Far East Journal of Mathematical Sciences (FJMS)

http://www.pphmj.com
http://dx.doi.org/10.17654/MS104020175
Volume 104, Number 2, 2018, Pages 175-205 **ISSN: 0972-0871**

THE EQUATIONS OF MATHEMATICAL PHYSICS AS A FOUNDATION OF THE FIELD-THEORY EQUATIONS

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Abstract

It is well known that the field-theory equations are equations for such functionals as wave function, the action functional, Pointing's vector, Einstein's tensor, and others.

It turns out that from the equations of mathematical physics describing material media, the evolutionary relation for the same functionals follows.

This fact demonstrates a connection between the field-theory equations and the equations of mathematical physics. Such a connection enables one to understand foundations of field-theory and the properties of physical fields.

Such results were obtained by investigation of the properties and specific features of the conservation laws of material media and physical fields carried out with the help of skew-symmetric forms, which describe the conservation laws.

Received: September 27, 2017; Accepted: March 25, 2018

Keywords and phrases: conservation laws, non-identical evolutionary relation, double solutions of equations of mathematical physics, realization of physical structures, linkage between field-theory equations and the equations of mathematical physics, foundations of field theory.

0. Introduction

In the end of 19th and the beginning of 20th century, the following problem arose in physics. It turns out that one cannot describe observed physical structures such as massless particles, electromagnetic waves, gravitational fields and others with the help of existing equations of mathematical physics.

As it is known, the attempts to solve this problems gave rise to creation of the field-theory equations such as the quantum mechanics equations (the equations by Schrödinger, Dirac, Heisenberg), the equations of Hamiltonian formalism, the equations of electromagnetic field (the Maxwell equations), and the equations of gravitational field (the Einstein equations).

In this case, it appears that the equations obtained possess a common peculiarity, namely, all equations of field theories contain terms that are potentials. That is, the field-theory equations, as opposed to differential equations, are relations between potentials and other terms. Only the equations that are of form of such relations (non-identical) may have the solutions that are differentials (rather then functions) and can describe physical structures.

Below it will be shown what from the mathematical physics equations one can obtain a relation that can generate closed exterior forms which are differentials and can describe observable physical structures. This fact discloses a connection between the field-theory equations and the equations of mathematical physics and proofs of the fact that the equations of mathematical physics can describe physical structures.

Some basic principles used when deriving the field-theory equations

It appears to be possible to obtain equations of field-theory by introducing functionals (which specify physical fields) such as the state function ψ, the action functional S, the strength vectors E and H that made up the functional [EH], i.e., the Pointing’s vector, and the metric tensors related to the curvature from which the functional of gravitation field (the

Einstein function) is obtained. The field-theory equations are equations for such functionals.

Moreover, to obtain the equations that enable one to describe physical structures, it is necessary to introduce the conditions of invariance and covariance which the physical structures have to be subject to. For introducing the conditions of invariance and covariance, the theories of transformations, the vector and tensor identical relations and others were used.

The existing equations of mathematical physics were used only for obtaining the sources of the field-theory equations. Thus, the Newton equations and wave equations were used in deriving the equations by Dirac and Schrödinger, the Lagrange equations were used within the framework of Hamilton formalisms. In derivation of the Maxwell equations, the laws by Gauss and Faraday for electric and magnetic inductions and the theorem on circulation of magnetic field were used. In the Einstein equation, only the energy-momentum tensor, which follows from the conservation law equations for energy and momentum of material media, was used.

But it appears that the equations of mathematical physics, which describe material media (material systems) such as the thermodynamic, gas-dynamic and cosmic systems, as well as the systems of charged particles and others, possess the properties which were laid in the basis of field-theory equations.

However, such properties of the equations of mathematical physics are hidden ones. They do not directly follow from differential equations. Such properties are connected to the conservation laws and reveal only in describing evolutionary processes that depend on the consistency between the conservation law equations made up the equations of mathematical physics. To discover these properties and disclose their roles in describing physical structures turn out to be possible only by using skew-symmetric forms [1-3], whose properties correspond to conservation laws. [In this case, besides the exterior skew-symmetric forms, there were applied skew-symmetric forms, which follow from differential equations and possess non-traditional mathematical apparatus that includes such elements as *degenerate transformations*, *non-identical relations*, *non-integrable manifolds* and so on. (In Appendix 1, some information concerning the skew-symmetric forms applied in this paper is presented.) Such skew-symmetric forms are evolutionary ones, and they possess the unique property that does not possess any existing mathematical formalism. Namely, *they can generate closed exterior skew-symmetric forms* that are invariants and, as it will be shown below, describe physical structures. *It should be emphasized once more that significance of skew-symmetric forms in mathematical physics and field-theory relates to the fact that they describe conservation laws*.]

1. Conservation Laws as a Foundation of the Equations of Mathematical Physics and the Field-theory Equations

1.1. Peculiarities of conservation laws

At present time, there exist many problems related to an interpretation of conservation laws. This is caused by the fact that the concept of ”conservation laws” assumes a different meaning in various branches of physics.

In the theory that describes physical fields (equations by Dirac, Schrödinger, Maxwell, Einstein and others), these are conservation laws according to which there *exist conservative quantities or objects*. Such conservation laws may be called as *exact* ones.

The exact conservation laws are described by closed exterior skew-symmetric forms. The example of a formulation of such conservation law is the Noether theorem that can be written in the form $d\omega = 0$, where ω is the closed exterior skew-symmetric form.

