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MODULES THAT HAVE A GENERALIZED δ-SUPPLEMENT IN EVERY COFINITE EXTENSION

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Abstract

In this paper, we define modules with the properties $(\delta\text{-}GSCE)$ and $(\delta\text{-}GSCEE)$ by adapting Zöschinger's modules with the properties (*E*) and (*EE*) and we investigate the structure of modules with these properties. It is shown that: (1) a module has the property $(\delta\text{-}GSCEE)$ iff every submodule has the property $(\delta\text{-}GSCE)$; (2) the property $(\delta\text{-}GSCE)$ is inherited by direct summands; (3) for an *R*-module *M* over a δ-*V*-ring, *M* has the property $(\delta\text{-}GSCE)$ iff *M* is cofinitely injective; (4) if *R* is a δ-semiperfect ring, then every left *R*-module has the property $(\delta\text{-}GSCE)$.

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1. Introduction

In this paper, all rings will be associative with an identity element. Unless otherwise stated, *R* denotes an arbitrary ring and all modules will be left unitary *R*-modules. Let *M* be an *R*-module by $N \leq M$ we mean that *N* is a submodule of *M*. Recall that a submodule $N \leq M$ is called *small in M* (denoted by $N << M$) if $M \neq N + T$ for every proper submodule *T* of *M*. Dually, a submodule $L \leq M$ is called *essential in M* (denoted by $L \trianglelefteq M$) if $L \cap X \neq 0$ for every nonzero submodule *X* of *M*. A module *M* is said to be *singular* if $M \cong \frac{N}{L}$ for some submodule *N* and a submodule $L \leq N$ with $L \trianglelefteq N$. $Rad(M)$ will indicate the Jacobson radical of *M*. For two submodules *N* and *K* of *M*, *N* is called a *supplement* of *K* in *M* if *N* is minimal with the property $M = K + N$; equivalently $M = K + N$ and $K \cap N << N$. A module *M* is called *supplemented* if every submodule of *M* has a supplement in *M*. Also, *M* is called *amply supplemented* if, for any two submodules *L*, *K* of *M* with $M = L + K$, there exists a supplement *P* of *L* such that $P \leq K$.

In [16], Zhou introduced the concept of δ-small submodules as a generalization of small submodules. A submodule *N* of *M* is said to be δ-*small* in *M* if whenever $M = N + K$ and $\frac{M}{K}$ is singular, we have $M = K$. The sum of all δ-small submodules of a module *M* is denoted by $\delta(M)$. It is easy to see that every small submodule of a module *M* is δ-small in *M*, so $Rad(M) \subseteq \delta(M)$. A submodule *L* of *M* is called a δ-*supplement* of *N* in *M* if $M = N + L$ and $N \cap L$ is δ-small in *L* and *M* is called δ-*supplemented* in case every submodule of *M* has a δ-supplement in *M*. Note that every supplemented module is δ-supplemented. For submodules *U* and *V* of a module *M*, *V* is said to be a *rad-supplement* of *U* in *M* if $U + V = M$ and $U \cap V \subseteq Rad(V)$. *M* is called a *rad-supplemented module* if every submodule of *M* has a rad-supplement in *M*. In [10], these modules are also called *generalized supplemented modules*. Let M be an R-module and let U and V be any submodules of M with $M = U + V$. If $U \cap V \leq \delta(V)$, then V is called a *generalized* δ*-supplement* of U in M. Following [10], M is called a *generalized* δ*-supplemented module* (or briefly δ-*GS module*) if every submodule of M has a generalized δ-supplement in M. Also, in [10], M is called a *generalized amply* δ*-supplemented* (or briefly a δ-*GAS module*) if whenever $M = U + V$ for submodules U, V of M, then U contains a generalized δ-supplement of V in M.

A submodule N of a module M is said to be *cofinite* if $\frac{M}{N}$ is finitely generated. M is called a *cofinitely generalized supplemented module* if every cofinite submodule of M has a generalized supplement (see [3]). Since every submodule of a finitely generated module is cofinite, a finitely generated module is generalized supplemented if and only if it is cofinitely generalized supplemented. M is called *cofinitely generalized* δ*-supplemented* or briefly δ-*CGS module* if each cofinite submodule of M has a generalized δ-supplement in M (see [15]).

