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ON A GENERALIZATION OF CONDITION (P')

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Abstract

In 2013, Golchin and Mohammadzadeh introduced a generalization of Condition (*P*) called Condition (P') and gave a characterization of monoids by this condition of their (Rees factor) acts. Also in 2012, Qiao and Wei introduced the *GP*-flatness property of acts, as a generalization of principal weak flatness. In this paper, first we introduce Condition (GP'), a generalization of Condition (P') of acts over monoids and then we will give general properties of it. We also give a characterization of monoids all of whose acts satisfy this condition. Furthermore, many known results are generalized.

1. Preliminaries

For a monoid *S*, a non-empty set *A* is called a *right S-act*, if *S* acts on *A* unitarily from the right, that is, there exists a mapping $A \times S \to A$, $(a, s) \mapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A$ and all $s, t \in S$. Left *S*-acts are defined similarly. The notations A_S

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and ${}_S B$ will often be used to denote a right or left *S*-act and $\Theta_S = \{\theta\}$ is the one-element right *S*-act. The study of flatness properties of *S*-acts in general began in the early 1970s, and a comprehensive survey of this research (up until the year 2000) is to be found in [8]. In [4], Golchin and Mohammadzadeh defined a generalization of Condition (*P*), which was called *Condition* (P'). In this paper, we introduce Condition (GP'), as a generalization of Condition (P'), and will give some general properties. We show that Condition (P') implies Condition (GP'), but the converse is not true in generally, and Condition (GP') implies *GP*-flat, but the converse is not true too. Also a characterization of monoids for which this condition transfers from products of acts to their components will be given. Then, we will give a characterization of a monoid *S* over which all right *S*-acts satisfy Condition (GP') and a characterization of a monoid *S* for which this condition of right *S*-acts has some other properties. Also we will a characterization of a monoid *S* for which all right *S*-acts satisfying some flatness properties have Condition (GP').

Throughout this article, *S* will always denote a monoid. For basic concepts and terminologies relating to semigroups and acts over monoids we refer the reader to [7] and [8].

2. General Properties

In this section, we introduce Condition (GP') and give some general properties of it. We show that Condition (P') implies Condition (GP'), but the converse is not true, also that Condition (GP') implies *GP*-flat, but the converse is not true. We show that Condition (GP') can be transferred from product of *S*-acts to their components, for commutative monoid *S*. Also we give a characterization of monoids for which Condition (GP') transfers from products of acts to their components.

Definition 2.1. We say that a right S-act A_S satisfies Condition (GP') if for every $a, a' \in A_S$, $t, t', z \in S$, $at = a't'$ and $tz = t'z$ imply that there exist $a'' \in A_S, u, v \in S$ and $n \in \mathbb{N}$ such that $a = a''u$, $a' = a''v$ and $ut^n = vt'^n$.

Theorem 2.2. *The following statements hold*:

(1) *If the right S-act* A_S *satisfies Condition* (GP'), *then every retract of* A_S *satisfies Condition* (GP').

(2) $A_S = \coprod_{i \in I} A_i$, *where* $A_i, i \in I$ *are right S-acts, satisfies Condition* (GP') *if and only if* A_i *satisfies Condition* (GP'), *for every* $i \in I$.

(3) *If* $\{B_i | i \in I\}$ *is a chain of subacts of* A_S *and every* B_i, $i \in I$ *satisfies Condition* (GP'), *then* $\bigcup_{i \in I} B_i$ *satisfies Condition* (GP').

(4) S_S *satisfies Condition* (GP').

Proof. Proofs are obvious, by definition. □

We recall from [4] that S-act A_S satisfies Condition (P'), if for every $a, a' \in A_S$, $t, t', z \in S$,

$$at = a't', \; tz = t'z \Rightarrow (\exists a'' \in A_S)(\exists u, v \in S)(a = a''u, \; a' = a''v \text{ and } ut = vt').$$

Clearly, Condition (P') implies Condition (GP'), but not the converse, see the following example.