In mechanics and physics of continuous media (material systems like thermodynamic, gas-dynamic, cosmological systems and so on), the concept of “conservation laws” is connected with *the conservation laws for energy*, *momentum*, *angular momentum*, *and mass*. Such conservation laws, which can be named as the *balance* ones (since they establish a balance between changes of physical quantities and external actions), *are described by differential equations*. (The Navier-Stokes and Euler equations for gas-dynamic systems are examples of such equations.)

The problems of interpretation conservation laws are induced by the fact that *exact and balance conservation laws are conservation laws of different physical systems*. And this not always accounted for in describing physical processes. (It should be emphasized once more that the balance conservation laws are conservation laws for material media, whereas the exact conservation laws are conservation laws for physical fields.)

The fact that exact and balance conservation laws relate to different physical systems is of unique significance. In spite that exact and balance conservation laws are conservation laws for different physical systems, there *exists a connection between the exact conservation laws and the balance ones*. However, this connection is not identical. This connection is realized discretely in evolutionary processes executed in material media. In the processes that obey the balance conservation laws, physical (observable) structures obey exact conservation laws that emerge.

Such interaction of exact and balance conservation laws lies at the basis of evolutionary processes accompanied by emergence of physical structures and discloses a connection between the field-theory equations and the equations of mathematical physics.

Conservation laws are described with the help of skew-symmetric differential forms. In this case, the properties of closed exterior skew-symmetric differential forms correspond to exact conservation laws. To the balance conservation laws, the evolutionary skew-symmetric forms obtained from differential equations are assigned.

1.2. Exact conservation laws as a basis of the field-theory equations

The exact conservation laws are conservation laws according to which there exist conservative physical quantities or objects.

The properties of closed exterior skew-symmetric differential forms correspond to exact conservation laws. (In Appendix 1, the properties of skew-symmetric differential forms corresponding to exact and balance conservation laws, that is, what is necessary for further presentation, are outlined. In more details, the properties of skew-symmetric differential forms are presented in the papers [2] and [3]: http://arxiv.org/pdf/math-ph/0310050v1.pdf.)

From the closure condition of exterior form (see Appendix 1) $d\theta^p = 0$, one can see that the closed exterior form is a conservative quantity. This means that such a form can correspond to the conservation law, i.e., it is a conservative quantity.

If the form be closed only on a certain structure (that in its metric properties it is a pseudo-structure and is described by its dual form), i.e., this form is a closed inexact one, then (see Appendix 1) from closure conditions of inexact form $d_\pi\theta^p = 0$ and corresponding dual form $d_\pi^*\theta^p = 0$, one can see that the dual form (pseudo-structure) and closed inexact form (conservative quantity) describe a conservative object that can also correspond to conservation law.

The exact conservation laws are such conservation laws.

The exact conservation laws described by a closed inexact exterior form and corresponding dual form are conservation laws for physical fields.

The closed dual form and associated closed inexact exterior form made up a differential-geometrical structure. Such differential-geometrical structure describes pseudo-structures with conservative quantity, that is, it describes a physical structure on which the exact conservation law is fulfilled. It is evident that physical structures, from which physical fields are formatted, are described by such differential-geometrical structures. This points out to the fact that the exact conservation laws are conservation laws for physical fields.

It can be shown that the closed inexact exterior forms corresponding to exact conservation laws are solutions to the equations, which describe physical fields, i.e., the field-theory equations such as the equations by Dirac, Schrödinger, Maxwell and Einstein. In this case

- closed exterior forms of zero degree follow from the Dirac and Schrödinger equations (in quantum mechanics);

- closed exterior and dual forms of first degree follow from the equations of Hamilton formalism;

- closed exterior and dual forms of second degree are obtained from the Maxwell equations;

- closed exterior and dual forms of third degree are the base of the gravitational field equations. (However, the forms of lower degrees also characterize the corresponding fields. In particular, the forms of first degree correspond to the Einstein equation as well.)

The connection of field theories with closed exterior forms corresponding to exact conservation laws shows that there exists an internal connection between theories describing physical fields of different types. In this case, the degree of closed exterior forms is a parameter that integrates field theories in unified theory.

(It can be noted that the closed exterior forms assigned to exact conservation laws were used, for example, in the paper by Wheeler [4] for the analysis of equations of electromagnetic field and ones of electromagnetic and weak interactions. In the papers by Weinberg [5] and Konopleva and Popov [6], closed exterior forms were applied in a study of gravitational fields.)

In Introduction, it was noted that when deriving the equations allowing to describe physical structures, it was used the conditions of invariance and covariance, to which physical structures have to obey.

These conditions are also connected with the properties of closed inexact forms. The closed exterior forms are invariants (and the dual forms corresponding to closed inexact exterior forms are covariants). This is explained by the fact that the closed exterior forms are differentials (total if the exterior form is exact, or interior if the exterior form is an inexact exterior form), and because of this they are invariants under all transformations that hold a differential. In particular, the gauge transformations, which are non-degenerate transformations of field-theory, are such transformations.

The connection of field theories with closed exterior forms corresponding to exact conservation laws shows that exact conservation laws are a basis of field-theory equations.

Below it will be shown that closed exterior forms corresponding to exact conservation laws (and on which the field-theory equations are based) are obtained from the equations of mathematical physics made up of the equations of the balance conservation laws.

1.3. Properties of the balance conservation law equations made up the equations of mathematical physics for material media

The equations of mathematical physics, which describe material media, consist of the equations of the conservation laws for energy, linear momentum, angular momentum, and mass [7-9].

Commonly the equations of mathematical physics are applied for a description of a change of physical quantities (such as energy, pressure, density) that characterize material media.