Let R be a ring and let M and N be R-modules. N is called a (*cofinite*) *extension* of M in case $M \leq N$ $\left(\frac{N}{M} \text{ is finitely generated}\right)$ [4]. Zöschinger generalized injective modules to modules with the property (E) such that a module M has the property (E) if M has a supplement in every extension. He also defined the structure of the modules which are called *modules* with the property (EE), that is if M has ample supplements in every extension, i.e., for $M \subseteq N$, if $N = M + K$, K contains a supplement of M in N [17]. Every left R-module has the property (E) iff R is left perfect.

By adapting Zöschinger's module with the properties (E) and (EE), Çalısıcı and Türkmen called a module M has the properties (CE) and (CEE) if M has a supplement (ample supplements) in every cofinite extension. Following this, in [8], Öztürk Sözen and Eren defined modules with the property $(\delta\text{-}E)$ as a generalization of modules with the property (*E*). In [11], the author introduced the properties (*CRE*) and (*CREE*) as a generalization of the properties (*CE*) and (*CEE*) and gave a characterization of rad supplemented rings which are a generalization of semiperfect rings.

In this paper, we investigate the properties $(\delta\text{-}GSCE)$ and $(\delta\text{-}GSCEE)$ for modules as a generalization of the properties (*CRE*) and (*CREE*), respectively. We show that a module has the property $(\delta\text{-}GSCEE)$ if and only if every submodule has the property $(\delta\text{-}GSCE)$. We prove that every direct summand of a module *M* with the property $(\delta\text{-}GSCE)$ has this property. We also prove that the class of modules with the property $(\delta\text{-}GSCE)$ is closed under extension with a special condition. By using the property $(\delta\text{-}GSCE)$, we give a characterization of generalized δ-supplemented rings. Since every generalized δ-supplement is a δ-supplement, modules that have a δ-supplement in every cofinite extension (briefly $(\delta\text{-}CE)$ modules) also have a generalized δ-supplement in every cofinite extension. It is clear that modules with the property $(\delta\text{-}CE)$ also have the property $(\delta\text{-}GSCE)$. So at the end, we give an example showing that a module with the property $(\delta\text{-}GSCE)$ need not be $(\delta\text{-}CE)$.

2. Preliminaries

We begin by stating the following lemmas which are contained in [16] for the completeness.

Lemma 1. *Let N be a submodule of M. The following are equivalent*:

(1) $N \ll_{\delta} M$;

(2) *if* $X + N = M$, *then* $M = X \oplus Y$ *for a projective semisimple submodule Y with* $Y \subseteq N$;

(3) *if* $X + N = M$ *with* $\frac{M}{X}$ *Goldie torsion, then* $X = M$.

Lemma 2. *Let M be a module. Then we have the following*:

(1) *For submodules N, K, L of M with* $K \subseteq N$, *we have*

(a) $N \ll_\delta M$ *if and only if* $K \ll_\delta M$ *and* $\frac{N}{K} \ll_\delta \frac{M}{K}$.

(b) $N + L \ll_\delta M$ *if and only if* $N \ll_\delta M$ *and* $L \ll_\delta M$.

(2) *If* $K \ll_\delta M$ *and* $f : M \to N$ *is a homomorphism, then* $f(K) \ll_\delta N$. *In particular, if* $K \ll_\delta M \subseteq N$, *then* $K \ll_\delta N$.

(3) *Let* $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ *and* $M = M_1 \oplus M_2$. *Then* $K_1 \oplus K_2 \ll_\delta M_1 \oplus M_2$ *if and only if* $K_1 \ll_\delta M_1$ *and* $K_2 \ll_\delta M_2$.

3. Main Results

Definition 1. Let *M* be an *R* module. *M* is called with the *property* $(\delta\text{-}GSCE)$ if *M* has a generalized δ-supplement in every cofinite extension and we say that *M* has the *property* $(\delta\text{-}GSCEE)$ if *M* has ample generalized δ-supplements in every cofinite extension.