Example 2.3. Let $T = \{0, e, f\}$ be a null semigroup, $S = T^1$ and let $A = \{a_1, a_2\}$ be regarded as an S-act in such a way that $a_i 1 = a_i$ and $a_i s = a_1$, for $i = 1, 2$ and every $s \in T$. Obviously A_S is a right S-act, which does not satisfy Condition (P'). Otherwise, $a_1 e = a_2 f$ and $e0 = f0$ imply that there exist $a'' \in A_S$ and $u, v \in S$ such that $a_1 = a''u$, $a_2 = a''v$ and $ue = vf$. Now $a_2 = a''v$ implies that $a'' = a_2$ and $v = 1$. Then $ue \neq f$, for every $u \in S$, which is a contradiction. Therefore, A_S does not satisfy Condition (P'). On the other hand $a_1 s = a_1 t$ and $s0 = t0$, for every

$s, t \in S$. Then $a_1 = a_1 1$ and $1s^2 = 1f^2$. Also $a_1 s = a_2 t$ and $s0 = t0$, for every $1 \neq t$, $s \in S$. Then $a_1 = a_2 e$, $a_2 = a_2 1$ and $es^2 = 1t^2$. Therefore, A_S satisfies Condition (GP').

The S-act A_S is called *GP-flat*, if for every $s \in S$, and $a, a' \in A_S$, $a \otimes s = a' \otimes s$ in $A_S \otimes_S S$ implies the existence of a natural number n such that $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes_S Ss^n$ (see [16]).

Proposition 2.4. *If the right S-act* A_S *satisfies Condition* (GP'), *then* A_S *is GP-flat.*

Proof. Suppose that A_S satisfies Condition (GP') and let $as = a's$, for $a, a' \in A_S$ and $s \in S$. Then there exist $a'' \in A_S$, $u, v \in S$ and $n \in \mathbb{N}$ such that $a = a''u$, $a' = a''v$ and $us^n = vs^n$. Therefore,

$$a \otimes s^n = a''u \otimes s^n = a'' \otimes us^n = a'' \otimes vs^n = a''v \otimes s^n = a' \otimes s^n,$$

in $A_S \otimes_S Ss^n$, and so A_S is GP-flat as required. □

The converse of Proposition 2.4 is not true, see the following example. Recall from [11] that a right ideal K of a monoid S is called *left stabilizing*, if for every $k \in K$ there exists $l \in K$, such that $lk = k$.

Example 2.5. Let $S = \{1, e, f, 0\}$ be a semilattice, where $ef = 0$. Consider the right ideal $K = eS = \{e, 0\}$ of S. Since K is left stabilizing and S is right reversible, S/K is flat by [8, III, Theorem 12.17], and so it is GP-flat. But it is easy to see that S-act S/K does not satisfy Condition (GP').

Now, for a commutative monoid we can show Condition (GP') can be transferred from products of acts to their components.

Proposition 2.6. *For a commutative monoid S, if* $\prod_{i \in I} A_i$, *where* A_i, $i \in I$ *are right S-acts, satisfies Condition* (GP'), *then* A_i *satisfies Condition* (GP'), *for every* $i \in I$.

Proof. This is clear. $\square$

The right *S*-act $S \times S$ equipped with the right *S*-action $(s, t)u = (su, tu)$, for $s, t, u \in S$ is called the *right diagonal act* of *S* and will be denoted by $D(S)$. Left diagonal act $_S(S \times S)$ is defined dually. By a similar argument as in the proof of [13, Theorems 3.7 and 3.8], we can show the following theorems, respectively.

Theorem 2.7. *The following statements are equivalent*:

(1) $D(S)$ *satisfies Condition* (GP') *and* $|E(S)| \leq 2$;

(2) *S is right cancellative*.

Theorem 2.8. *For an idempotent monoid S, the following statements are equivalent*:

(1) $D(S)$ *satisfies Condition* (GP');

(2) $S = \{1\}$.

Corollary 2.9. *For an idempotent monoid S, the following statements are equivalent*:

(1) $D(S)$ *satisfies Condition* (P');

(2) $D(S)$ *satisfies Condition* (GP');

(3) $S = \{1\}$.

Here we give a characterization of monoids for which property (GP') transfer from products of acts to their components. First we recall from [4] that a submonoid *P* of *S* is called *weakly right reversible* if $(\forall s, s' \in P)$ $(\forall z \in S)(sz = s'z \Rightarrow (\exists u, v \in P)(us = vs'))$.

