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BAYESIAN PANEL DATA MODEL AS A MIXED MODEL

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Abstract

In this paper, we consider the random effect panel data model which has fixed and random effects as well as the experimental error term. By using the properties of mixed model, we can represent random panel data model as a mixed model. Bayesian approach is employed to make inferences on the resulting mixed model coefficients. We investigate the posterior density and identify the analytic form of the Bayes factor.

1. Introduction

Statistical data is important to study most of phenomena in economical, social, psychological phenomena, etc. The analysis of this data via the statistical methods gives the researcher or the decision maker more information about the studied phenomenon to make the suitable decision. The data availability needs to limit a mathematical model and put into consideration the type of the available data. One of these data is panel which can be represented by one of the models (fixed effect model or random effect model). One of the aims of science is to describe and predict events in the

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world in which we live. One way this is accomplished by finding a formula or equation that relates quantities in the real world. The linear model involves the simplest and seemingly most restrictive statistical properties: independence, normality, constancy of variance, and linearity. However, the model and the statistical methods associated with it are surprisingly versatile and robust. More importantly, mastery of linear model is a prerequisite to work with advanced statistical tools because most advanced tools are generalizations of the linear model. The linear model is thus central to the training of any statistician, applied or theoretical. Panel (or longitudinal) data are cross-sectional and time-series. There are multiple entities, each of which has repeated measurements at different time periods. Panel data have a cross-sectional (entity or subject) variable and a time-series variable. Panel data usually give the researcher a large number of data points, increasing the degrees of freedom and reducing the collinearity among explanatory variables. Panel data models have become increasingly popular among applied researchers due to their heightened capacity for capturing the complexity of human behaviour as compared to cross-sectional or time-series data models. As a consequence, more and richer panel data sets also have become increasingly available. Linear mixed effects modelling is a widely used statistical method for analyzing repeated measures or longitudinal data. Such longitudinal studies typically aim to investigate and describe the trajectory of a desired outcome. Longitudinal data have the advantage over cross-sectional data by providing more accuracy for the model. Linear mixed effects models are important class of statistical models that can be used to analyze correlated data. Such data include clustered observations repeated measurements, longitudinal measurements, multivariate observations, etc. Linear mixed effects models allow researchers to account for random variation among individuals and between individuals, [1-8, 11-14].

In this paper, we consider the random effect panel data model which has fixed and random effects as well as the experimental error term. By using the properties of mixed model, we can represent random panel data model as a mixed model. Bayesian approach is employed to make inferences on the resulting mixed model coefficients. We investigate the posterior density and identify the analytic form of the Bayes factor.

2. Panel Data Model and the Prior Distribution

Consider the model:

$$y_{it} = \mu + \sum_{j=1}^{K} \beta_j x_{jit} + \varepsilon_{it}, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T, \tag{1}$$

where y_{it} is the value of response variable for *i*th unit at time *t*, X_{ijt} are the explanatory variables, $\mu, \beta_j, \; j = 1, \ldots, K$ are fixed parameters and ε_{it} is an error term with $\varepsilon_{it} \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$, [9].

Now, if the parameter μ is specified as:

$$\mu = \beta_0 + u_i, \tag{2}$$

where $u_i \sim N(0, \sigma_u^2)$, then the model (1) is

$$y_{it} = \beta_0 + \sum_{j=1}^{K} \beta_j x_{jit} + u_i + \varepsilon_{it}. \tag{3}$$

The model (3) is rewrite as follows:

$$y_{it} = \beta_0 + \sum_{j=1}^{K} \beta_j x_{jit} + \omega_{it}, \tag{4}$$

where $\omega_{it} = u_i + \varepsilon_{it}$, $\omega_{it} \sim N(0, \sigma_\omega^2)$, $\sigma_\omega^2 = \sigma_\varepsilon^2 + \sigma_u^2$, thus by using the properties of mixed model, we can represent the model (4) as follows:

$$Y = F\theta + Zu + \varepsilon, \tag{5}$$

where $Y = [Y_{11}, \ldots, Y_{1T}, Y_{21}, \ldots, Y_{N1}, \ldots, Y_{NT}]^T$ has length NT, $F = [e, X]$

is an $NT \times (K + 1)$ design matrix of fixed effects, $e = [1, 1, ..., 1]^T$ has length NT and $X = [X_1, X_2, ..., X_N]^T$ is an $NT \times K$ matrix, $\theta = [\beta_0, \beta_1, ..., \beta_K]^T$ has length $K + 1$, $u = [u_1, ..., u_N]^T$ has length $N \times 1$, Z is an $NT \times N$ design matrix of random effects, $\varepsilon = [\varepsilon_{11}, ..., \varepsilon_{1T}, \varepsilon_{21}, ..., \varepsilon_{2T}, ..., \varepsilon_{N1}, ..., \varepsilon_{NT}]^T$ has length NT with $\varepsilon \sim N(0, \sigma_\varepsilon^2 I_{NT})$ and $u \sim N(0, \sigma_u^2 I_N)$. The model (5) is rewritten as follows:

$$Y = C\theta^* + \omega, \tag{6}$$

then we have

$$Y \mid \theta^*, \sigma_\varepsilon^2, \sigma_u^2 \sim N(C\theta^*, \sigma_\varepsilon^2 I_{NT}) \quad \text{and} \quad u \sim N(0, \sigma_u^2 I_N),$$

where $\theta^* = [\theta, u]^T$ and $C = [F, Z]$.