It is easy to see that every module with the property $(\delta\text{-}CE)$ has the property $(\delta\text{-}GSCE)$. But the converse may not be true.

Proposition 1. *Every δ-radical module has the property* $(\delta\text{-}GSCE)$.

Proof. Let *M* be a module and *N* be any cofinite extension of *M*. $N = M + N$ since $\delta(M) = M$ and $M \cap N = M = \delta(M) \subseteq \delta(N)$. So *N* is a generalized δ-supplement of *M* in *N*. □

We denote the sum of all δ-radical submodules of a module *M* by "$P_\delta(M)$", that is, $P_\delta(M) = \sum \{U \subseteq M \mid \delta(U) = U\}$.

A module *M* is called δ-*reduced* if $P_\delta(M) = 0$.

Since $P_\delta(M)$ is a δ-radical submodule of *M*, we have the next result as a clear consequence of Proposition 1.

Corollary 1. *For a module M,* $P_\delta(M)$ *has the property* $(\delta\text{-}GSCE)$.

The following proposition shows that the property $(\delta\text{-}GSCE)$ is preserved by direct summands.

Proposition 2. *Every direct summand of a module with the property* $(\delta\text{-}GSCE)$ *has the property* $(\delta\text{-}GSCE)$.

Proof. Let *U* be a direct summand of *M* and *N* be any cofinite extension of $U \subseteq N$. Then $M = A \oplus U$ for some submodule $A \subseteq M$. Let N' be the external direct sum $A \oplus N$ and $\varphi : M \to N'$ be the canonical embedding. Then $M \cong \varphi(M)$ has the property $(\delta\text{-}GSCE)$. We have

$$\frac{N}{U} \cong \frac{A \oplus N}{\varphi(M)} = \frac{N'}{\varphi(M)}$$

is finitely generated. Since $\varphi(M)$ has the property $(\delta\text{-}GSCE)$, there exists a submodule V of N' such that $N' = \varphi(M) + V$ and $\varphi(M) \cap V \subseteq \delta(V)$. Consider the projection $\pi : N' \to N$. So we have $U + \pi(V) = N$. Also since $Ker(\pi) \subseteq \varphi(M)$,

$$\pi(\varphi(M) \cap V) \subseteq \pi(\varphi(M)) \cap \pi(V) = U \cap \pi(V) \subseteq \delta(\pi(V)).$$

Finally, $\pi(V)$ is a generalized δ-supplement of *U* in *N*. □

A module *M* is called *cofinitely injective* if *M* is a direct summand of every cofinite extension [4].

In [12], the authors defined δ-*V*-ring, that is, for any left (or right) *R*-module *M*, $\delta(M) = 0$.

Proposition 3. *Let R be any* δ-*V-ring and M be an R-module. Then the following statements are equivalent*:

(1) *M has the property* $(\delta\text{-}GSCE)$.

(2) *M is cofinitely injective*.

(3) *M has the property* $(\delta\text{-}CE)$.

Proof. (1) $\Rightarrow$ (2) Let M be a module with the property $(\delta\text{-}GSCE)$ and N be any cofinite extension of M. By hypothesis, M has a generalized δ-supplement V in N. Since R is a δ-V-ring, $N = M \oplus V$ is obtained.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are clear. □

In [17], Zöschinger proved that a module has the property (EE) $\Leftrightarrow$ every submodule has the property (E).

In the following theorem, we give an analogous characterization that establishes a similar relation between our modules.

Theorem 1. *A module M has the property* $(\delta\text{-}GSCEE)$ $\Leftrightarrow$ *every submodule of M has the property* $(\delta\text{-}GSCE)$.

