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CHARACTERIZATIONS OF REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM WITH RESPECT TO STRUCTURE TENSOR FIELD

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Abstract

Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. In this paper, we prove that if the structure tensor field is (i) Lie ξ-parallel and (ii) ξ-parallel, then M is a Hopf hypersurface. We characterize such Hopf hypersurfaces of $M_n(c)$.

1. Introduction

A complex n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. As is well-known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n\mathbf{C}$, a complex Euclidean space $\mathbf{C}^n$ or a complex hyperbolic space $H_n\mathbf{C}$, according to $c > 0$, $c = 0$ or $c < 0$.

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In this paper, we consider a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kaehler metric and complex structure J on $M_n(c)$. The Reeb vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant [4] and that M is called a *Hopf hypersurface*.

Typical examples of Hopf hypersurfaces in $P_n\mathbf{C}$ are homogeneous ones, namely those real hypersurfaces are given as orbits under subgroup of the projective unitary groups $PU(n+1)$. Takagi [10] completely classified homogeneous real hypersurfaces in such hypersurfaces as six model spaces A_1, A_2, B, C, D and E. On the other hand, real hypersurfaces in $H_n\mathbf{C}$ have been investigated by Berndt [1], Montiel and Romero [6] and so on. Berndt [1] classified all homogeneous Hopf hyersurfaces in $H_n\mathbf{C}$ as four model spaces which are said to be A_0, A_1, A_2 and B. A real hypersurface of A_1 or A_2 in $P_n\mathbf{C}$ or A_0, A_1, A_2 in $H_n\mathbf{C}$, is said to be a *type A* for simplicity.

As a typical characterization of real hypersurfaces of type A, the following is due to Okumura [8] for $c > 0$ and Montiel and Romero [6] for $c < 0$.

Theorem 1.1 [6, 8]**.** *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $A\phi - \phi A = 0$ on M if and only if M is locally congruent to one of the model spaces of type A.*

For the structure tensor field ϕ on M, we define the Lie derivative $\mathcal{L}_\xi$ by $(\mathcal{L}_\xi\phi)X = [\xi, \phi X] - \phi[\xi, X]$, and $(\nabla_\xi\phi)X$ with respect to a unit vector field X. We call the Lie derivative and covariant derivative in the Reeb vector field ξ direction of the structure tensor field as ξ-*Lie parallel* and ξ-*parallel*. Several workers have studied real hypersurfaces with certain conditions and obtained results on the classification of real hypersurfaces in complex space form $M_n(c)$.

As for the differential operator, Maeda and Udagawa [5] and Lim and Jun [4] have proved the following theorems:

Theorem 1.2 [5]**.** *Let M be a real hypersurface of* $P_n(c)$. *Then the following are equivalent*:

(i) $\mathcal{L}_\xi \phi = 0$, *where* $\mathcal{L}_\xi$ *is the Lie derivative on M*, *namely*, ξ *is an infinitesimal automorphism of* ϕ.

(ii) *M is locally congruent to one of homogeneous real hypersurfaces of types* A_1 *and* A_2.

Theorem 1.3 [4]**.** *Let M be a real hypersurface of* $M_n(c)$. *Then we have* $(\mathcal{L}_\xi L_\xi)X = (\nabla_\xi L_\xi)X$ *on M if and only if M is locally congruent to one of the model spaces of type A.*

Theorem A is a generalization of Theorem 1.2, the proof of this theorem is also different from Maeda and Udagawa [5] paper because it classifies the real hypersurface according to the kind of principal curvature under the Hopf hypersurface hypothesis. Also, from Theorem 1.3, it is found that the different derivatives of structure Lie operator have the geometrical properties of the real hypersurface. Therefore, Theorem B investigates the characteristics of the real hypersurfaces through these differentiations of the structure tensor field. In other words, we prove the following theorems:

Theorem A. *Let M be a real hypersurface satisfying* $\mathcal{L}_\xi \phi = 0$ *in a nonflat complex space form* $M_n(c)$. *Then M is a Hopf hypersurface and it is locally c congruent to one of the model spaces of type A in* $M_n(c)$.

Theorem B. *Let M be a real hypersurface satisfying* $\mathcal{L}_\xi \phi = \nabla_\xi \phi$ *in a nonflat complex space form* $M_n(c)$. *Then M is a Hopf hypersurface and it is locally congruent to one of the model spaces of type A in* $M_n(c)$.

All manifolds in the present paper are assumed to be connected and of class C^{∞} and the real hypersurfaces supposed to be orientable.