The likelihood function of $Y \mid \theta^*, \sigma_\varepsilon^2, \sigma_u^2$ can be expressed as

$$L(Y \mid \theta^*, \sigma_\varepsilon^2, \sigma_u^2)$$

$$= (2\pi\sigma_\varepsilon^2 | I_{NT} |)^{\frac{-NT}{2}} \exp\left\{-\frac{1}{2}(Y - C\theta^*)^T (\sigma_\varepsilon^2 I_{NT})^{-1}(Y - C\theta^*)\right\}.$$

We assume that the prior distribution on the model (5) is $\theta \sim U(0, 1)$, the prior distribution of u is $N(0, \sigma_u^2 I_N)$, and the prior distributions of σ_ε^2 and σ_u^2 are inverse gamma parameters $\alpha_\varepsilon, \beta_\varepsilon, \alpha_u$ and β_u, respectively, i.e.,

$$\pi_0(\sigma_\varepsilon^2) = \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)} (\sigma_\varepsilon^2)^{-(\alpha_\varepsilon + 1)} \exp\left(-\frac{\beta_\varepsilon}{\sigma_\varepsilon^2}\right),$$

$$\pi_0(\sigma_u^2) = \frac{\beta_u^{\alpha_u}}{\Gamma(\alpha_u)} (\sigma_u^2)^{-(\alpha_u + 1)} \exp\left(-\frac{\beta_u}{\sigma_u^2}\right),$$

where α_ε, β_ε are hyperparameters that determine the priors and must be chosen by the statistician, [10]. Then the prior distribution of θ^* is

$$\pi_0(\theta^*) = \pi_0(u) \times \pi_0(\theta) = \theta^{*^T} (\sigma_u^2 I_N)^{-1} \theta^*$$

$$= \theta^{*^T} (\sigma_u^2)^{-1} \Gamma \theta^*, \text{where } \Gamma = \begin{pmatrix} 0_{K+1} & 0 \\ 0 & I_N \end{pmatrix}.$$

3. Posterior Distribution

The posterior distribution of θ^* is

$$\pi_1(\theta^* | Y, \sigma_\varepsilon^2, \sigma_u^2)$$

$$\propto L(Y | \theta^*, \sigma_\varepsilon^2, \sigma_u^2) \times \pi_0(\theta^*)$$

$$\propto (2\pi)^{\frac{-NT}{2}} | \sigma_\varepsilon^2 I_{NT} |^{\frac{-1}{2}} \exp\left\{ -\frac{1}{2} (Y - C\theta^*)^T (\sigma_\varepsilon^2 I_{NT})^{-1} (Y - C\theta^*) + \theta^{*T} (\sigma_u^2)^{-1} \Gamma \theta^* \right\}. \quad (7)$$

Also, we note that the exponent of (7) is

$$(Y - C\theta^*)^T (\sigma_\varepsilon^2 I_{NT})^{-1} (Y - C\theta^*) + \theta^{*^T} (\sigma_u^2)^{-1} \Gamma \theta^*$$

$$= Y^T (\sigma_\varepsilon^2 I_{NT})^{-1} Y - 2\theta^{*^T} C^T (\sigma_\varepsilon^2 I_{NT})^{-1} Y$$

$$+ \theta^{*^T} C^T (\sigma_\varepsilon^2 I_{NT})^{-1} C\theta^* + \theta^{*^T} (\sigma_u^2)^{-1} \Gamma \theta^*$$

$$= Y^T (\sigma_\varepsilon^2 I_{NT})^{-1} Y + \theta^{*^T} [C^T (\sigma_\varepsilon^2 I_{NT})^{-1} C + (\sigma_u^2)^{-1} \Gamma] \theta^*$$

$$- 2\theta^{*^T} [C^T (\sigma_\varepsilon^2 I_{NT})^{-1} C + (\sigma_u^2)^{-1} \Gamma][C^T (\sigma_\varepsilon^2 I_{NT})^{-1} C$$

$$+ (\sigma_u^2)^{-1} \Gamma]^{-1} C^T (\sigma_\varepsilon^2 I_{NT})^{-1} Y$$

$$+ \hat{\theta}^{*^T} [C^T (\sigma_\varepsilon^2 I_{NT})^{-1} C + (\sigma_u^2)^{-1} \Gamma] \hat{\theta}^* - \hat{\theta}^{*^T} [C^T (\sigma_\varepsilon^2 I_{NT})^{-1} C + (\sigma_u^2)^{-1} \Gamma] \hat{\theta}^*$$

$$= Y^T(\sigma_\varepsilon^2 I_{NT})^{-1}Y + \theta^{*^T}[C^T(\sigma_\varepsilon^2 I_{NT})^{-1}C + (\sigma_u^2)^{-1}\Gamma]\theta^*$$

$$- 2\theta^{*^T}[C^T(\sigma_\varepsilon^2 I_{NT})^{-1}C + (\sigma_u^2)^{-1}\Gamma]\hat{\theta}^* + \hat{\theta}^{*^T}[C^T(\sigma_\varepsilon^2 I_{NT})^{-1}C + (\sigma_u^2)^{-1}\Gamma]\hat{\theta}^*$$

$$- \hat{\theta}^{*^T}[C^T(\sigma_\varepsilon^2 I_{NT})^{-1}C + (\sigma_u^2)^{-1}\Gamma]\hat{\theta}^*$$

$$= (\theta^* - \hat{\theta}^*)^T[C^T(\sigma_\varepsilon^2 I_{NT})^{-1}C + (\sigma_u^2)^{-1}\Gamma](\theta^* - \hat{\theta}^*) + \gamma^*. \qquad (8)$$

Then

$$\pi_1(\theta^* \mid \sigma_\varepsilon^2, \sigma_u^2, Y)$$

$$\propto \exp\left\{-\frac{1}{2}[(\theta^* - \hat{\theta}^*)^T[C^T(\sigma_\varepsilon^2 I_{NT})^{-1}C + (\sigma_u^2)^{-1}\Gamma](\theta^* - \hat{\theta}^*)]\right\},$$

where $\gamma^* = Y^T(\sigma_\varepsilon^2 I_{NT})^{-1}Y - \hat{\theta}^{*^T}[C^T(\sigma_\varepsilon^2 I_{NT})^{-1}C + (\sigma_u^2)^{-1}\Gamma]\hat{\theta}^*$.