Proof. ($\Rightarrow$) Let M be a module with the property $(\delta\text{-}GSCEE)$ and T be any submodule of M. For a cofinite extension N of T, let $F = \frac{M \oplus N}{H}$, where the submodule H is the set of all elements $(x, -x)$ of F with $x \in T$ and let $\alpha : M \to F$ via $\alpha(m) = (m, 0) + H$, $\beta : N \to F$ via $\beta(n) = (0, n) + H$ for all $m \in M$, $n \in N$. It is clear that α and β are monomorphisms. So we have the following pushout where μ_1 and μ_2 are inclusion mappings.

$$\begin{tikzcd}
M \arrow[r, dashed, "\alpha"] & F \\
T \arrow[u, hook, "\mu_1"] \arrow[r, hook, "\mu_2"'] & N \arrow[u, dashed, "\beta"']
\end{tikzcd}$$

It is easy to prove that $F = \mathrm{Im}(\alpha) + \mathrm{Im}(\beta)$. Now we define $\gamma : F \to \frac{N}{T}$ by $\gamma((m, n) + H) = n + T$ for all $(m, n) + H \in F$. Then γ is an epimorphism. Note that $Ker(\gamma) = \mathrm{Im}(\alpha)$ and so $\frac{N}{T} \cong \frac{F}{\mathrm{Im}(\alpha)}$ is finitely generated. Since α is a monomorphism, we have $M \cong \mathrm{Im}(\alpha)$. By hypothesis, $\mathrm{Im}(\alpha)$ has the property $(\delta\text{-}GSCEE)$ so $\mathrm{Im}(\alpha)$ has a generalized δ-supplement of V in F with $V \leq \mathrm{Im}(\beta)$, i.e.,

$$F = \mathrm{Im}(\alpha) + V \quad \text{and} \quad \mathrm{Im}(\alpha) \cap V \subseteq \delta(V).$$

Then $N = \beta^{-1}(\mathrm{Im}(\alpha)) + \beta^{-1}(V) = T + \beta^{-1}(V)$ and $T \cap \beta^{-1}(V) \subseteq \delta(\beta^{-1}(V))$. Hence $\beta^{-1}(V)$ is a generalized δ-supplement of T in N.

$(\Rightarrow)$ Suppose that every submodule of M has the property $(\delta\text{-}GSCE)$. For a cofinite extension N of M, let $N = M + K$ for some submodule K of N. Then $\frac{N}{M} \cong \frac{K}{M \cap K}$ is finitely generated and so $M \cap K$ is a cofinite submodule of K. By hypothesis, there exists a submodule V of K such that

$$K = (M \cap K) + V,$$

$$(M \cap K) \cap V = M \cap V \subseteq \delta(V).$$

Note that $N = M + V$. So V is a generalized δ-supplement of M in N which is contained in K. So M has the property $(\delta\text{-}GSCEE)$. □

Corollary 2. *Every submodule of a module with the property* $(\delta\text{-}GSCEE)$ *has the property* $(\delta\text{-}GSCE)$. *Moreover, a module with the property* $(\delta\text{-}GSCEE)$ *is cofinitely generalized* δ*-supplemented.*

Proposition 4. *Let M be a module and A be a submodule of B such that* $\frac{B}{A}$ *is Noetherian. If A and* $\frac{B}{A}$ *have the property* $(\delta\text{-}GSCE)$, *then so is B.*

Proof. Let $B \subseteq N$ be any cofinite extension of B. By hypothesis, there is a generalized δ-supplement $\frac{V}{A}$ of $\frac{B}{A}$ in $\frac{N}{A}$. Note that $\frac{N}{B} \cong \frac{\frac{V}{A}}{\frac{B \cap V}{A}}$.

Since $\frac{B}{A}$ is Noetherian, $\frac{V}{A}$ is finitely generated. So, V is a cofinite extension of A. Since A has the property $(\delta\text{-}GSCE)$, A has a generalized δ-supplement

W in V. We claim that M is a generalized δ-supplement of B in N. We have epimorphisms $f : W \to \frac{V}{A}$ and $g : \frac{V}{A} \to \frac{N}{B}$ such that $Ker\, f = W \cap A \subseteq \delta(W)$ and $Ker\, g = \frac{V}{A} \cap \frac{B}{A} \subseteq \delta\left(\frac{V}{A}\right)$. Then $g \circ f : W \to \frac{N}{B}$ is an epimorphism such that $W \cap B = Ker(g \circ f) \subseteq \delta(W)$. Finally, $N = V + B = (W + A) + B = W + B$. □