Theorem 2.10. *The following statements are equivalent*:

(1) *If* $\prod_{i \in I} A_i$ *satisfies Condition* (GP'), *then* A_i *satisfies Condition* (GP'), *for every* $i \in I$;

(2) Θ_S *satisfies Condition* (GP');

(3) *S is weakly right reversible*.

Proof. This is similar to the proof of [13, Theorem 3.10]. □

We recall from [6] that for $a, b \in S,$ the set $L(a, b) := \{(s, t) \in S \times S; sa = tb\},$ is either empty or else a left *S*-act.

Also, we recall that if *S* is a monoid and *I* is a non-empty set, then the set of all maps from *I* to *S*, equipped with the right *S*-action $(\alpha s) x = (\alpha(x)) s,$ for mapping $\alpha : I \to S,$ $s \in S$ and $x \in I,$ is a right *S*-act, denoted by $(S^I)_S$.

Theorem 2.11. *The following statements are equivalent*:

(1) S_S^I *satisfies Condition* $(GP'),$ *for every non-empty set I*;

(2) *for any* $a, b \in S,$ $L(a, b)$ *is either empty or else*, *if there exists* $z \in S$ *such that* $(a, b) \in \ker \rho_z,$ *then there exist* $u, v \in S$ *and* $n \in \mathbb{N}$ *such that* $L(a, b) \subseteq S(u, v) \subseteq L(a^n, b^n)$.

Proof. (1) $\Rightarrow$ (2). Suppose $L(a, b) \neq \emptyset,$ for $a, b \in S$ and suppose there exists $z \in S$ such that $(a, b) \in \ker \rho_z$. Suppose also that $L(a, b)$ is indexed by *I*, that is, $L(a, b) := \{(x_i, y_i) \mid i \in I\}$. Let $\underline{x}, \underline{y}$ be the elements of S^I, whose *i*-th components are $x_i, y_i,$ respectively. Then $\underline{x}a = \underline{y}b$ and $az = bz$. By assumption there exist $\underline{z} \in S^I, u, v \in S$ and $n \in \mathbb{N}$ such that $\underline{x} = \underline{z}u, \underline{y} = \underline{z}v$ and $ua^n = vb^n$. Thus, $(u, v) \in L(a^n, b^n), (x_i, y_i) = z_i(u, v),$ where z_i is the *i*-th component of $\underline{z}$. Therefore, $L(a, b) \subseteq S(u, v) \subseteq L(a^n, b^n),$ as required.

(2) $\Rightarrow$ (1). Let $(x_i)_I a = (y_i)_I b$ and $az = bz,$ for $(x_i)_I, (y_i)_I \in (S^I)_S$ and $a, b, z \in S$. Then $(x_i, y_i) \in L(a, b),$ for every $i \in I$. Thus, by assumption there exist $u, v \in S$ and $n \in \mathbb{N}$ such that $L(a, b) \subseteq S(u, v) \subseteq L(a^n, b^n)$. Since $(x_i, y_i) \in L(a, b),$ for every $i \in I,$ thus for every $i \in I,$

there exists $w_i \in S$ such that $(x_i, y_i) = w_i(u, v)$, that is $x_i = w_i u$ and $y_i = w_i v$, for every $i \in I$. Hence, $(x_i)_I = (w_i)_I u$, $(y_i)_I = (w_i)_I v$ and since $(u, v) \in S(u, v) \subseteq L(a^n, b^n)$, we have $ua^n = vb^n$. Thus, the right S-act $(S^I)_S$ satisfies Condition (GP'), as required. □

Theorem 2.12. *The following statements are equivalent*:

(1) $D(S)$ *satisfies Condition* (GP');

(2) *for any* $a, b \in S$, $L(a, b)$ *is either empty or else, if there exists* $z \in S$ *such that* $(a, b) \in \ker \rho_z$, *then for every* $(x_1, y_1), (x_2, y_2) \in L(a, b)$ *there exist* $u, v \in S$ *and* $n \in \mathbb{N}$ *such that* $(x_i, y_i) \in S(u, v) \subseteq L(a^n, b^n)$, *for* $i = 1, 2$.