Therefore, it follows that

$$\theta^* \mid Y, \sigma_\varepsilon^2, \sigma_u^2 \sim N(\hat{\theta}^*, [C^T(\sigma_\varepsilon^2 I_{NT})^{-1}C + (\sigma_u^2)^{-1}\Gamma]^{-1}), \qquad (9)$$

where

$$\hat{\theta}^* = [C^T(\sigma_\varepsilon^2 I_{NT})^{-1}C + (\sigma_u^2)^{-1}\Gamma]^{-1}C^T(\sigma_\varepsilon^2 I_{NT})^{-1}Y$$

$$= \left(C^T C + \frac{\sigma_\varepsilon^2}{\sigma_u^2}\Gamma\right)^{-1} C^T Y. \qquad (10)$$

Proposition 1. *The posterior distribution of* θ^* *is*

$$\theta^* \mid Y, \sigma_\varepsilon^2, \sigma_u^2 \sim N[\sigma_u^2\Gamma C^T(\sigma_\varepsilon^2 I_{NT} + C\sigma_u^2\Gamma C^T)^{-1}Y,$$

$$\sigma_u^2\Gamma - (\sigma_u^2)^2\Gamma C^T(C^T\sigma_u^2\Gamma C + \sigma_\varepsilon^2 I_{NT})^{-1}C\Gamma].$$

Proof. Since $E(\theta^* \mid Y, \sigma_\varepsilon^2, \sigma_u^2) = \left(C^T C + \frac{\sigma_\varepsilon^2}{\sigma_u^2} \Gamma \right)^{-1} C^T Y,$ where

$$C^T C + \frac{\sigma_\varepsilon^2}{\sigma_u^2} \Gamma = C^T \left(I_{NT} + C^{T^{-1}} \frac{\sigma_\varepsilon^2}{\sigma_u^2} \Gamma C^{-1} \right) C = C^T \left(I_{NT} + C \frac{\sigma_u^2}{\sigma_\varepsilon^2} \Gamma C^T \right)^{-1} C,$$

$$\begin{aligned}
\left(C^T C + \frac{\sigma_\varepsilon^2}{\sigma_u^2} \Gamma \right)^{-1} C^T Y &= \left[C^T \left(I_{NT} + C \frac{\sigma_u^2}{\sigma_\varepsilon^2} \Gamma C^T \right)^{-1} C \right]^{-1} C^T Y \\
&= \left[C^{-1} \left(I_{NT} + C \frac{\sigma_u^2}{\sigma_\varepsilon^2} \Gamma C^T \right) C^{T^{-1}} \right] C^T Y \\
&= \left(C^{-1} C^{T^{-1}} + \frac{\sigma_u^2}{\sigma_\varepsilon^2} \Gamma \right) C^T Y \\
&= \frac{\sigma_u^2}{\sigma_\varepsilon^2} \Gamma \left(\frac{\sigma_\varepsilon^2}{\sigma_u^2} \Gamma C^{-1} C^{T^{-1}} C^T + C^T \right) Y \\
&= \frac{\sigma_u^2}{\sigma_\varepsilon^2} \Gamma \left(C^T C^{T^{-1}} \frac{\sigma_\varepsilon^2}{\sigma_u^2} \Gamma C^{-1} C C^{-1} + C^T C^{T^{-1}} C^T C C^{-1} \right) Y \\
&= \frac{\sigma_u^2}{\sigma_\varepsilon^2} \Gamma C^T \left(C^{T^{-1}} \frac{\sigma_\varepsilon^2}{\sigma_u^2} \Gamma C^{-1} + I_{NT} \right) Y \\
&= \sigma_u^2 \Gamma C^T \left(C^{T^{-1}} \frac{1}{\sigma_u^2} \Gamma C^{-1} + \frac{1}{\sigma_\varepsilon^2} I_{NT} \right) Y \\
&= \sigma_u^2 \Gamma C^T (C \sigma_u^2 \Gamma C^T + \sigma_\varepsilon^2 I_{NT})^{-1} Y. \qquad (11)
\end{aligned}$$

We now prove the form of covariance.

Since

$$[C^T (\sigma_\varepsilon^2 I_{NT})^{-1} C + (\sigma_u^2)^{-1} \Gamma]^{-1} = \sigma_\varepsilon^2 \left(C^T C + \frac{\sigma_\varepsilon^2}{\sigma_u^2} \Gamma \right)^{-1},$$

we have

$$C^T C + \frac{\sigma_\varepsilon^2}{\sigma_u^2}\Gamma = C^T\left(I_{NT} + C^{T^{-1}}\frac{\sigma_\varepsilon^2}{\sigma_u^2}\Gamma C^{-1}\right)C$$

$$\rightarrow \left(C^T C + \frac{\sigma_\varepsilon^2}{\sigma_u^2}\Gamma\right)^{-1} = \left[C^T\left(I_{NT} + C^{T^{-1}}\frac{\sigma_\varepsilon^2}{\sigma_u^2}\Gamma C^{-1}\right)C\right]^{-1}$$

$$= C^{-1}\left(I_{NT} + C^{T^{-1}}\frac{\sigma_\varepsilon^2}{\sigma_u^2}\Gamma C^{-1}\right)^{-1} C^{T^{-1}}$$

$$= C^{-1}\left[\left(C^{T^{-1}}\frac{\sigma_\varepsilon^2}{\sigma_u^2}\Gamma C^{-1}\right)^{-1} - \left(C^{T^{-1}}\frac{\sigma_\varepsilon^2}{\sigma_u^2}\Gamma C^{-1}\right)^{-1}\left(I_{NT} + C^{T^{-1}}\frac{\sigma_\varepsilon^2}{\sigma_u^2}\Gamma C^{-1}\right)^{-1}\right]C^{T^{-1}}$$

$$= \frac{\sigma_u^2}{\sigma_\varepsilon^2}\Gamma - \frac{\sigma_u^2}{\sigma_\varepsilon^2}\Gamma\left(I_{NT} + C^{T^{-1}}\frac{\sigma_\varepsilon^2}{\sigma_u^2}\Gamma C^{-1}\right)^{-1}$$

$$\rightarrow \sigma_\varepsilon^2\left(C^T C + \frac{\sigma_\varepsilon^2}{\sigma_u^2}\Gamma\right)^{-1} = \sigma_u^2\Gamma - \sigma_u^2\Gamma\left(I_{NT} + C^{T^{-1}}\frac{\sigma_\varepsilon^2}{\sigma_u^2}\Gamma C^{-1}\right)^{-1}$$

$$= \sigma_u^2\Gamma - (\sigma_u^2)^2\Gamma C^T(C^T\sigma_u^2\Gamma C + \sigma_\varepsilon^2 I_{NT})^{-1}C\Gamma. \tag{12}$$