Proposition 5. *Let M be a module and K be a δ-radical submodule of M. If* $\frac{M}{K}$ *has the property* $(\delta\text{-}GSCE)$, *then so does M.*

Proof. Let N be any cofinite extension of M. Then $\frac{N}{M}$ is finitely generated and so $\frac{N}{M} \cong \frac{\frac{N}{K}}{\frac{M}{K}}$ is finitely generated. That is, $\frac{N}{K}$ is a cofinite extension of $\frac{M}{K}$. By hypothesis, there exists a submodule $\frac{V}{K}$ in $\frac{N}{K}$ such that $\frac{N}{K} = \frac{M}{K} + \frac{V}{K}$ and $\frac{M}{K} \cap \frac{V}{K} \subseteq \delta\left(\frac{V}{K}\right)$. Then we have $U = M + V$. Since K is δ-radical, $M \cap V \subseteq \delta(V)$. So V is a generalized δ-supplement of M in N. □

Theorem 2. *Let*

$$0 \to K \xrightarrow{f} M \xrightarrow{g} L \to 0$$

be a short exact sequence. Suppose that K is δ-radical. If K and L have the property $(\delta\text{-}GSCE)$, *so is M. If the sequence splits, then the converse holds.*

Proof. Without loss of generality, we assume that $K \subseteq M$. Since $\frac{M}{K} \cong L$ has the property $(\delta\text{-}GSCE)$ and K is δ-radical, we have M has the property $(\delta\text{-}GSCE)$ by the previous proposition.

Conversely, if the sequence splits, then $M \cong K \oplus L$. So, K and L have the property $(\delta\text{-}GSCE)$ by Proposition 2. □

Corollary 3. *Let M_i $(i = 1, 2, \ldots, n)$ be any finite collection of* δ*-radical modules and $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$. Then M has the property $(\delta\text{-}GSCE)$ iff M_i has the property $(\delta\text{-}GSCE)$ for each $i = 1, 2, \ldots, n$.*

Proof. The necessity follows from Proposition 2. Conversely, to prove that M has the property $(\delta\text{-}GSCE)$, it is sufficient by induction on n to prove that this is the case when $n = 2$. Thus suppose $M = M_1 \oplus M_2$. By using the following short exact sequence:

$$0 \to M_1 \to M \to M_2 \to 0,$$

we have that M has the property $(\delta\text{-}GSCE)$ by Theorem 2, since M_1 is δ-radical and M_2 has the property $(\delta\text{-}GSCE)$. □

Proposition 6. *Let M be a module. Then the following are equivalent*:

(1) *M has a generalized* δ*-supplement in every cofinite essential extension.*

(2) *M has a generalized* δ*-supplement in its cofinitely injective envelope $E(M)$.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Let N be any cofinite essential extension of M, and let $f : M \to N$ and $g : M \to E(M)$ be inclusion mappings. Then we have the following commutative diagram with h as necessarily monic.

$$\begin{tikzcd} M \arrow[r, hook, "f"] \arrow[d, hook, "g"'] & N \arrow[dl, dashed, "h"] \\ E(M) & \end{tikzcd}$$

By hypothesis, M has a generalized δ-supplement in $E(M)$, say K, that is, $M + K = E(M)$ and $M \cap K \subseteq \delta(K)$. Since $M \subseteq h(N)$, we obtain that

$$h(N) \cap E(M) = h(N) \cap (M + K) = M + (h(N) \cap K).$$

Now, taking any $n \in N$, we have $h(n) = m + h_1(n) = h(m + n_1)$, where $m \in M$ and $h(n_1) \in h(N) \cap K$. So, $n = m + n_1 \in M + h^{-1}(K)$ since h is monic, and thus $M + h^{-1}(K) = N$. Moreover, $M \cap h^{-1}(K) = h^{-1}(M \cap K)$ $\subseteq h^{-1}(\delta(K)) \subseteq \delta(h^{-1}(K))$ since $h^{-1}(M) = M$ as h is monic. Hence $h^{-1}(K)$ is a generalized δ-supplement of M in N. □

Proposition 7. *Let M be a module with the property* $(\delta\text{-}GSCE)$. *If M is Noetherian, then M has the property* $(\delta\text{-}CE)$.