However, it turns out that such equations can describe not only changes of physical quantities, but also describe evolutionary processes happening in material media accompanied by the emergence of physical structures. That is, they can describe physical structures.

Such possibilities of the mathematical physics equations are caused by the properties of balance conservation laws that reveal under investigation of the consistency of conservation law equations.

Investigation of the consistency of the balance conservation law equations - Evolutionary relation for the state functionals

For investigation of the consistency of the balance conservation law equations, it is necessary to use two non-equivalent frames of reference.

Usually the equations of mathematical physics are written in the inertial frame of reference (the Euler frame of reference is an example of such frame) (see also Appendix 2).

In the case under consideration, in addition to the equations in inertial frame of reference, the equations obtained after transition from inertial frame of reference to accompanying one were used (see also Appendix 2). The accompanying frame of reference is a frame of reference connected with the manifold made up by the trajectories of the material system elements (the Lagrange frame of reference is an example of such a frame).

The other peculiarity of present investigation consists in the fact that the conservation law equations are transformed into equations expressed in terms of state functional [10].

As it was noted, the field-theory equations, which enable one to describe physical structures, were obtained due to the introduction of functional that specifies physical field made up by physical structures. such as wave function, the action functional, the Pointing tensor, the Einstein's tensor and so on. It appears that the equations of mathematical physics, which describe material media, also possess similar functionals. This is connected with the properties of the physical quantities of material media. Since the physical quantities (like temperature, energy, pressure, density) relate to a single material medium, a certain connection between them should exist. Such a connection is described by state functionals. The functionals such as wave function, entropy, the action functional, the Pointing tensor, the Einstein's tensor and so on, which are the field-theory functionals, are also functionals of the equations describing material media [10]. (Below it will be shown that from the equations of mathematical physics, a relation is obtained for such functionals that disclose specific features of the equations of mathematical physics.)

Let us analyze the correlation of the equations that describe conservation laws for energy and linear momentum.

In the inertial frame of reference, the energy equation can be reduced to the form:

$$\frac{D\psi}{Dt} = A_1, \tag{1}$$

where D/Dt is the total derivative with respect to time, A_1 is a quantity that depends on specific features of material system (material medium) and on external energy actions onto the system, ψ is the functional that specifies a material medium (and which is a state functional).

In the accompanying frame of reference, which is tied to the manifold built by the trajectories of elements of material system (particles of material medium), the total derivative with respect to time is a derivative along the trajectory. Since in the accompanying frame of reference, the equation of energy is written in the form

$$\frac{\partial\psi}{\partial\xi^1} = A_1. \tag{2}$$

Here ξ^1 are the coordinates along the trajectory. [Thus, the equation for energy expressed in terms of the action functional S has a similar form: $\partial S/\partial\xi^1 = DS/Dt = L$. And the equation for the energy of ideal gas can be written in the form [7]: $\partial s/\partial\xi^1 = Ds/Dt = 0$, where s is entropy.]

In a similar way, in the accompanying frame of reference, the equation for linear momentum appears to be reduced to the equation of the form

$$\frac{\partial\psi}{\partial\xi^\nu} = A_\nu, \quad \nu = 2, \ldots, \tag{3}$$

where ξ^ν are the coordinates in the direction normal to the trajectory, A_ν are the quantities that depend on the specific features of material medium and on force actions. (In Appendix 2 and in papers [7, 11], the equations of gas-dynamics for entropy in accompanying frame of reference are presented.)

Here it should be called an *attention* to a certain peculiarity. For a single quantity (the functional ψ), one has two equations.

How to study such overdetermined set of equations?

Since equations (2) and (3) are expressions for derivatives along different directions, they can be convoluted into the relation

$$d\psi = A_\mu d\xi^\mu \quad (\mu = 1, \nu), \tag{4}$$

where $d\psi$ is the differential expression $d\psi = (\partial\xi/\partial\xi^\mu)d\xi^\mu$.

Relation (4) can be rewritten as

$$d\psi = \omega, \tag{5}$$

here $\omega = A_\mu d\xi^\mu$ is a skew-symmetric differential form of the first degree.

Relation (5) has been obtained from the equation of the balance conservation laws for energy and linear momentum. In this relation, the form ω is that of the first degree. If the equations of the balance conservation laws for angular momentum be added to the equations for energy and linear momentum, then this form will be a form of the second degree. And, in combination with the equation of the balance conservation law for mass, this form will be a form of degree 3. In general case, the evolutionary relation can be written as

$$d\psi = \omega^p, \tag{6}$$

where the form degree p takes the values $p = 0, 1, 2, 3.$ (The relation for $p = 0$ is an analog to that in the differential forms, and it has been obtained from the interaction of energy and time.)

This relation is an evolutionary one since the original equations are evolutionary ones.

[A concrete form of relation (6) for $p = 2$ was considered for electromagnetic field in the paper http://arxiv.org/pdf/math-ph/0310050v1.pdf. In this case, the functional ψ is Pointing's vector. The relation for Einstein's tensor is obtained after integrating the evolutionary relation for $p = 3$.]

The evolutionary relation obtained from the equations of the balance conservation law possesses the properties which disclose the connection of the field-theory equations with the equations of mathematical physics.

Specific peculiarities of the evolutionary relation

The evolutionary relation possesses a peculiarity, namely, it appears to be non-identical. The evolutionary relation was obtained in the accompanying frame of reference that is connected with the manifold built up by the trajectories of the material system elements. Such a manifold is a deforming non-integrable one. The skew-symmetric form defined on non-integrable manifold cannot be closed since the commutator of skew-symmetric form defined on such manifold includes an additional term, namely, the commutator of metric form, which is nonzero because the metric form of non-integrable manifold is not closed one. (See Appendix 1 and [3]: http://arxiv.org/pdf/math-ph/0310050v1.pdf.)