□

It is also clear from (11)

$$C\sigma_u^2\Gamma C^T + \sigma_\varepsilon^2 I_{NT} = \sigma_\varepsilon^2\left(I_{NT} + C\frac{\sigma_u^2}{\sigma_\varepsilon^2}\Gamma C^T\right) = \sigma_\varepsilon^2 I_{NT} + [C\sigma_u^2\Gamma C^T],$$

by spectral decomposition, we obtain $C\Gamma C^T = P^* D^* {P^*}^T$, where $D^* = diag(d_1^*, d_1^*, ..., d_{N^*}^*)$, $N^* = NT$ is the matrix of eigenvalues and P^* is the orthogonal matrix of eigenvectors. Thus

$$\sigma_\varepsilon^2 I_{NT} + [C\sigma_u^2\Gamma C^T] = \sigma_\varepsilon^2 I_{NT} + P^*\sigma_u^2 D^* {P^*}^T$$

$$= P^*\sigma_\varepsilon^2 I_{NT}{P^*}^T + P^*\sigma_u^2 D^* {P^*}^T$$

$$= P^* \sigma_\varepsilon^2 \left(I_{NT} + \frac{\sigma_u^2}{\sigma_\varepsilon^2} D^* \right) P^{*^T}$$

$$= P^* \sigma_\varepsilon^2 (I_{NT} + vD^*) P^{*^T}, \text{ where } v = \frac{\sigma_u^2}{\sigma_\varepsilon^2}.$$

Then the conditional density of Y given σ_ε^2 and v can be written as:

$$m(Y | \sigma_\varepsilon^2, v) = \frac{1}{(2\pi\sigma_\varepsilon^2)^{\frac{N^*}{2}}} \frac{1}{\det[I_{NT} + vD^*]^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2\sigma_\varepsilon^2} Y^T P^* (I_{NT} + vD^*)^{-1} P^{*^T} Y \right\}$$

$$= \frac{1}{(2\pi\sigma_\varepsilon^2)^{\frac{N^*}{2}}} \frac{1}{\left[\prod_{t=1}^{T} \prod_{i=1}^{N} [1 + vd_{it}^*] \right]^{\frac{1}{2}}}$$

$$\cdot \exp\left\{ -\frac{1}{2\sigma_\varepsilon^2} \left(\sum_{t=1}^{T} \sum_{i=1}^{N} \frac{a_{it}^2}{1 + vd_{it}^*} \right) \right\}, \tag{13}$$

where $a = (a_1, \ldots, a_{N^*})^T = P^{*^T} Y$.

We take $\pi_1 = (\sigma_\varepsilon^2, v)$ to be proportional to the product of an inverse gamma density for σ_ε^2 and the density of an $F(b, a)$ distribution for v (for suitable choice of β_ε, α_ε, b and a, $a, b > 0$). The posterior density of v given Y, and the posterior mean and covariance matrix of θ^* are described in the following theorems:

Theorem 1. *The posterior density of v given Y is*

$$\pi_1(v | Y) = \frac{v^{\left(\frac{b}{2}\right) - 1}}{(a + vb)^{-\frac{a+b}{2}}} \left[\prod_{t=1}^{T} \prod_{i=1}^{N} [1 + vd_{it}^*] \right]^{\frac{-1}{2}}$$

$$\cdot \left(2\beta_\varepsilon + \sum_{t=1}^{T} \sum_{i=1}^{N} \frac{a_{it}^2}{1 + vd_{it}^*} \right)^{-(N^* + 2\alpha_\varepsilon + 2)/2}.$$

Proof.

$$\pi_1(v|Y) = \int m(Y|\sigma_\varepsilon^2, v)\, f(v, b, a)\, f(\sigma_\varepsilon^2, \alpha_\varepsilon, \beta_\varepsilon)\, d\sigma_\varepsilon^2$$

$$= \int \frac{1}{(2\pi\sigma_\varepsilon^2)^{\frac{N^*}{2}}}\left[\prod_{t=1}^{T}\prod_{i=1}^{N}[1 + vd_{it}^*]\right]^{\frac{-1}{2}} \frac{b^{b/2}a^{a/2}}{B(b, a)} \frac{v^{\left(\frac{b}{2}\right)-1}}{(a + vb)^{-\frac{a+b}{2}}}$$

$$\cdot \exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\left(\sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1 + vd_{it}^*}\right)\right\} \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)}(\sigma_\varepsilon^2)^{-(\alpha_\varepsilon+1)} \exp\left\{\frac{-\beta_\varepsilon}{\sigma_\varepsilon^2}\right\} d\sigma_\varepsilon^2$$

$$= \frac{(2\pi)^{-\frac{N^*}{2}}}{\Gamma(\alpha_\varepsilon)} \cdot \frac{b^{b/2}}{B(b, a)} \frac{v^{\left(\frac{b}{2}\right)-1}}{(a + vb)^{-\frac{a+b}{2}}} \int\left[\prod_{t=1}^{T}\prod_{i=1}^{N}[1 + vd_{it}^*]\right]^{\frac{-1}{2}}$$

$$\cdot \exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\left(2\beta_\varepsilon + \sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1 + vd_{it}^*}\right)\right\}(\sigma_\varepsilon^2)^{-(N^*+2\alpha_\varepsilon+2)/2} d\sigma_\varepsilon^2$$

$$= \frac{(2\pi)^{-\frac{N^*}{2}}}{\Gamma(\alpha_\varepsilon)} \cdot \frac{b^{b/2}}{B(b, a)} \frac{v^{\left(\frac{b}{2}\right)-1}}{(a + vb)^{-\frac{a+b}{2}}} \int\left[\prod_{t=1}^{T}\prod_{i=1}^{N}[1 + vd_{it}^*]\right]^{\frac{-1}{2}}$$

$$\cdot \exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\left(2\beta_\varepsilon + \sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1 + vd_{it}^*}\right)\right\}\left(2\beta_\varepsilon + \sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1 + vd_{it}^*}\right)^{\frac{N^*+2\alpha_\varepsilon+2}{2}}$$

$$\cdot\left(2\beta_\varepsilon + \sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1 + vd_{it}^*}\right)^{-\frac{N^*+2\alpha_\varepsilon+2}{2}} (\sigma_\varepsilon^2)^{-\frac{N^*+2\alpha_\varepsilon+2}{2}}$$

$$\cdot (2)^{\left(\frac{N^*+2\alpha_\varepsilon+2}{2}\right)}(2)^{-\left(\frac{N^*+2\alpha_\varepsilon+2}{2}\right)} d\sigma_\varepsilon^2$$

$$= \frac{(2\pi)^{-\frac{N^*}{2}}}{\Gamma(\alpha_\varepsilon)} \frac{b^{b/2}}{B(b, a)} \frac{v^{\left(\frac{b}{2}\right)-1}}{(a + vb)^{-\frac{a+b}{2}}} \int\left[\prod_{t=1}^{T}\prod_{i=1}^{N}[1 + vd_{it}^*]\right]^{\frac{-1}{2}}$$