Proof. (1) $\Rightarrow$ (2). For $a, b \in S$, let $L(a, b) \neq \varnothing$ and $(a, b) \in \ker \rho_z$, for $z \in S$. Now let $(x_1, y_1), (x_2, y_2) \in L(a, b)$, for $x_1, y_1, x_2, y_2 \in S$. Then we have $x_1 a = y_1 b$ and $x_2 a = y_2 b$, and so $(x_1, x_2)a = (y_1, y_2)b$, which with $(a, b) \in \ker \rho_z$ imply that there exist $w_1, w_2, u, v \in S$ and $n \in \mathbb{N}$ such that $(x_1, x_2) = (w_1, w_2)u, (y_1, y_2) = (w_1, w_2)v$ and $ua^n = vb^n$, that is $x_1 = w_1 u, y_1 = w_1 v, x_2 = w_2 u$ and $y_2 = w_2 v$. Therefore,

$$(x_i, y_i) = w_i(u, v) \in S(u, v) \subseteq L(a^n, b^n),$$

for $i = 1, 2$, as required.

(2) $\Rightarrow$ (1). Let $(x_1, x_2)a = (y_1, y_2)b$ and $az = bz$, for (x_1, x_2), $(y_1, y_2) \in D(S)$ and $a, b, z \in S$. Then $x_1 a = y_1 b, x_2 a = y_2 b$, and so $(x_1, y_1), (x_2, y_2) \in L(a, b)$ and $(a, b) \in \ker \rho_z$. Thus, by assumption there exist $u, v \in S$ and $n \in \mathbb{N}$, such that $(x_i, y_i) \in S(u, v) \subseteq L(a^n, b^n)$, for $i = 1, 2$, and so, there exist $w_1, w_2 \in S$ such that $(x_i, y_i) = w_i(u, v)$, for $i = 1, 2$, that is, $x_i = w_i u$ and $y_i = w_i v$, for $i = 1, 2$. Hence, $(x_1, x_2) = (w_1, w_2)u$ and $(y_1, y_2) = (w_1, w_2)v$, and since $(u, v) \in S(u, v) \subseteq$

$L(a^n, b^n)$, we have $ua^n = vb^n$. Therefore, $D(S)$ satisfies Condition (GP'). □

3. A Characterization of Right Acts by the Condition (GP')

In this section, we give a characterization of monoid S by Condition (GP') of right S-acts.

Let J be a proper right ideal of a monoid S. If x, y, z are different symbols not belonging to S, define $A(J) := (\{x, y\} \times (S \setminus J)) \cup (\{z\} \times J)$, and a right S-action on $A(J)$ by

$$(x, u)s = \begin{cases} (x, us), & \text{if } us \notin J, \\ (z, us), & \text{if } us \in J, \end{cases}$$

$$(y, u)s = \begin{cases} (y, us), & \text{if } us \notin J, \\ (z, us), & \text{if } us \in J, \end{cases}$$

$$(z, u)s = (z, us).$$

Clearly, $A(J)$ is a right S-act.

Lemma 3.1. *Let J be a proper right ideal of S. Then $A(J)$ does not satisfy Condition (GP').*

Proof. For every $j \in J$, we have $(x, 1)j = (y, 1)j$ and $jj = jj$. If there exist $(w, r) \in A(J)$, $u, v \in S$ and $n \in \mathbb{N}$ such that $(x, 1) = (w, r)u$, $(y, 1) = (w, r)v$ and $uj^n = vj^n$, then the equality $(x, 1) = (w, r)u$ implies that $w = x$, and the equality $(y, 1) = (w, r)v$ implies that $w = y$. Thus, $x = y$, which is a contradiction. □

Recall from [2] and [3] that the S-act A_S satisfies Condition $(E'P)$, if for every $a \in A_S$, $s, s', z \in S$,

$$as = as', sz = s'z \Rightarrow (\exists a' \in A_S)(\exists u, u' \in S)(a = a'u = a'u' \text{ and } us = u's').$$

The S-act A_S satisfies Condition (EP), if for every $a \in A_S$, $s, s' \in S$,

$$as = as' \Rightarrow (\exists a' \in A_S)(\exists u, u' \in S)(a = a'u = a'u' \text{ and } us = u's').$$

Theorem 3.2. *The following statements are equivalent*:

(1) *All right S-acts satisfy Condition* (GP');

(2) *all* $\mathfrak{R}$*-torsion free right S-acts satisfy Condition* (GP');

(3) *all right S-acts satisfying Condition* $(E'P)$ *satisfy Condition* (GP');

(4) *all right S-acts satisfying Condition* (EP) *satisfy Condition* (GP');

(5) *all right S-acts satisfying Condition* (E') *satisfy Condition* (GP');

(6) *all right S-acts satisfying Condition* (E) *satisfy Condition* (GP');

(7) *S is a group*.