Since the evolutionary form in the right-hand side of evolutionary relation is unclosed and is not a differential, the evolutionary relation (6) turns out to be non-identical.

(It can be noted that the first principle of thermodynamics is a non-identical evolutionary relation.)

The evolutionary relation possesses one more peculiarity, namely, this relation is a self-varying relation.

The evolutionary non-identical relation is a self-varying one, because, firstly, it is a non-identical, namely, it contains two objects one of which appears to be unmeasurable, and, secondly, it is an evolutionary relation, that is, the variation of any object of the relation in some process leads to a variation of another object; and, in turn, the variation of the latter leads to variation of the former. Since one of the objects is an unmeasurable quantity, the other cannot be compared with the first one, and hence, the process of mutual variation cannot terminate.

2. Mathematical and Physical Properties of Evolutionary Relation - Realization of Physical Structures

The evolutionary relation obtained from the equations of mathematical physics possesses unique properties. That relation reveals hidden properties of the solutions to the equations of mathematical physics (double solutions) which enable one to describe evolutionary processes happening in material media and processing the emergence of physical structures.

2.1. Properties of solutions to the mathematical physics equations

From evolutionary relation, it follows that the mathematical physics equations has a double solutions, namely, the solutions that are not functions (their derivatives do not made up a differential) and the solutions that are discrete functions.

Inexact solution to mathematical physics equation (which is not a function)

The non-identical evolutionary relation $d\psi = \omega^p$ cannot be integrated directly since its right-hand side contains unclosed skew-symmetric form which is not a differential. This means that the original equations of mathematical physics prove to be non-integrable (they cannot be convoluted into identical relation for differentials and be integrated). In this case, the solutions to original equations of mathematical physics are not functions (their derivatives do not make up a differential). Such solutions will depend on the commutator of the evolutionary skew-symmetric form ω which is nonzero. *If the commutator is equal to zero, the evolutionary relation would be identical and this would point out to integrability of original equations*.

[Here it should be emphasized that one considers equations on which no additional conditions are imposed. It will be shown later that the equations of mathematical physics can become locally integrable under additional conditions.]

Since the derivatives of inexact solutions do not make up a differential, this points out to the fact that in this case, there is no closed exterior form. This means that such solutions cannot describe physical structures.

Exact solutions to equations of mathematical physics (discrete functions)

The realization of closed inexact exterior forms that describe physical structures

From the evolutionary unclosed skew-symmetric form ω^p (in right-hand side of evolutionary relation), whose differential is nonzero, one can obtain a closed exterior form with a differential being equal to zero only under *degenerate transformation*, namely, under a transformation that does not conserve differential. (The Legendre transformation is an example of such a transformation. The transition from Legendre manifold to Hamiltonian one is a degenerate transformation.)

Such degenerate transformations can take place under additional conditions, which are due to degrees of freedom. *The vanishing of such functional expressions as determinants, Jacobians, Poisson's brackets, residues, and others correspond to these additional conditions*. These conditions can be realized (spontaneously) under a change of non-identical evolutionary relation, which, as it was noted, appears to be a self-varying relation.

If the conditions of degenerate transformation are realized, from the unclosed evolutionary form ω^p (see evolutionary relation (6)) with non-vanishing differential $d\omega^p \neq 0$, one can obtain a closed inexact (only on some pseudo-structure) exterior form with vanishing (interior) differential.

That is, it is realized as the transition

$$d\omega^p \neq 0 \to \text{(degenerate transformation)} \to d_\pi \omega^p = 0, \quad d_\pi^* \omega^p = 0.$$

The realization of the conditions $d_{\pi}^{*}\omega^{p} = 0$ and $d_{\pi}\omega^{p} = 0$ means that it is realized the closed dual form ${}^{*}\omega^{p}$, which describes some structure π (this is a pseudo-structure with respect to its metric properties), and the closed exterior (inexact) form ω_{π}^{p}, whose basis is a pseudo-structure, is obtained.

The closed dual form and associated closed inexact exterior form make up a differential-geometrical structure.

Such a differential-geometrical structure, as it was already noted, describes a pseudo-structure with conservative quantity, i.e., a physical structure on which the exact conservation law is fulfilled.

This points out to the fact that *from the equations of mathematical physics, one can obtain closed inexact exterior forms that describe physical structures on which are fulfilled exact conservation laws*, *that is*, *conservation laws for physical fields*. (Below the process of emergence of physical structures will be described.)

The realization of integrability of the mathematical physics equations

The realization of closed inexact exterior form points also to the fact that the mathematical physics equations become locally integrable.

The realization of closed inexact exterior form leads to that on a pseudo-structure from evolutionary relation (6), it follows the relation

$$d\psi_{\pi} = \omega_{\pi}^{p} \tag{7}$$

that occurs to be an identical one, since the form ω_{π}^{p} is a differential.

Since the identical relation can be integrated (because it contains only of differentials), this means that on the pseudo-structure, the equations of mathematical physics become locally integrable (only on pseudo-structure). In this case, the pseudo-structure is an integrable structure. The solutions to the mathematical physics equations on integrable structures are generalized solutions, which are discrete functions, since they are realized only under additional conditions (on the integrable structures).