$$\cdot \exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\left(2\beta_\varepsilon + \sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1 + vd_{it}^*}\right)\right\}$$

$$\cdot \frac{\left(2\beta_\varepsilon + \sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1+vd_{it}^*}\right)^{(N^*+2\alpha_\varepsilon+2)/2}}{(2\sigma_\varepsilon^2)^{(N^*+2\alpha_\varepsilon+2)/2}}$$

$$\cdot\left(2\beta_\varepsilon + \sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1+vd_{it}^*}\right)^{-(N^*+2\alpha_\varepsilon+2)/2}(2)^{\frac{N^*+2\alpha_\varepsilon+2}{2}}d\sigma_\varepsilon^2$$

$$\propto \frac{(2\pi)^{-\frac{N^*}{2}}}{\Gamma(\alpha_\varepsilon)}\cdot\frac{b^{b/2}}{B(b,\,a)}\frac{v^{\left(\frac{b}{2}\right)-1}}{(a+vb)^{-\frac{a+b}{2}}}\int\left[\prod_{t=1}^{T}\prod_{i=1}^{N}[1+vd_{it}^*]\right]^{\frac{-1}{2}}$$

$$\cdot\exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\left(2\beta_\varepsilon + \sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1+vd_{it}^*}\right)\right\}$$

$$\cdot\left(\frac{2\beta_\varepsilon + \sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1+vd_{it}^*}}{2\sigma_\varepsilon^2}\right)^{(N^*+2\alpha_\varepsilon+2)/2}$$

$$\cdot\left(2\beta_\varepsilon + \sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1+vd_{it}^*}\right)^{-(N^*+2\alpha_\varepsilon+2)/2}d\sigma_\varepsilon^2$$

$$\propto \frac{v^{\left(\frac{b}{2}\right)-1}}{(a+vb)^{-\frac{a+b}{2}}}\int\left[\prod_{t=1}^{T}\prod_{i=1}^{N}[1+vd_{it}^*]\right]^{\frac{-1}{2}}$$

$$\cdot\exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\left(2\beta_\varepsilon + \sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1+vd_{it}^*}\right)\right\}$$

$$\cdot\left(\frac{2\beta_\varepsilon + \sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1+vd_{it}^*}}{2\sigma_\varepsilon^2}\right)^{((N^*+2\alpha_\varepsilon+4)/2)-1}$$

$$\cdot\left(2\beta_\varepsilon + \sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1+vd_{it}^*}\right)^{-(N^*+2\alpha_\varepsilon+2)/2}d\sigma_\varepsilon^2$$

$$\propto \frac{v^{\left(\frac{b}{2}\right)-1}}{(a+vb)^{-\frac{a+b}{2}}} \Gamma\left(\frac{N^* + 2\alpha_\varepsilon + 4}{2}\right)\left[\prod_{t=1}^T \prod_{i=1}^N [1 + vd_{it}^*]\right]^{\frac{-1}{2}}$$

$$\cdot \left(2\beta_\varepsilon + \sum_{t=1}^T \sum_{i=1}^N \frac{a_{it}^2}{1 + vd_{it}^*}\right)^{-\left(\frac{N^* + 2\alpha_\varepsilon + 2}{2}\right)}$$

$$\propto \frac{v^{\left(\frac{b}{2}\right)-1}}{(a+vb)^{-\frac{a+b}{2}}} \left[\prod_{t=1}^T \prod_{i=1}^N [1 + vd_{it}^*]\right]^{\frac{-1}{2}}$$

$$\cdot \left(2\beta_\varepsilon + \sum_{t=1}^T \sum_{i=1}^N \frac{a_{it}^2}{1 + vd_{it}^*}\right)^{-\left(\frac{N^* + 2\alpha_\varepsilon + 2}{2}\right)}. \qquad \square$$

Theorem 2. *The posterior mean and variance-covariance matrix of* θ^* *are* $E(\theta^* | Y) = \Gamma C^T P^* E\{(I_{NT} + vD^*)^{-1} | Y\} a,$ *and*

$$\operatorname{var}(\theta^* | Y) = \frac{1}{\alpha_\varepsilon + \frac{N^*}{2} + 1} E\left(2\beta_\varepsilon + \sum_{t=1}^T \sum_{i=1}^N \frac{a_{it}^2}{1 + vd_{it}^*}\right)\Gamma - \frac{1}{\alpha_\varepsilon + \frac{N^*}{2} + 1} \Gamma C^T P^*$$

$$\cdot E\left[\left(2\beta_\varepsilon + \sum_{t=1}^T \sum_{i=1}^N \frac{a_{it}^2}{1 + vd_{it}^*}\right)(I_{NT} + vD^*)^{-1} | Y\right]$$

$$+ E[R^*(v) R^*(v)^T | Y].$$

Proof. From (11), we have

$$E(\theta^* | Y) = \sigma_u^2 \Gamma C^T (\sigma_\varepsilon^2 I_{NT} + C\sigma_u^2 \Gamma C^T)^{-1} Y$$

$$= \sigma_u^2 \Gamma C^T \left(\sigma_\varepsilon^2 \left(I_{NT} + \frac{\sigma_u^2}{\sigma_\varepsilon^2} C\Gamma C^T\right)\right)^{-1} Y$$

$$= \frac{\sigma_u^2}{\sigma_\varepsilon^2} \Gamma C^T (I_{NT} + P^* vD^* P^{*^T})^{-1} Y$$

$$= \frac{\sigma_u^2}{\sigma_\varepsilon^2}\Gamma C^T {P^*}^{T^{-1}}(I_{NT} + vD^*)^{-1}{P^*}^{-1}Y$$

$$= \frac{\sigma_u^2}{\sigma_\varepsilon^2}\Gamma C^T P^*(I_{NT} + vD^*)^{-1}{P^*}^T Y.$$

Since P^* is the orthogonal matrix of eigenvectors, ${P^*}^{-1} = {P^*}^T$ and ${P^*}^{T^{-1}} = P^*$.

