 + x^3 dx^1 \\ \omega_2 & = & dx^3 + x^4 dx^1 \\ \omega_3 & = & dx^4 + x^5 dx^1 \\ \omega_4 & = & dx^1 + x^6 dx^5 \\ \omega_5 & = & dx^6 + X_7 dx^5 \\ \vdots & & \vdots \\ \omega_{n-2} & = & dx^{n-1} + X_n dx^5, \end{cases}$$

where $X_i = C_i + x^i$, *with* $C_i \in \mathbb{R}$.

Proof. Since E satisfies the Goursat condition, $E^{\perp}$ is generated by ([2])

$$\begin{cases} \omega_1 & = & dx^2 + x^3 dx^1 \\ \omega_2 & = & dx^3 + x^4 dx^1 \\ \omega_3 & = & dx^{i_3} + x^5 dx^{j_3}; (i_3, j_3) \in \{(1, 4), (4, 1)\} \\ \omega_4 & = & dx^{i_4} + X_6 dx^{j_4}; (i_4, j_4) \in \{(5, j_3), (j_3, 5)\} \\ \vdots & & \\ \omega_{n-2} & = & dx^{i_{n-2}} + X_n dx^{j_{n-2}}; (i_{n-2}, j_{n-2}) \in \{(n-1, j_{n-3}), (j_{n-3}, n-1)\}, \end{cases}$$

where $X_i = C_i + x^i$, with $C_i \in \mathbb{R}$.

To prove the theorem we adopt following steps:

(1) Obtain that $\omega_3 = dx^4 + x^5 dx^1$.

(2) Obtain that necessarily $\omega_4 = dx^1 + x^6 dx^5$.

(3) Finally, obtain that $\omega_i = dx^{i+1} + X_{i+2} dx^5$ for every $i = 5, 6, \ldots, n-2$.

Step 1. We have two possibilities for ω_3:

$$\omega_3 = dx^4 + x^5 dx^1 \quad \text{or} \quad \omega_3 = dx^1 + x^5 dx^4.$$

Suppose that $\omega_3 = dx^1 + x^5 dx^4$. In this case $E^{\perp}$ is generated by

$$\begin{cases} \omega_1 = dx^2 + x^3 dx^1 \\ \omega_2 = dx^3 + x^4 dx^1 \\ \omega_3 = dx^1 + x^5 dx^4 \\ \omega_4 = dx^{i_4} + X_6 dx^{j_4};\ (i_4, j_4) \in \{(5, 4), (4, 5)\} \\ \vdots \\ \omega_{n-2} = dx^{i_{n-2}} + X_n dx^{j_{n-2}};\ (i_{n-2}, j_{n-2}) \in \{(n-1, j_{n-3}), (j_{n-3}, n-1)\}. \end{cases}$$

It means E is generated by ∂_n and

$$Y = p_4(-x^5\partial_1 + x^3x^5\partial_2 + x^4x^5\partial_3 + \partial_4 + p_5\partial_5 + \cdots + p_{n-1}\partial_{n-1}),$$

where $p_4 = p_4(x^6, ..., x^n)$; $p_5 = p_5(x^6, ..., x^n)$ and $p_i = p_i(x^{i+1}, ..., x^n)$, for $i = 6, 7, ..., n-1$.

We have that $(E^{n-4})^{\perp}$ is generated by $\omega_1, \omega_2, \omega_3$ and $\dim(E^{n-4}) = \dim(E_{n-4}) = n-3$. Therefore, E^{n-4} is generated by

$$v_1 = -x^5\partial_1 + x^3x^5\partial_2 + x^4x^5\partial_3 + \partial_4;\ \partial_5;\ \cdots;\ \partial_n$$

but $E_{n-3} = [E, E_{n-4}]$. $[\partial_5, Y]|_0 = p_4(-\partial_1 + x^3\partial_2 + x^4\partial_3)|_0 = -p_4(0)\partial_1$, so necessarily $p_4(0) \neq 0$. Now,

$$E_{n-3} = E^{n-3} = \begin{cases} \omega_1 = dx^2 + x^3dx^1 = 0 \\ \omega_2 = dx^3 + x^4dx^1 = 0 \end{cases}$$

so, $E_{n-3} = \mathrm{span}\{Z = \partial_1 - x^3\partial_2 - x^4\partial_3, \partial_4; \partial_5; \cdots; \partial_n\}$ and $\dim E_{n-3} = n-2$. Since

$$[\partial_4, Y] = p_4x^5\partial_3$$

$$[Z, Y]|_0 = p_4(0)\partial_3 \neq 0,$$

$E_{n-1}|_0$ has a new dimension but $\dim E_{n-2}|_0 = n-2$ which is not impossible.