On integrable structures, the desired quantities of the material system, such as temperature, pressure, density, become functions of only independent variables and do not depend on the commutator (and on the path of integrating). Such functions may be found by means of integrating (on integrable structures) the equations of mathematical physics. The solutions on characteristics or on potential surfaces are examples of such generalized solutions.

Here one can see a following peculiarity.

Realized closed dual form and associated closed inexact exterior form make up a differential-geometrical structure that possesses a duality. On one hand, differential-geometrical structure describes a pseudo-structure with conservative quantity, namely, a physical structure, on which the exact conservation laws are fulfilled. On the other hand, differential-geometrical structure describes a pseudo-structure with generalized solution, namely, the integrable structure. Such a duality of differential-geometrical structures has a physical meaning. (Below it will be shown that observed formations, which are described by generalized solution, arise in material medium. The duality of differential-geometrical structures discloses a connection between physical structure and observable formations.)

2.2. Description of evolutionary processes in material media - The processes of physical structure emergence

It was shown that the equations of mathematical physics possess double solutions, namely, solutions on non-integrable initial manifold and ones on integral structures. This has a physical meaning.

It turns out that inexact solution describes non-equilibrium state of material medium, whereas exact solution describes locally-equilibrium state. In this case, the transition from inexact solution to exact one, which corresponds to a transition of material medium from non-equilibrium state to locally-equilibrium state, describes the process of emergence of physical structures.

This follows from the evolutionary relation and is connected with the property of the functional ψ.

Non-stable state of material media

The functional ψ in the left-hand side of the evolutionary relation $d\psi = \omega^p$ possesses a unique property, namely, this functional specifies a state of material media. The differential of functional availability means that there exists a state function, and this fact points out to equilibrium state of material medium.

However, since the evolutionary relation turns out to be not identical, from this relation, one cannot get the differential $d\psi$. The absence of differential $d\psi$ means that the material medium state is non-equilibrium.

This means that an internal force acts in material medium.

The internal force originates at the expense of some quantity described by the evolutionary form commutator. (If the evolutionary form commutator is zero, the evolutionary relation would be identical, and this would point to the equilibrium state, i.e., the absence of internal forces.) Everything that gives a contribution into the evolutionary form commutator leads to emergence of internal force.

One more peculiarity of evolutionary relation, namely, a self-variation of the non-identical evolutionary relation points to the fact that the non-equilibrium state of material medium turns out to be self-varying. The state of material medium varies. In this case, the state of material medium changes, but it remains to be non-equilibrium since the evolutionary relation remains to be non-identical under self-variation.

It is evident that inexact solutions, which depend on the evolutionary form commutator (and are not functions), describes a non-equilibrium state of material medium. (*That is, the inexact solution has a physical meaning*.)

Now it arises a question of whether the material medium can get rid of the internal force and to transfer into the equilibrium state?

Transition of material medium into locally-equilibrium state - Advent of observable formations

From the evolutionary relation, one found that inexact solutions to the mathematical physics equations describe non-equilibrium state of material medium.

It turns out that the generalized solutions, which are discrete functions, describe a locally-equilibrium state of material medium. This follows from the physical properties of identical relation obtained.

From identical relation (7) $d\psi_\pi = \omega_\pi^p$, one can find the differential of the state functional $d\psi_\pi$, and this points out to a presence of the state function and the transition of material medium from non-equilibrium state into equilibrium one. However, such a state of material medium turns out to be realized only locally that relates to the fact that differential of the state functional obtained is a differential interior (only on pseudo-structure). Yet the total state of material medium remains to be non-equilibrium state because the evolutionary relation, which describes the material medium state, remains non-identical one. (That is, there exists a duality. Non-identical evolutionary relation goes on to act simultaneously with identical relation.) [It can be noted that these results point out to the fact that the functionals of evolutionary relation, i.e., functionals of the mathematical physics equations, are really state functionals, that is, they characterize a state of material medium.]

The transition from non-equilibrium state to locally-equilibrium state means that unmeasurable quantity, which is described by the evolutionary form commutator and acts as internal force, converts into a measurable quantity of material medium.

The transition of unmeasurable quantity into a measurable quantity of material medium reveals in emergence in material medium of some observed formations. Waves, vortices, fluctuations, turbulent pulsations [11] and so on are examples of such formations. The intensity of such formations is controlled by a quantity accumulated by the evolutionary form commutator. (This discloses a mechanism of such processes like an origin of vortices and turbulence [11].)

Such emerged formations are described by generalized solutions to the equations of mathematical physics. The functions that correspond to generalized solutions, as it is known, are discrete functions that have breaks of functions itself or its derivatives [12].

Relationship between physical structures and observed formations

The advent in material media of observable formations relates to emergence of physical structures.

From the evolutionary relation, it follows that under realization of degenerate transformation (that is conditioned by any degrees of freedom), one obtains a closed dual form (which describes pseudo-structure) and a closed inexact exterior form (conservative quantity) and this points to emergence of physical structure on which exact conservation law is satisfied. On another hand, the realization of closed inexact exterior form leads to the fact from non-identical evolutionary relation, one obtains identical relation that describes the transition of material medium from non-equilibrium state to locally-equilibrium state that is accompanied by the emergence in material medium of observable formation.

Thus, one can see that the emergence of physical structures reveals in material medium as an advent of certain observable formations, which develop spontaneously. Such formations and their manifestations, as it was noted, are fluctuations, turbulent pulsations, waves, vortices, and others. It appears that physical structures and the observed formations of material medium observed are a manifestation of the same phenomena. The light is an example of such a duality, namely, as a massless particle (photon) and as a wave.

However, physical structures and observed formations are not identical objects. Whereas the wave is an observable formation, the element of wave front made up the physical structure in the process of its motion.