Proof. Let N be any cofinite extension of M. There exists a submodule V of N such that $N = M + V$ and $M \cap V \subseteq \delta(V)$ since M has the property $(\delta\text{-}GSCE)$. Since M is Noetherian, $M \cap V$ is finitely generated. So it is easy to see that $M \cap V \ll_{\delta} V$. So M has the property $(\delta\text{-}CE)$. □

Proposition 8. *If R is a* δ*-semiperfect ring, then every left R-module has the property* $(\delta\text{-}GSCE)$.

Proof. Let M be an R-module and N be any cofinite extension of M. Then there exists a finitely generated submodule K of N such that $N = M + K$. Since R is δ-semiperfect, $\frac{N}{M}$ has a projective δ-cover and M has a δ-supplement and so generalized δ-supplement in N. Hence M has the property $(\delta\text{-}GSCE)$. □

Theorem 3. *The following statements are equivalent for a ring R*:

(1) *R is generalized* δ*-supplemented*.

(2) *R-module R has ample generalized* δ*-supplements in every cofinite extension*.

(3) *R-module R has ample generalized δ-supplements in every finitely generated extension*.

(4) *Every left R-module has the property* $(\delta\text{-}GSCE)$.

(5) *Every left R-module has the property* $(\delta\text{-}GSCEE)$.

Proof. (1) $\Rightarrow$ (4) Let M be an R-module and N be a cofinite extension of M. Then there exists a finitely generated submodule K of N such that $N = M + K$. Since R is a generalized δ-supplemented ring, $M \cap K$ has a generalized δ-supplement T in K. And so, $N = M + T$ and $M \cap T \subseteq \delta(T)$. Hence T is a generalized δ-supplement of M in N.

(4) $\Rightarrow$ (1) Suppose that every left R-module has the property $(\delta\text{-}GSCE)$. Since R is a cofinite extension of ideal of itself, every ideal of R has a generalized δ-supplement in R as a left R-module. Hence, R is generalized δ-supplemented.

(2) $\Rightarrow$ (1) If R-module R has the property $(\delta\text{-}GSCEE)$, then it is generalized δ-supplemented by Corollary 2.

(5) $\Rightarrow$ (3) $\Rightarrow$ (2) are clear. (4) $\Rightarrow$ (5) follows from Theorem 1. □

A module M over a ring R is called *unserial* if the lattice of submodules of M is a chain and M is called *serial* if M is a direct sum of uniserial modules. A ring R is called *left* (*right*) *unserial* if the module ${}_RR$ (R_R) is unserial. R is called a *uniserial ring* if R is both right and left uniserial. A uniserial domain R is called *nearly simple* if $Rad(R)$ is the unique nonzero two sided ideal of R and $Rad^2(R) \neq 0$.

In the following example, we show that a module with the property $(\delta\text{-}GSCE)$ need not be $(\delta\text{-}CE)$.

Example 1 (See [3])**.** Let $G = \{f : Q \to Q \mid f(t) = at + b$ for $a, b \in Q$ and $a > 0\}$ be the group of affine linear functions on the field of rational numbers Q. Choose any irrational number $\varepsilon \in \mathbb{R}$ and set $P =$

$\{f \in G \mid \varepsilon \leq f(\varepsilon)\}$ and $P^+ = \{f \in G \mid \varepsilon < f(\varepsilon)\}$. Note that P, resp. P^+ defines a left order on G. Take an arbitrary field F and consider the semigroup group ring $F[P]$ in which the right ideal $M = \sum_{g \in P^+} gF[P]$ is maximal. The set $F[P]/M$ is a left and right ore set and the corresponding localization R is a nearly simple uniserial domain. Taking any nonzero element $r \in R$, $S = End(R/rR)$ is a *rad*-supplemented ring and so a semilocal generalized δ-supplemented ring which is not δ-semiperfect [3]. Since S is a generalized δ-supplemented ring, ${}_S S$ has the property $(\delta\text{-}GSCE)$ by Theorem [3]. However, it can be clearly seen that ${}_S S$ does not have a δ-supplement in every cofinite extension since it is not δ-semiperfect.

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