Therefore,

$$E(\theta^* | Y) = \Gamma C^T vP^*\{(I_{NT} + vD^*)^{-1}\}{P^*}^T Y$$

$$= \Gamma C^T P^* E\{(I_{NT} + vD^*)^{-1} | Y\} a,$$

where the expectation $E\{(I_{NT} + vD^*)^{-1} | Y\}$ is taken with respect to $\pi_1(v | Y)$ (see Theorem 1 above).

Furthermore, the posterior variance-covariance matrix of θ^* is given by

$$E[Var(\theta^* | Y)] + Var[E(\theta^* | Y)]$$

$$= E[\sigma_u^2\Gamma - (\sigma_u^2)^2\Gamma C^T(\sigma_\varepsilon^2 I_{NT} + C\sigma_u^2\Gamma C^T)^{-1}C\Gamma] + Var\{E[R^*(v) | Y]\},$$

where

$$R^*(v) = E(\theta^* | Y) = \Gamma C^T P^* E\{(I_{NT} + vD^*)^{-1} | Y\} a$$

$$Var(\theta^* | Y) = E\left[\sigma_u^2\Gamma - (\sigma_u^2)^2\Gamma C^T \frac{1}{\sigma_\varepsilon^2}\left(I_{NT} + C\frac{\sigma_u^2}{\sigma_\varepsilon^2}\Gamma C^T\right)C\Gamma\right]$$

$$+ E[R^*(v)R^*(v)^T | Y]$$

$$= E\left[\sigma_u^2\Gamma - \frac{(\sigma_u^2)^2}{\sigma_\varepsilon^2}\Gamma C^T\left(I_{NT} + C\frac{\sigma_u^2}{\sigma_\varepsilon^2}\Gamma C^T\right)^{-1}C\Gamma\right]$$

$$+ E[R^*(v)R^*(v)^T | Y]$$

$$= E\left[\sigma_u^2\Gamma - \frac{(\sigma_u^2)^2}{\sigma_\varepsilon^2}\Gamma C^T\left(I_{NT} + P^* \frac{\sigma_u^2}{\sigma_\varepsilon^2} D^* P^{*^T}\right)^{-1} C\Gamma\right]$$

$$+ E[R^*(v)R^*(v)^T \mid Y]$$

$$= E\left[\sigma_u^2\Gamma - \frac{\sigma_u^2}{\sigma_\varepsilon^2}\sigma_u^2\Gamma C^T (P^{*^T})^{-1}\left(I_{NT} + \frac{\sigma_u^2}{\sigma_\varepsilon^2} D^*\right)^{-1}(P^*)^{-1}C\Gamma\right]$$

$$+ E[R^*(v)R^*(v)^T \mid Y]$$

$$= E[\sigma_u^2\Gamma - v\sigma_u^2\Gamma C^T (P^{*^T})^{-1}(I_{NT} + vD^*)^{-1}(P^*)^{-1}C\Gamma]$$

$$+ E[R^*(v)R^*(v)^T \mid Y]$$

$$= E[\sigma_u^2\Gamma - v\sigma_u^2\Gamma C^T P^*(I_{NT} + vD^*)^{-1}P^{*^T} C\Gamma] + E[R^*(v)R^*(v)^T \mid Y],$$

since $P^{*^{-1}} = P^{*^T}$ and $(P^{*^T})^{-1} = P^*$.

Therefore, the posterior variance-covariance matrix of θ^* is given by

$$\frac{1}{\alpha_\varepsilon + \frac{N^*}{2} + 1} E\left(2\beta_\varepsilon + \sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1 + vd_{it}^*}\right)\Gamma$$

$$- \frac{1}{\alpha_\varepsilon + \frac{N^*}{2} + 1}\Gamma C^T P^* E\left[\left(2\beta_\varepsilon + \sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1 + vd_{it}^*}\right)(I_{NT} + vD^*)^{-1} \mid Y\right]$$

$$+ E[R^*(v)R^*(v)^T \mid Y]. \qquad \square$$

4. Bayes Factor

We would like to choose between a Bayesian mixed panel data model and its fixed counterpart by the criterion of the Bayes factor for two hypotheses:

$$H_0 = F\theta + \varepsilon \text{ versus } H_1 = F\theta + Zu + \varepsilon. \tag{14}$$

We compute the Bayes factor B_{01}, of H_0 relative to H_1 for testing problem (14) as follows:

$$B_{01}(Y) = \frac{m(Y \mid H_0)}{m(Y \mid H_1)}, \tag{15}$$

where $m(Y \mid H_i)$ is predictive density of Y under the model H_i, $i = 0, 1$. We have

$$m(Y \mid H_0) = \int f(Y \mid \theta, \sigma_\varepsilon^2)\pi_0(\sigma_\varepsilon^2)d\sigma_\varepsilon^2,$$

where under H_0, $\pi_0(\sigma_\varepsilon^2)$ induced by $\pi_0(\theta^*, \sigma_u^2, \sigma_\varepsilon^2)$ is the only part needed, and

$$m(Y \mid H_1) = \int f(Y \mid \theta^*, \sigma_u^2, \sigma_\varepsilon^2), \pi_0(\theta^*, \sigma_u^2, \sigma_\varepsilon^2)d\theta^* d\sigma_u^2 d\sigma_\varepsilon^2,$$

where $\pi_0(\sigma_u^2, \sigma_\varepsilon^2)$ will be constant in θ^*. Therefore,

$$m(Y \mid H_0)$$

$$= \int f(Y \mid \theta, \sigma_\varepsilon^2)\pi_0(\sigma_\varepsilon^2)d\sigma_\varepsilon^2$$

$$= \int \left[(2\pi)^{\frac{-N^*}{2}} (\sigma_\varepsilon^2)^{\frac{-N^*}{2}} \exp\left\{ -\frac{1}{2\sigma_\varepsilon^2}(Y - F\theta)^T (Y - F\theta) \right\} \right.$$

$$\left. \cdot \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)} (\sigma_\varepsilon^2)^{-(\alpha_\varepsilon + 1)} \exp\left(-\frac{\beta_\varepsilon}{\sigma_\varepsilon^2} \right) \right] d\sigma_\varepsilon^2$$

$$= (2\pi)^{\frac{-N^*}{2}} \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)} \int (\sigma_\varepsilon^2)^{-\left(\alpha_\varepsilon + \frac{N^*}{2} + 1\right)} \exp -\left(\frac{\beta_\varepsilon + \frac{1}{2}(Y - F\theta)^T (Y - F\theta)}{\sigma_\varepsilon^2} \right) d\sigma_\varepsilon^2$$