Proof. Implications $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$ are obvious and implications $(3) \Rightarrow (4) \Rightarrow (6)$, $(3) \Rightarrow (5) \Rightarrow (6)$ are obvious, because $(E) \Rightarrow (EP) \Rightarrow (E'P)$, and $(E) \Rightarrow (E') \Rightarrow (E'P)$.

$(2) \Rightarrow (6)$. This is obvious, since all right S-acts satisfying Condition (E) are $\mathfrak{R}$-torsion free, by [17, Proposition 1.2].

$(6) \Rightarrow (7)$. Suppose $sS \neq S$, for $s \in S$. Then the right S-act $A(sS)$ satisfies Condition (E), by [8, III, Exercise 14.3(3)], and so by the assumption that $A(sS)$ satisfies Condition (GP'), which is a contradiction, by Lemma 3.1. Hence, $sS = S$, for every $s \in S$, and so S is a group.

$(7) \Rightarrow (1)$. This is obvious, by [4, Theorem 2.5], since Condition (P') implies Condition (GP'). □

Theorem 3.3. *The following statements are equivalent*:

(1) *All right S-acts satisfy Condition* (GP');

(2) *all generator right S-acts satisfy Condition* (GP');

(3) *all generator finitely generated right S-acts satisfy Condition* (GP');

(4) *all generator right S-acts generated by at most three elements satisfy Condition* (GP');

(5) $S \times A_S$ *satisfies Condition* (GP'), *for every generator right S-act* A_S;

(6) $S \times A_S$ *satisfies Condition* (GP'), *for every generator finitely generated right S-act* A_S;

(7) $S \times A_S$ *satisfies Condition* (GP'), *for every generator right S-act* A_S *generated by at most three elements*;

(8) $S \times A_S$ *satisfies Condition* (GP'), *for every right S-act* A_S;

(9) $S \times A_S$ *satisfies Condition* (GP'), *for every finitely generated right S-act* A_S;

(10) $S \times A_S$ *satisfies Condition* (GP'), *for every right S-act* A_S *generated by at most two elements*;

(11) *the right S-act* A_S *satisfies Condition* (GP'), *if* $Hom(A_S, S_S) \neq \emptyset$;

(12) *the finitely generated right S-act* A_S *satisfies Condition* (GP'), *if* $Hom(A_S, S_S) \neq \emptyset$;

(13) *the right S-act* A_S, *generated by at most two elements*, *satisfies Condition* (GP'), *if* $Hom(A_S, S_S) \neq \emptyset$;

(14) *S is a group*.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), (1) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7), (1) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10) and (1) $\Rightarrow$ (11) $\Rightarrow$ (12) $\Rightarrow$ (13) are obvious.

(7) $\Rightarrow$ (2). Let A_S be a generator right S-act, $as = a't$ and $sz = tz$, for $a, a' \in A_S$, $s, t, z \in S$. Since A_S is generator, thus there exists epimorphism $\pi : A_S \to S_S$. Let $\pi(a'') = 1$ and $B_S = aS \cup a'S \cup a''S$. Since $\pi|_{B_S} : B_S \to S_S$

is an epimorphism, thus B_S is a generator right S-act, which generated by at most three elements, and so by the assumption that $S \times B_S$ satisfies Condition (GP'). Now $(\pi(a), a)s = (\pi(a'), a')t$ and $sz = tz$ imply that there exist $(w, b) \in S \times B_S, u, v \in S$ and $n \in \mathbb{N}$ such that $(\pi(a), a) = (w, b)u, (\pi(a'), a') = (w, b)v$ and $us^n = vt^n$. Therefore, $a = bu,\ a' = bv$ and $us^n = vt^n$, and so A_S satisfies Condition (GP'), as required.