It should be emphasized that physical structures are elements of physical fields, whereas observed formations are elements of material system.

This duality also follows from identical relation. The closed inexact exterior form (right-hand side of identical relation (7)) points to a presence of physical structure (physical fields are made up from such physical structures). The existence of state differential (left-hand side of relation (7)) points to transition of material medium state in locally-equilibrium state that is related to emergence of observed formations.

[Note that the existence of *double solutions and degenerate transformation of the evolutionary skew-symmetric form enables to describe the emergence of discrete formations*. This cannot be described within the framework of another mathematical formalism.]

[Here it should be emphasized that not every unmeasurable quantity gets converted into measurable quantities of material system itself and into physical structures, since the process of transition into equilibrium state executes only locally. In this case, the total state of material medium keeps to be in a non-equilibrium state. This means that a certain non-measurable quantity remains. The dark energy and dark matter are such non-measurable and non-observable quantities (essence) that emerge due to various non-potential actions and, because of the non-commutativity of conservation laws, cannot directly convert into own quantities of material medium.]

2.3. State functionals of the equations of mathematical physics

From a description of the processes of emerging physical structures and observable formations, one can see a role of functionals of mathematical physics equations in a description of these processes. It is evident that functionals of mathematical physics equations are state functionals. They characterize a state of material medium. Such a role of functionals of mathematical physics equations relates to a specific peculiarity of functionals. They can at once be both functionals and state functions or potentials. As functionals, they describe a non-equilibrium state of material medium, and as state functionals, they point out to a locally-equilibrium state of material medium. The transition from functionals to state functions describes a transition of material medium from non-equilibrium state to locally-equilibrium one that is accompanied by emergence of physical structures and observed formations. That is, the transition from functionals to state functions describes the mechanism of physical structure origination. This discloses functional properties of functionals as state functionals of mathematical physics equations and as field-theory functionals connected with physical structures.

2.4. Correspondence between the evolutionary relation and field-theory equations - The linkage between field-theory equations and equations of mathematical physics

The correspondence between the evolutionary relation and the field-theory equations emphasizes the fact that the functionals of field-theory equations and functional of equations of mathematical physics are identical. The field-theory equations, which describe physical fields, are equations for functionals such as wave function, the action functional, Pointing's vector, Einstein's tensor, and others. The non-identical evolutionary relations derived from the equations of mathematical physics, which describe material media, are relations for all these functionals.

The correspondence between the field-theory equations and the evolutionary relation also points out to the fact that all equations of field theories, as well as the evolutionary relation, are non-identical relations in differential forms or in the forms of their tensor or differential (i.e., expressed in terms of derivatives) analogs. For example,

- the Einstein equation is a relation in differential forms;

- the Dirac equation relates Dirac's *brac*- and *ket*- vectors, which made up a differential form of zero degree;

- the Maxwell equations have the form of tensor relations;

- the Schrödinger equations have the form of relations expressed in terms of derivatives and their analogs.

[The field-theory equations are those whose solutions have to be not functions but differentials (closed inexact exterior forms that must describe physical structures). Only the equations that have the form of relations (non-identical) may have the solutions that are differentials rather then functions.]

From the field-theory equations, as well as from the non-identical evolutionary relation, the identical relation, which contains the closed exterior form, is obtained. As one can see, from the field-theory equations, it follows such identical relation as

- the Poincaré invariant that relates closed exterior forms of first degree;

- the relations $d\theta^2 = 0,\ d^*\theta^2 = 0$ are those for closed exterior forms of second degree obtained from the Maxwell equations;

- the Bianchi identity for gravitational field.

Here it should be emphasized that the correspondence between the evolutionary relation and the field-theory equations is not identical. From the field-theory equations, one gets solutions that correspond to observed physical structures, and this enables to discover and describe such structures.

Unlike the field-theory equations, the evolutionary relation gives a possibility not only to obtain closed exterior forms, and hence to find physical structures, but to describe the process of emerging physical structures as well.

[Such a distinction relates to the fact that the right-hand side of field-theory equation contains a potential, i.e., an invariant quantity. (Thus, the right-hand side of the Einstein equation contains the energy-momentum tensor, which is an invariant quantity. The potential enters into the right-hand side of the Schrödinger equation. Only in this case, solutions to equations are meaningful.) The right-hand side of evolutionary relation contains skew-symmetric form, which is not a differential, i.e., not an invariant, whereas the invariant (closed exterior form) is realized only discretely. In this case, all process of physical structures emergence is described.]

The correspondence between the evolutionary relation obtained from the mathematical physics equations and the field-theory equations gives proof to the fact that the physical structures, which made up physical fields, are physical structures that are generated by material media and are described by the equations of mathematical physics.

Thus the mathematical physics equations can describe observable physical structures from which physical fields are formatted.

The linkage between physical structures, which made up physical fields, and physical structures of material media means that physical fields are generated by material media.

Linkage between field-theory equations and the equations of mathematical physics

Thus, one can see that there exists a correspondence between the field-theory equations, which describe physical fields, and the evolutionary relation obtained from the equations of mathematical physics for a material medium.

Such a correspondence between the evolutionary relation and the field-theory equations points out to a linkage of the field-theory equations, which describe physical fields, with the equations of mathematical physics for material media.

It appears that the field-theory equations derived by employment some postulates (for describing observed physical structures) are related with the equations of mathematical physics for material media, which are based on the properties of conservation laws. It was shown that the postulates, which were used when deriving the field-theory equations, are also based on conservation laws.