$$= (2\pi)^{\frac{-N^*}{2}} \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)} \int (\sigma_\varepsilon^2)^{-\left(\alpha_\varepsilon + \frac{N^*}{2} + 1\right)} \left(\beta_\varepsilon + \frac{1}{2}(Y - F\theta)^T (Y - F\theta) \right)^{\left(\alpha_\varepsilon + \frac{N^*}{2} + 1\right)}$$

$$\cdot\left(\beta_\varepsilon + \frac{1}{2}(Y - F\theta)^T (Y - F\theta)\right)^{-\left(\alpha_\varepsilon + \frac{N^*}{2} + 1\right)} \exp -\left(\frac{\beta_\varepsilon + \frac{1}{2}(Y - F\theta)^T (Y - F\theta)}{\sigma_\varepsilon^2}\right) d\sigma_\varepsilon^2$$

$$= (2\pi)^{\frac{-N^*}{2}} \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)} \left\{ \int \frac{\left(\beta_\varepsilon + \frac{1}{2}(Y - F\theta)^T (Y - F\theta)\right)^{\left(\alpha_\varepsilon + \frac{N^*}{2} + 1\right)}}{(\sigma_\varepsilon^2)^{\left(\alpha_\varepsilon + \frac{N^*}{2} + 1\right)}} \exp \right.$$

$$-\left(\frac{\beta_\varepsilon + \frac{1}{2}(Y - F\theta)^T (Y - F\theta)}{\sigma_\varepsilon^2}\right)$$

$$\left. \cdot\left(\beta_\varepsilon + \frac{1}{2}(Y - F\theta)^T (Y - F\theta)\right)^{-\left(\alpha_\varepsilon + \frac{N^*}{2} + 1\right)} \right\} d\sigma_\varepsilon^2$$

$$= (2\pi)^{\frac{-N^*}{2}} \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)} \left\{ \int \left(\frac{\beta_\varepsilon + \frac{1}{2}(Y - F\theta)^T (Y - F\theta)}{\sigma_\varepsilon^2}\right)^{\left(\alpha_\varepsilon + \frac{N^*}{2} + 1\right)} \exp \right.$$

$$-\left(\frac{2\beta_\varepsilon + (Y - F\theta)^T (Y - F\theta)}{2\sigma_\varepsilon^2}\right)$$

$$\left. \cdot\,(2\beta_\varepsilon + (Y - F\theta)^T (Y - F\theta))^{-\left(\alpha_\varepsilon + \frac{N^*}{2} + 1\right)} d\sigma_\varepsilon^2 \right\}$$

$$= (2\pi)^{\frac{-N^*}{2}} \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)} \left\{ \int \left(\frac{\beta_\varepsilon + \frac{1}{2}(Y - F\theta)^T (Y - F\theta)}{\sigma_\varepsilon^2}\right)^{\left(\alpha_\varepsilon + \frac{N^*}{2} + 2\right) - 1} \exp \right.$$

$$-\left(\frac{\beta_\varepsilon + \frac{1}{2}(Y - F\theta)^T(Y - F\theta)}{\sigma_\varepsilon^2}\right)$$

$$\left.\cdot\left[\beta_\varepsilon + \frac{1}{2}(Y - F\theta)^T(Y - F\theta)\right]^{-\left(\alpha_\varepsilon + \frac{N^*}{2} + 1\right)} d\sigma_\varepsilon^2\right\}$$

$$= (2\pi)^{\frac{-N^*}{2}} \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)} \Gamma\left(\alpha_\varepsilon + \frac{N^*}{2} + 2\right)\left(\beta_\varepsilon + \frac{1}{2}(Y - F\theta)^T(Y - F\theta)\right)^{-\left(\alpha_\varepsilon + \frac{N^*}{2} + 1\right)}. \quad (16)$$

Further, using (13), it follows that

$$m(Y \mid H_1, \sigma_\varepsilon^2, v)$$

$$= (2\pi\sigma_\varepsilon^2)^{\frac{-N^*}{2}} \left[\prod_{t=1}^{T}\prod_{i=1}^{N}[1 + vd_{it}^*]\right]^{\frac{-1}{2}} \exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\left(\sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1 + vd_{it}^*}\right)\right\}.$$

Therefore,

$$m(Y|H_1)$$

$$= \int m(Y \mid H_1, \sigma_\varepsilon^2, v)\pi_0(\sigma_\varepsilon^2, v)d\sigma_\varepsilon^2 dv$$

$$= \int (2\pi)^{\frac{-N^*}{2}} (\sigma_\varepsilon^2)^{\frac{-N^*}{2}} \left[\prod_{t=1}^{T}\prod_{i=1}^{N}[1 + vd_{it}^*]\right]^{\frac{-1}{2}} \exp\left\{-\frac{1}{2\sigma_\varepsilon^2}\left(\sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1 + vd_{it}^*}\right)\right\}\pi_0(v)$$

$$\cdot \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)}(\sigma_\varepsilon^2)^{-(\alpha_\varepsilon + 1)} \exp\left(-\frac{\beta_\varepsilon}{\sigma_\varepsilon^2}\right) d\sigma_\varepsilon^2 dv$$

$$= \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)}(2\pi)^{\frac{-N^*}{2}} \int (\sigma_\varepsilon^2)^{-\left(\frac{N^*}{2} + \alpha_\varepsilon + 1\right)} \left[\prod_{t=1}^{T}\prod_{i=1}^{N}[1 + vd_{it}^*]\right]^{\frac{-1}{2}} \pi_0(v)$$

$$\cdot \left\{\int \exp\left[-\frac{1}{\sigma_\varepsilon^2}\left(\beta_\varepsilon + \frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1 + vd_{it}^*}\right)\right] d\sigma_\varepsilon^2\right\} dv$$

$$= \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)}(2\pi)^{\frac{-N^*}{2}} \int (\sigma_\varepsilon^2)^{-\left(\frac{N^*}{2}+\alpha_\varepsilon+1\right)} \left[\prod_{t=1}^{T}\prod_{i=1}^{N}[1+vd_{it}^*]\right]^{\frac{-1}{2}} \pi_0(v)$$

$$\cdot \left(\beta_\varepsilon + \frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1+vd_{it}^*}\right)^{\left(\frac{N^*}{2}+\alpha_\varepsilon+1\right)}$$