(13) $\Rightarrow$ (2). Let A_S be a generator right S-act, $as = a't$ and $sz = tz$, for $a, a' \in A_S,\ s, t, z \in S$. Since A_S is generator, thus there exists epimorphism $\pi : A_S \to S_S$. Let $B_S = aS \cup a'S$. Since $\pi|_{B_S} : B_S \to S_S$ is a homomorphism, thus $Hom(B_S, S_S) \neq \varnothing$. Also B_S is generated by at most two elements and so by the assumption that B_S satisfies Condition (GP'). Now $as = a't$ and $sz = tz$ imply that there exist $a'' \in B_S \subseteq A_S,\ u, v \in S$ and $n \in \mathbb{N}$ such that $a = a''u,\ a' = a''v$ and $us^n = vt^n$. Therefore, A_S satisfies Condition (GP'), as required.

(10) $\Rightarrow$ (14). Suppose $sS \neq S$, for $s \in S$. Then the right S-act $A(sS)$ is generated by two different elements $(1, x)$ and $(1, y)$, and so by the assumption that $S \times A(sS)$ satisfies Condition (GP'). Since $(1, (1, x))s = (1, (1, y))s$ and $s.1 = s.1$, thus there exist $u, v, t, l \in S,\ w \in \{x, y\}$ and $n \in \mathbb{N}$ such that $(1, (1, x)) = (l, (t, w))u, (1, (1, y)) = (l, (t, w))v$ and $us^n = vs^n$. Therefore, $(1, x) = (t, w)u$ and $(1, y) = (t, w)v$, and so $x = w = y$, which is a contradiction. Hence, $sS = S$, for every $s \in S$, and so S is a group.

(2) $\Rightarrow$ (10). Let A_S be a right S-act, generated by at most two elements. It is obvious that the mapping $\varphi : S \times A_S \to S_S$ with $\varphi(s, a) = s$, for all $s \in S$ and $a \in A_S$ is an epimorphism in **Act**-S, thus $S \times A_S$ is a generator, by [8, II, Theorem 3.16], and so by the assumption that $S \times A_S$ satisfies Condition (GP').

(4) $\Rightarrow$ (2). Let A_S be a generator right S-act, $as = a't$ and $sz = tz$, for $a, a' \in A_S$, $s, t, z \in S$. Since A_S is generator, thus there exists epimorphism $\pi : A_S \to S_S$. Let $\pi(a'') = 1$ and $B_S = aS \cup a'S \cup a''S$. Since $\pi|_{B_S} : B_S \to S_S$ is an epimorphism, thus B_S is a generator right S-act, which generated by at most three elements, and so by the assumption that B_S satisfies Condition (GP'). Now $as = a't$ and $sz = tz$ imply that there exist $a'' \in B_S \subseteq A_S, u, v \in S$ and $n \in \mathbb{N}$ such that $a = a''u$, $a' = a''v$ and $us^n = vt^n$. Therefore, A_S satisfies Condition (GP'), as required.

(14) $\Rightarrow$ (1). This follows from Theorem 3.2. □

We recall from [8, I, Definition 5.7], that an S-act A_S is called *indecomposable* if there exist no subacts $B_S, C_S \subseteq A_S$ such that $A_S = B_S \cup C_S$ and $B_S \cap C_S = \varnothing$.

Theorem 3.4. *The following statements are equivalent*:

(1) *All indecomposable right S-acts satisfy Condition* (GP');

(2) *all finitely generated indecomposable right S-acts satisfy Condition* (GP');

(3) *all indecomposable right S-acts generated by two elements satisfy Condition* (GP');

(4) *S is a group*.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Suppose that $sS \neq S$, for $s \in S$, and let $A_S = S \coprod^{sS} S$. As we know A_S is an indecomposable right S-act generated by two elements, and so by the assumption that A_S satisfies Condition (GP'), which is a contradiction, by Lemma 3.1. Hence, $sS = S$, for every $s \in S$, and so S is a group.

(4) $\Rightarrow$ (1). This is obvious, by Theorem 3.2. $\square$

For every weakly right reversible monoid S, there exists a right S-act A_S, such that A_S is not indecomposable, but satisfies Condition (GP'). For this let S be a weakly right reversible monoid and $A = \{\theta_1, \theta_2\}$, which $\theta_1 \neq \theta_2$, $\theta_i s = \theta_i$, for every $s \in S$, $i = 1, 2$.