The linkage between field-theory equations and the equations of mathematical physics gives a possibility to prove a validity of many foundations of field-theory and the properties of physical fields.

3. Some Foundations of Field-theory - Characteristics of Physical Structures

Some characteristics of physical structures

The physical structure is an object obtained by conjugating the conserved physical quantity, which is described by inexact closed exterior form, and the pseudo-structure, which is described by relevant dual form.

Conserved physical quantity describes a certain charge.

Spin is an example of the second characteristic. *Spin is a characteristic that determines a character of the manifold deformation before origination of physical structure*. (The spin value depends on the skew-symmetric form degree.)

The connection of the physical structures with the skew-symmetric differential forms allows to introduce a classification of these structures and corresponding physical fields in their dependence on parameters that specify the skew-symmetric differential forms.

Closed forms that correspond to physical structures are generated by the evolutionary relation containing the parameter p that defines a number of interacting balance conservation laws. Therefore, the physical structures can be classified by the parameter p. The other parameter is a degree of k closed forms generated by the evolutionary relation. The dimension n of the initial inertial space is one more parameter since the dimension of pseudo-structures, on which closed forms are defined, depends on the dimension of space.

By entering a classification in parameters p, k and n, one can understand an internal connection between various fields and interactions. It may be seen the correspondence between the degree k of the closed forms realized and the type of interactions. Thus, $k = 0$ corresponds to strong interaction, $k = 1$ corresponds to weak interaction, $k = 2$ corresponds to electromagnetic interaction, and $k = 3$ corresponds to gravitational interaction. (In the paper [2], it is presented the table of elementary particles, where physical fields and interactions in their dependence on the parameters *p*, *k* and *n* of evolutionary and exterior closed forms are demonstrated.)

Some foundations of field-theory

The connection of the field-theory equations with the equations of mathematical physics for material media enables one to understand some foundations of field-theory and the properties of physical fields.

The closed exterior forms, which describe conservation laws for physical fields, are generated by evolutionary forms obtained from the equations of mathematical physics for material media.

The functionals of field-theory equations are functionals that specify the state of relevant material medium.

The emergence of physical structures occurs discretely under realization of the degrees of freedom of material media. This explains the quantum character of field theories.

The mechanism of origination the physical structures elucidates the mechanism of forming physical fields and manifolds [2]. (The dual forms, which are metric forms of the manifold, describe pseudo-structures, from which pseudo-metric and metric manifolds are formatted.)

The symmetries of field theories are conditioned by the degrees of freedom of material media. (In this case, the internal symmetries of field theories relate to the conservation laws for physical fields, whereas the external symmetries of field theories relate to the equations of the conservation laws for material media.)

The constants and characteristics of field theories are connected with characteristics of relevant material media. (But this connection is indirect. This connection is realized in evolutionary process.)

The solutions of field-theory equations are connected with closed exterior forms of a certain degree. This points out to an internal connection between field theories, which describe physical fields of various types.

Foundations of unified and general field theories

The connection of the solutions of field-theory equations with closed exterior forms, which correspond to exact conservation laws, that is common for all field-theory equations, shows that there exists an internal connection between theories describing physical fields of various types. In this case, the degree of closed exterior forms is a parameter that integrates the field-theory equations into a unified theory. As it was noted, there exists a correspondence between the degree k of the closed forms realized, which describe exact conservation laws, and the type of interactions.

Thus, the exact conservation laws, to which physical fields obey, made up the basis of unified field-theory.

The correspondence between the evolutionary relation followed from the equations of mathematical physics for material media and the field-theory equations for physical fields discloses a connection between the field-theory equations and the equations of mathematical physics. This demonstrates that the basis of field-theory equations obtained with the help of postulates are the equations of mathematical physics that are made up of the balance conservation laws for material media, i.e., the conservations laws for energy, linear momentum, angular momentum, and mass.

The evolutionary relation obtained from the equations of mathematical physics combines the field-theory equations, discloses their internal connection and justifies postulates on which the field-theory bases.

This points out to the fact that the field-theory equations for various physical fields obey common principles regardless to the type of physical fields and material media that generate physical fields.

Appendix 1

Some properties of skew-symmetric forms corresponding to the conservation laws

Closed inexact exterior forms: differential-geometrical structures. The external differential form of the degree p (p-form) can be written as [16]:

$$\theta^p = \sum_{i_1 \dots i_p} a_{i_1 \dots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}, \quad 0 \le p \le n \cdot a_{i_1 \dots i_p}. \tag{1}$$

Here $a_{i_1 \dots i_p}$ is the function of independent variables $x^1, \dots, x^n$, n is the space dimension, and dx^i, $dx^i \wedge dx^j$, $dx^i \wedge dx^j \wedge dx^k$, $\dots$ is the local basis subject to the condition of skew-symmetry:

$$dx^i \wedge dx^i = 0,$$

$$dx^i \wedge dx^j = -dx^j \wedge dx^i, \quad i \ne j. \tag{2}$$

The exterior form differential θ^p is expressed by the formula

$$d\theta^p = \sum_{i_1 \dots i_p} da_{i_1 \dots i_p} \wedge dx^{i_1} \wedge dx^{i_2} \dots \wedge dx^{i_p}. \tag{3}$$

The form is called as a *closed one* if its differential equals to zero:

$$d\theta^p = 0. \tag{4}$$

From condition (4), one can see that the closed form is a conservative quantity. This means that such a form can correspond to the conservation law (for physical fields), i.e., a conservative quantity.