$$\cdot \left\{\int \exp\left[-\frac{1}{\sigma_\varepsilon^2}\left(\beta_\varepsilon + \frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1+vd_{it}^*}\right)\right] d\sigma_\varepsilon^2\right\} dv$$

$$\cdot \left(\beta_\varepsilon + \frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1+vd_{it}^*}\right)^{-\left(\frac{N^*}{2}+\alpha_\varepsilon+1\right)}$$

$$= \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)}(2\pi)^{\frac{-N^*}{2}} \int \frac{\left(\beta_\varepsilon + \frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1+vd_{it}^*}\right)^{\left(\frac{N^*}{2}+\alpha_\varepsilon+1\right)}}{(\sigma_\varepsilon^2)^{\left(\frac{NT}{2}+\alpha_\varepsilon+1\right)}}$$

$$\cdot \exp\left[-\frac{1}{\sigma_\varepsilon^2}\left(\beta_\varepsilon + \frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1+vd_{it}^*}\right)\right] d\sigma_\varepsilon^2 \int \left[\prod_{t=1}^{T}\prod_{i=1}^{N}[1+vd_{it}^*]\right]^{\frac{-1}{2}} \pi_0(v)$$

$$\cdot \left(\beta_\varepsilon + \frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1+vd_{it}^*}\right)^{-\left(\frac{N^*}{2}+\alpha_\varepsilon+1\right)} dv$$

$$= \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)}(2\pi)^{\frac{-N^*}{2}} \int \left(\frac{\beta_\varepsilon + \frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1+vd_{it}^*}}{\sigma_\varepsilon^2}\right)^{\left(\frac{N^*}{2}+\alpha_\varepsilon+2\right)-1}$$

$$\cdot \exp\left[-\frac{1}{\sigma_\varepsilon^2}\left(\beta_\varepsilon + \frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{N}\frac{a_{it}^2}{1+vd_{it}^*}\right)\right] d\sigma_\varepsilon^2$$

$$\cdot \int \left[\prod_{t=1}^{T} \prod_{i=1}^{N} [1 + vd_{it}^{*}]\right]^{\frac{-1}{2}} \pi_0(v)$$

$$\cdot \left(\beta_\varepsilon + \frac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{N} \frac{a_{it}^2}{1 + vd_{it}^{*}}\right)^{-\left(\frac{N^*}{2} + \alpha_\varepsilon + 1\right)} dv$$

$$= (2\pi)^{\frac{-N^*}{2}} \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)} \Gamma\left(\frac{N^*}{2} + \alpha_\varepsilon + 2\right)$$

$$\cdot \int \left[\prod_{t=1}^{T} \prod_{i=1}^{N} [1 + vd_{it}^{*}]\right]^{\frac{-1}{2}}$$

$$\cdot\, \pi_0(v) \left(\beta_\varepsilon + \frac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{N} \frac{a_{it}^2}{1 + vd_{it}^{*}}\right)^{-\left(\frac{N^*}{2} + \alpha_\varepsilon + 1\right)} dv. \qquad (17)$$

5. Conclusion

(1) The posterior density of v given Y, where $v = \frac{\sigma_u^2}{\sigma_\varepsilon^2}$ is:

$$\pi_1(v \mid Y) = \frac{v^{\left(\frac{b}{2}\right) - 1}}{(a + vb)^{-\frac{a+b}{2}}} \left[\prod_{t=1}^{T} \prod_{i=1}^{N} [1 + vd_{it}^{*}]\right]^{\frac{-1}{2}}$$

$$\cdot \left(2\beta_\varepsilon + \sum_{t=1}^{T} \sum_{i=1}^{N} \frac{a_{it}^2}{1 + vd_{it}^{*}}\right)^{-(N^* + 2\alpha_\varepsilon + 2)/2}.$$

(2) The posterior mean of θ^* given Y is:

$$E(\theta^* \mid Y) = \Gamma C^T P^* E\{(I_{NT} + vD^*)^{-1} \mid Y\} a.$$

(3) The posterior covariance matrix of θ^* given Y is:

$$Var(\theta^* | Y) = \frac{1}{\alpha_\varepsilon + \frac{N^*}{2} + 1} E\left(2\beta_\varepsilon + \sum_{t=1}^{T} \sum_{i=1}^{N} \frac{a_{it}^2}{1 + vd_{it}^*}\right)\Gamma - \frac{1}{\alpha_\varepsilon + \frac{N^*}{2} + 1} \Gamma C^T P^*$$

$$\cdot E\left[\left(2\beta_\varepsilon + \sum_{t=1}^{T} \sum_{i=1}^{N} \frac{a_{it}^2}{1 + vd_{it}^*}\right)(I_{NT} + vD^*)^{-1} | Y\right]$$

$$+ E[R^*(v) R^*(v)^T | Y].$$

(4) Bayes factor for testing the two models $H_0 = F\theta + \varepsilon$ versus $H_1 = F\theta + Zu + \varepsilon$ is:

$$B_{01}(Y) = \frac{m(Y | H_0)}{m(Y | H_1)},$$

where $m(Y | H_i)$ is predictive density of Y under the model H_i, $i = 0, 1.$ We have

$$m(Y | H_0)$$

$$= (2\pi)^{\frac{-N^*}{2}} \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)} \Gamma\left(\alpha_\varepsilon + \frac{N^*}{2} + 2\right)\left[\beta_\varepsilon + \frac{1}{2}(Y - F\theta)^T (Y - F\theta)\right]^{-\left(\alpha_\varepsilon + \frac{N^*}{2} + 1\right)}$$

and

$$m(Y | H_1) = (2\pi)^{\frac{-N^*}{2}} \frac{\beta_\varepsilon^{\alpha_\varepsilon}}{\Gamma(\alpha_\varepsilon)} \Gamma\left(\frac{N^*}{2} + \alpha_\varepsilon + 2\right)$$

$$\cdot \int \left[\prod_{t=1}^{T} \prod_{i=1}^{N} [1 + vd_{it}^*]\right]^{\frac{-1}{2}} \pi_0(v)$$

$$\cdot \left[\beta_\varepsilon + \frac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{N} \frac{a_{it}^2}{1 + vd_{it}^*}\right]^{-\left(\frac{N^*}{2} + \alpha_\varepsilon + 1\right)} dv.$$

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