Obviously, A_S is a right S-act that satisfies Condition (GP'), but since $A_S = \{\theta_1\} \dot{\cup} \{\theta_2\}$, $\{\theta_1\}$ and $\{\theta_2\}$ are subacts of A_S, thus A_S is not indecomposable.

We recall from [8] that A_S is (strongly) faithful, if for $s, t \in S$ the equality $as = at$, for (some) all $a \in A_S$ implies that $s = t$. Now by using a similar argument as in the proof of Theorem 3.4, we can show the following theorem.

Theorem 3.5. *The following statements are equivalent*:

(1) *All faithful right S-acts satisfy Condition* (GP');

(2) *all finitely generated faithful right S-acts satisfy Condition* (GP');

(3) *all faithful right S-acts generated by two elements satisfy Condition* (GP');

(4) *S is a group*.

Notation: C_l (C_r) is the set of all left (right) cancellable elements of S.

Lemma 3.6. *The following statements are equivalent*:

(1) *There exists at least one strongly faithful* (*left*) *right S-act*;

(2) *there exists at least one strongly faithful cyclic* (*left*) *right S-act*;

(3) *there exists at least one strongly faithful monocyclic* (*left*) *right S-act*;

(4) *there exists at least one strongly faithful finitely generated* (*left*) *right S-act*;

(5) $(Ss)\ sS$, *as a* (*left*) *right* S-*act*, *is strongly faithful*, *for every* $s \in S$;

(6) *there exists* $s \in S$, *such that* $(Ss)\ sS$, *as a* (*left*) *right* S-*act*, *is strongly faithful*;

(7) $({}_S S)\ S_S$, *as a* (*left*) *right* S-*act*, *is strongly faithful*;

(8) $(Ss \subseteq C_r)\ sS \subseteq C_l$, *for every* $s \in S$;

(9) *there exists* $s \in S$, *such that* $(Ss \subseteq C_r)\ sS \subseteq C_l$;

(10) S *is* (*right*) *left cancellative*.

Proof. Implications (3) ⇒ (2) ⇒ (4) ⇒ (1), (5) ⇒ (6) ⇒ (1), (10) ⇒ (8) ⇒ (9) and (7) ⇒ (1) are obvious.

(7) ⇒ (3). This is obvious, since $S_S \cong S_S/\Delta_S \cong S/\rho(s, s)$, for every s.

(1) ⇒ (10). Let A_S be a strongly faithful right S-act and $sl = st$, for $s, l, t \in S$. Then $asl = ast$, for $a \in A_S$. Since A_S is strongly faithful and $as \in A_S$, thus $l = t$. Therefore, S is left cancellative.

(10) ⇒ (7). Let S be left cancellative and $sl = st$, for $l, t \in S$ and $s \in S_S$. By assumption $l = t$, and so S_S is strongly faithful, as a right S-act.

(9) ⇒ (10). Let $sS \subseteq C_l$, for $s \in S$ and $rt = rl$, for $r, t, l \in S$. Then $(sr)t = (sr)l$. By assumption $t = l$, and so S is left cancellative.

(10) ⇒ (5). Let $skt = skl$, for $sk \in sS$ and $t, l \in S$. By assumption $t = l$, and so sS is strongly faithful, as a right S-act. □

Theorem 3.7. *The following statements are equivalent*:

(1) *All strongly faithful right* S-*acts satisfy Condition* (GP');

(2) *all finitely generated strongly faithful right* S-*acts satisfy Condition* (GP');

(3) *all strongly faithful right S-acts generated by two elements satisfy Condition* (GP');

(4) *either S is not left cancellative or, S is a group*.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). If S is not left cancellative, then we are done. Now let S be left cancellative, and let there exist $s \in S$ such that $sS \neq S$. Put $A_S = A(sS)$. We have:

$$B_S = \{(l, x) | l \in S \backslash sS\} \dot{\cup} sS \cong S_S \cong \{(t, y) | t \in S \backslash sS\} \dot{\cup} sS = C_S$$

and $A_S = B_S \cup C_S$. Since S is left cancellative, S_S is strongly faithful, by Lemma 3.6, and so B_S and C_S are strongly faithful. That is, A_S is strongly faithful, and so by the assumption that A_S satisfies Condition (GP'), which is a contradiction, by Lemma 3.1. So, $sS = S$, for all $s \in S$, and S is a group.