If the form be closed only on pseudo-structure, i.e., this form is a closed inexact one, then the closure condition can be written as

$$d_\pi \theta^p = 0. \tag{5}$$

In this case, the pseudo-structure π obeys the condition

$$d_\pi {}^*\theta^p = 0, \tag{6}$$

here ${}^*\theta^p$ is the dual form.

From conditions (5) and (6), one can see that the dual form (pseudo-structure) and closed inexact form (conservative quantity) describe a conservative object that can also correspond to some conservation law. (It appears that the closed inexact exterior and dual forms describe a structure with conservative quantity. Such structures made up physical fields and pseudo-metric and metric manifolds.)

It turns out that the closed inexact exterior forms are obtained from the skew-symmetric differential forms, whose basis is non-integrable manifolds (in contrast to exterior skew-symmetric forms). Such forms, which possess the evolutionary properties, are obtained from the equations which describe any processes.

Distinction of evolutionary forms from exterior forms. The evolutionary form can be written in a manner similar for exterior differential form [2, 3]. However, in distinction from the exterior form differential, an additional term will appear in the evolutionary form differential. This is due to the fact that the evolutionary form basis changes since such a form is defined on non-integrable manifold.

The evolutionary form differential takes the form

$$\begin{aligned} d\theta^p = {} & \sum_{i_1 \ldots i_p} da_{i_1 \ldots 1_p} \wedge dx^{i_1} \wedge dx^{i_2} \ldots \wedge dx^{i_p} \\ & + \sum_{i_1 \ldots i_p} a_{i_1 \ldots i_p} d(dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}), \end{aligned} \tag{7}$$

where the second term is connected with the basis differential being nonzero: $d(dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}) \neq 0$. (For the exterior form defined on integrable manifold, one has $d(dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}) = 0$.)

The peculiarity of skew-symmetric forms defined on non-integrable manifold can be demonstrated by the example of a skew-symmetric form of first degree. Let us consider the first degree form $\omega = a_\alpha dx^\alpha$. The differential of this form can be written as $d\omega = K_{\alpha\beta} dx^\alpha dx^\beta$, where $K_{\alpha\beta} = a_{\beta;\alpha} - a_{\alpha;\beta}$ are components of the commutator of the form ω, and $a_{\beta;\alpha}$, $a_{\alpha;\beta}$ are covariant derivatives. If we express the covariant derivatives in terms of connectedness (if it is possible), they can be written as $a_{\beta;\alpha} = \partial a_\beta / \partial x^\alpha + \Gamma^\sigma_{\beta\alpha} a_\sigma$, where the first term results from differentiating the form coefficients, and the second term results from differentiating the basis. If we substitute the expressions for covariant derivatives into the formula for commutator components, we obtain the following expression for commutator components of the form ω:

$$K_{\alpha\beta} = \left(\frac{\partial a_\beta}{\partial x^\alpha} - \frac{\partial a_\alpha}{\partial x^\beta}\right) + (\Gamma^\sigma_{\beta\alpha} - \Gamma^\sigma_{\alpha\beta}) a_\sigma. \qquad (8)$$

Here the expressions $(\Gamma^\sigma_{\beta\alpha} - \Gamma^\sigma_{\alpha\beta})$ entered into the second term are just components of the commutator of the first degree metric form that specifies the manifold deformation and hence is nonzero. (It is well-known that the metric form commutators of the first, second and third degrees specify, respectively, torsion, rotation and curvature.)

(In the commutator of exterior form, which is defined on integrable manifold, the second term absents: the connectednesses are symmetric, that is, the expression $(\Gamma^\sigma_{\beta\alpha} - \Gamma^\sigma_{\alpha\beta})$ vanishes).

Since the commutator, and hence the differential, of skew-symmetric form defined on non-integrable manifold are nonzero, this means that such a form cannot be closed one.

Appendix 2

The equations of mathematical physics describing material media

In an inertial frame of reference such equations are presented, for example, in monographs:

R. Courant, Partial Differential Equations, New York London, 1962 (Chapter 6, paragraph 3a).

W. Pauli, Theory of Relativity, Pergamon Press, 1958 (paragraphs 30, 37).

A. Einstein, The Meaning of Relativity, Princeton, 1953 (paragraph 1).

R. C. Tolman, Relativity, Thermodynamics, and Cosmology, Clarendon Press, Oxford, UK, 1969.

J. F. Clark and M. Machesney, The Dynamics of Real Gases, Butterworths, London, 1964.

In the accompanying frame of reference (this frame of reference is connected with the manifold made up by the trajectories of the material system elements), the equations of gas-dynamics are presented, for example, in monograph J. F. Clark and M. Machesney. In this case, the conservation law equations for energy can be written as (see Chapter 6):

$$\frac{\partial s}{\partial \xi^1} = \frac{1}{\rho}\frac{\partial}{\partial x_i}\left(-\frac{q_i}{T}\right) - \frac{q_i}{\rho T}\frac{\partial T}{\partial x_i} + \frac{\tau_{ki}}{\rho}\frac{\partial u_i}{\partial x_k}$$

(here s is *entropy*, ξ^1 is *the coordinate along the trajectory*).

In the case of two-dimensional flow of ideal gas, the conservation law equations for linear momentum can be written in the following form (see Chapter 6):

$$\frac{\partial s}{\xi \partial^\nu} = \frac{\partial h_0}{\partial \nu} + (u_1^2 + u_2^2)^{1/2}\zeta - F_\nu + \frac{\partial U_\nu}{\partial t},$$

where $\zeta = \partial u_2/\partial x - \partial u_1/\partial y$. Here ξ^ν is *the coordinate in the direction normal to the trajectory*.

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