Step 2. We prove that $\omega_4 = dx^1 + x^6 dx^5$. In contrary, suppose that $\omega_4 = dx^5 + X_6 dx^1$, then

$$E_{n-5} = E^{n-5} = \begin{cases} \omega_1 = dx^2 + x^3 dx^1 = 0 \\ \omega_2 = dx^3 + x^4 dx^1 = 0 \\ \omega_3 = dx^4 + x^5 dx^1 = 0 \\ \omega_4 = dx^5 + X_6 dx^1 = 0 \end{cases}.$$

Therefore, $E^{n-5} = \operatorname{Span}\{\partial_1 - x^3\partial_2 - x^4\partial_3 - x^5\partial_4 - X_6\partial_5;\ \partial_6, \ldots;\ \partial_n\}$ and

$$E^{\perp} = \begin{cases} \omega_1 & = & dx^2 + x^3 dx^1 \\ \omega_2 & = & dx^3 + x^4 dx^1 \\ \omega_3 & = & dx^4 + x^5 dx^1 \\ \omega_4 & = & dx^5 + X_6 dx^1 \\ \omega_5 & = & dx^{i_5} + X_7 dx^{j_5}; (i_5, j_5) \in \{(6,1),(1,6)\} \\ \vdots & \vdots & \vdots \\ \omega_{n-2} & = & dx^{i_{n-2}} + X_n dx^{j_{n-2}}; (i_{n-2}, j_{n-2}) \in \{(n-1, j_{n-3}), (j_{n-3}, n-1)\}, \end{cases}$$

and hence, E is generated by $Y = p_5(-\partial_1 - x^3\partial_2 - x^4\partial_3 - x^5\partial_4 - X_6\partial_5) + p_6\partial_6 + \cdots + p_{n-1}\partial_{n-1}$ and ∂_n, where $p_5 = p_5(x^7, \ldots, x^n)$; $p_6 = p_6(x^7, \ldots, x^n)$ and $p_i = p_i(x^{i+1}, \ldots, x^n)$, for $i = 7, \ldots, n-1$.

Thus $(E^{n-4})^{\perp}$ is generated by $\omega_1, \omega_2, \omega_3$ and $\dim(E^{n-4}) = \dim(E_{n-4}) = n-3$, $\dim(E^{n-5}) = n-4$. Since $[\partial_6, Y]|_0 = -p_5(0)\partial_5 \neq 0$, $p_5(0) \neq 0$. It thus follows that $E^{n-4} = [E, E^{n-5}]$ is generated by

$$\partial_1 - x^3\partial_2 - x^4\partial_3 - x^5\partial_4;\ \partial_5;\ \cdots;\ \partial_n$$

but $E_{n-3} = [E, E_{n-4}]$. $[\partial_5, Y] = -p_5\partial_4$. Since

$$E_{n-3} = E^{n-3} = \begin{cases} \omega_1 = dx^2 + x^3 dx^1 = 0 \\ \omega_2 = dx^3 + x^4 dx^1 = 0 \end{cases}$$

so, $E_{n-3} = \operatorname{span}\{\partial_1 - x^3\partial_2 - x^4\partial_3,\ \partial_4;\ \partial_5;\ \cdots;\ \partial_n\}$ and $\dim E_{n-3} = n-2$.

As $[\partial_4, Y]|_0 = -p_5(0)\partial_3$, we have a new dimension for E_{n-2} but $\dim E_{n-2}|_0 = n-2$, which is not possible.

Step 3. Now, we prove $\omega_i = dx^{i+1} + X_{i+2}dx^5$, for every $i = 5, 6, \ldots, n-2$. In contrary, suppose that there exists i, $5 \le i \le n-2$, such that $\omega_i = dx^5 + x^{i+2}dx^{i+1}$, then

$$E^{\perp} = \begin{cases} \omega_1 & = & dx^2 + x^3 dx^1 \\ \omega_2 & = & dx^3 + x^4 dx^1 \\ \omega_3 & = & dx^4 + x^5 dx^1 \\ \omega_4 & = & dx^1 + x^6 dx^5 \\ \omega_5 & = & dx^6 + X_7 dx^5 \\ \vdots & & \vdots \\ \omega_{i-1} & = & dx^i + X_{i+1}dx^5 \\ \omega_i & = & dx^5 + x^{i+2}dx^{i+1} \\ \vdots & \vdots & \vdots \\ \omega_{n-2} & = & dx^{i_{n-2}} + X_n dx^{j_{n-2}}; (i_{n-2}, j_{n-2}) \in \{(n-1, j_{n-3}), (j_{n-3}, n-1)\} \end{cases}$$

and hence E is generated by

$$\begin{aligned} Y = {} & p_{i+1}(x^6x^{i+2}\partial_1 - x^3x^6x^{i+2}\partial_2 - x^4x^6x^{i+2}\partial_3 - x^5x^6x^{i+2}\partial_4 \\ & - x^{i+2}\partial_5 + X_7x^{i+2}\partial_6 + \cdots + X_{i+1}x^{i+2}\partial_i + \partial_{i+1}) \\ & + p_{i+2}\partial_{i+2} + \cdots + p_{n-1}\partial_{n-1} \end{aligned}$$

and ∂_n, where $p_{i+1} = p_{i+1}(x^{i+3}, \ldots, x^n)$; $p_{i+2} = p_{i+2}(x^{i+3}, \ldots, x^n)$ and $p_j = p_j(x^{j+1}, \ldots, x^n)$, for $j = i+2, \ldots, n-1$.