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c)$, and N be a unit normal vector field of M. By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\tilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX,$$

for any vector fields X and Y tangent to M, where g denotes the Riemannian metric tensor of M induced from $\tilde{g}$, and A is the shape operator of M in $M_n(c)$. For any vector field X on M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi,$$

where J is the almost complex structure of $M_n(c)$. Then we see that M induces an almost contact metric structure (ϕ, g, ξ, η), that is,

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{1}$$

for any vector fields X and Y on M. Since the almost complex structure J is parallel, we can verify from the Gauss and Weingarten formulas that

$$\nabla_X \xi = \phi AX, \tag{2}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{3}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the following Gauss, Codazzi equations and operator of Lie derivative, respectively:

$$R(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y$$
$$- 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \quad (4)$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \quad (5)$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

Let Ω be the open subset of M defined by

$$\Omega = \{p \in M \mid A\xi - \alpha\xi \neq 0\}, \quad (6)$$

where $\alpha = \eta(A\xi)$. We put

$$A\xi = \alpha\xi + \mu W, \quad (7)$$

where W is a unit vector field orthogonal to ξ and μ does not vanish on Ω.

3. Some Lemmas and Proof of the Theorems

In this section, we prove Theorems A and B. Now we state the following without proof:

Lemma 3.1 [4]**.** *If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

Lemma 3.2 [9]**.** *Assume that ξ is a principal curvature vector and the corresponding principal is α. Then*

$$A\phi A - \frac{\alpha}{2}(A\phi + \phi A) - \frac{c}{4}\phi = 0. \quad (8)$$

Proof of Theorem A. We assume that $\mathcal{L}_\xi\phi = 0$ for any vector field X. Then we have

$$(\mathcal{L}_\xi\phi)_X = [\xi, \phi X] - \phi[\xi, X] = \nabla_\xi(\phi X) - \nabla_{\phi X}\xi - \phi(\nabla_\xi X - \nabla_X\xi)$$
$$= (\nabla_\xi\phi)X - \phi A\phi X - AX + \eta(AX)\xi,$$

for any vector field X. Since Lie derivative of structure tensor field is zero, the above equation can be expressed as

$$(\nabla_\xi \phi)X = \phi A \phi X + AX - \eta(AX)\xi. \tag{9}$$

If we now use equation (3), then we obtain

$$\phi A \phi X + AX = \eta(X)A\xi. \tag{10}$$

If we put $X = W$ into (10), then we have

$$\phi A \phi W + AW = O. \tag{11}$$

If we take inner product of ξ into (11), then we immediately obtain $\mu = 0$ on Ω and it is a contradiction. Thus the set Ω is empty, and hence M is a Hopf hyersurface in $M_n(c)$.

The assumption $\mathcal{L}_\xi \phi = 0$ is equivalent to

$$\phi A \phi X + AX = 0 \tag{12}$$

by use of (1) and (3).

For any vector field $X(\perp \xi)$ on M such that $AX = \lambda X$, it follows from (8) that

$$\left(\lambda - \frac{\alpha}{2}\right)A\phi X = \frac{1}{2}\left(\alpha\lambda + \frac{c}{2}\right)\phi X. \tag{13}$$

If $\lambda \neq \frac{\alpha}{2}$, then we see from (13) that ϕX is also principal direction, say $A\phi X = \mu\phi X$.

From equation (12) and using the first equation (1), we have $\lambda = \mu$ and hence $A\phi X = \phi AX$. If $\lambda = \frac{\alpha}{2}$, then it is easily seen that $A\phi X = \phi AX$. Therefore, we have $\phi A - A\phi = 0$ on M. Thus follows Theorem A. □

Now, we will characterize the real hypersurfaces of a nonflat complex space form satisfying $\mathcal{L}_\xi \phi = \nabla_\xi \phi$ for any vector field X on M, we can state

Proof of Theorem B. By Theorem A, the real hypersurface *M* satisfying $\mathcal{L}_\xi\phi = \nabla_\xi\phi$ is equivalent to

$$\phi A\phi X + AX = \eta(AX)\xi, \tag{14}$$

for any vector field *X* on *M*.

If we put $X = \xi$ into (14) and make use of (6), then we obtain $A\xi = \alpha\xi$. Hence, by virtue of (6), the set Ω is empty and hence *M* is a Hopf hypersurface.

Since *M* is a Hopf hypersurface and satisfies (12), we conclude that *M* is locally congruent to one of the model spaces of type *A*. □

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