Thus $(E^{n-i})^{\perp}$ is generated by $\omega_1, \omega_2, \ldots, \omega_{i-1}$ and $\dim(E^{n-i}) = n-i+1$, $\dim(E^{n-i+1}) = n-i+2$, and hence $(E^{n-4})^{\perp}$ is generated by ω_1, ω_2, ω_3. It means $E^{n-4} = \operatorname{span}\{T; \partial_4; \cdots; \partial_n\}$, with $T = \partial_1 - x^3\partial_2 - x^4\partial_3 - x^5\partial_4$.

For E_{n-3}, we have the possible new dimension:

$$[T, Y] = -p_{i+1}x^{i+2}\partial_4$$

with $i + 2 \geq 7$. Then $[T, Y]|_0 = 0$, and we deduce that $\dim(E_{n-3})_0 = \dim(E_{n-4})_0 = n - 3$, which is not possible.

Step 4. By the previous steps we obtain

$$E^{\perp} = \begin{cases} \omega_1 &= dx^2 + x^3dx^1 \\ \omega_2 &= dx^3 + x^4dx^1 \\ \omega_3 &= dx^4 + x^5dx^1 \\ \omega_4 &= dx^1 + x^6dx^5 \\ \omega_5 &= dx^6 + X_7dx^5 \\ \vdots & \qquad \vdots \\ \omega_{n-2} &= dx^{n-1} + X_ndx^5 \end{cases}$$

so, E is generated by $Y = x^6(-\partial_1 + x^3\partial_2 + x^4\partial_3 + x^5\partial_4) + \partial_5 - X_7\partial_6 - \cdots + X_n\partial_{n-1}$ and ∂_n.

Thus $E^{n-3} = \operatorname{span}\{L = \partial_1 - x^3\partial_2 - x^4\partial_3, \partial_4; \partial_5; \cdots; \partial_n\}$ and $\dim E_{n-3} = n - 2$. Since

$$[\partial_4, Y] = x^6\partial_3 \text{ and } [L, Y] = x^5x^6\partial_3,$$

we have

$$E_{n-2} = \operatorname{span}\{x^6\partial_3, L, \partial_4; \partial_5; \cdots; \partial_n\}, \text{ and } \dim E_{n-2}|_0 = n - 2.$$

For

$$E_{n-1} : [x^6\partial_3, Y] = (x^6)^2\partial_2 + X_7\partial_3; [x^6\partial_3, Y]_0 = C_7\partial_3|_0,$$

$\dim E_{n-1}|_0 = n - 1$. Then necessarily $C_7 \neq 0$.

Remark. Since $M = [(x^6)^2\partial_2 + X_7\partial_3, Y] = 3X_7x^6\partial_2 + X_8\partial_3$ and

$[M, Y] = 3(-x^6 X_8 - (x_7)^2)\partial_2 - X_9\partial_3$, $[M, Y]_0 = -C_7\partial_2$ and $\dim E_n|_0 = n-1$, $\dim E_{n+1}|_0 = n$.

Step 5. By the previous steps, we obtain

$$E^{\perp} = \begin{cases} \omega_1 & = & dx^2 + x^3 dx^1 \\ \omega_2 & = & dx^3 + x^4 dx^1 \\ \omega_3 & = & dx^4 + x^5 dx^1 \\ \omega_4 & = & dx^1 + x^6 dx^5 \\ \omega_5 & = & dx^6 + (C_7 + x^7)dx^5; \; C_7 \neq 0 \\ \vdots & & \vdots \\ \omega_{n-2} & = & dx^{n-1} + (C_n + x^n)dx^5. \end{cases}$$

Let $C = C_7$. Then using the change in variables

$$x^1 = Cy^1,\; x^2 = C^3 y^2,\; x^3 = C^2 y^3,\; x^4 = Cy^4,$$

$$x^5 = y^5,\; x^6 = Cy^6, \ldots, x^n = Cy^n,$$

we obtain

$$E^{\perp} = \begin{cases} \omega_1 & = & dx^2 + x^3 dx^1 \\ \omega_2 & = & dx^3 + x^4 dx^1 \\ \omega_3 & = & dx^4 + x^5 dx^1 \\ \omega_4 & = & dx^1 + x^6 dx^5 \\ \omega_5 & = & dx^6 + (1 + x^7)dx^5 \\ \omega_6 & = & dx^7 + (k_n + x^n)dx^5 \\ \vdots & & \vdots \\ \omega_{n-2} & = & dx^{n-1} + (k_n + x^n)dx^5 \end{cases}$$

with $k_i \in \mathbb{R}$.

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