(4) $\Rightarrow$ (1). Let S be not left cancellative. Then there exists no strongly faithful right S-act, by Lemma 3.6, and so (1) is satisfied. Otherwise, S is a group, and all right S-acts satisfy Condition (GP'), by Theorem 3.2, as required. □

Theorem 3.8. *Every strongly faithful cyclic right S-act satisfies Condition* (GP').

Proof. Let $A_S = \alpha S$ be a strongly faithful cyclic right S-act. Then S is left cancellative, by Lemma 3.6, and so $\ker \lambda_\alpha = \Delta_S$. That is, $A_S = \alpha S = S/\ker \lambda_\alpha = S/\Delta_S \cong S_S$. So $A_S = \alpha S$ satisfies Condition (GP'), by Theorem 2.2. □

We say that S is right *PCP*, if all principal right ideals of S satisfy Condition (P). The S-act A_S is called *strongly* (P)-*cyclic*, if for every $a \in A_S$ there exists $z \in S$ such that $\ker \lambda_a = \ker \lambda_z$ and zS satisfies Condition (P) (see [5]). The proof of the following theorem is similar to that of [5, Theorem 2.6].

Theorem 3.9. *The following statements are equivalent*:

(1) *All strongly* (P)-*cyclic right S-acts satisfy Condition* (GP');

(2) *all finitely generated strongly* (P)-*cyclic right S-acts satisfy Condition* (GP');

(3) *all finitely generated strongly* (P)-*cyclic right S-acts generated by two elements satisfy Condition* (GP');

(4) *eS is a minimal right ideal of S, for every idempotent* $e \in T$, *where T is the greatest strongly* (P)-*cyclic right ideal of S.*

Corollary 3.10. *All cyclic strongly* (P)-*cyclic right S-acts satisfy Condition* (GP').

Proof. This is obvious, by definition. □

Lemma 3.11 [5, Lemma 2.3]**.** *If all cyclic subacts of a right S-act* A_S *are simple, then for every* $a, a' \in A_S$, *either* $aS \cap a'S = \emptyset$ *or* $aS = a'S$.

Theorem 3.12. *The following statements are equivalent*:

(1) *All regular right S-acts satisfy Condition* (GP');

(2) *all finitely generated regular right S-acts satisfy Condition* (GP');

(3) *all regular right S-acts generated by two elements satisfy Condition* (GP');

(4) *eS is a minimal right ideal of S, for any idempotent* $e \in T$, *where T is the greatest regular right ideal of S.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Let $e \in T$ be an idempotent. If *eS* is a minimal right ideal, then we are done. Otherwise, there exists a right ideal K_S of *S* such that $K_S \subset eS$. Let $A_S = eS \coprod^{K} eS$. Then A_S is regular by [8, III, Proposition 19.11], and so by the assumption that A_S satisfies Condition (GP'). Now let $eu \in K_S$. Then $eu = (e, x)u = (e, y)u$ and $uu = uu$, imply that there exist

$a \in A_S$, $v_1, v_2 \in S$ and $n \in \mathbb{N}$ such that $(e, x) = av_1$, $(e, y) = av_2$ and $v_1u^n = v_2u^n$. Then equality $(e, x) = av_1$ implies that $a = (t, x)$, for some $t \in eS \backslash K_S$, similarly, $a = (t', y)$, for some $t' \in eS \backslash K_S$, that is $x = y$, which is a contradiction.

(4) $\Rightarrow$ (1). Let A_S be a regular right S-act, and let $a \in A_S$. Then aS is projective, and so there exists $e \in E(S)$ such that $aS \cong eS$. Since T is the greatest regular right ideal of S, we have $eS \subseteq T$ and so $e \in T$. By the assumption that eS is a minimal right ideal of S, and so aS is too. Therefore, aS is simple. Now, suppose $at = a't'$ and $tz = t'z$, for $a, a' \in A_S$, $t, t', z \in S$. Then $aS \bigcap a'S \neq \emptyset$, and so $aS = a'S$, by Lemma 3.11. Thus, $a' = as_1$, for some $s_1 \in S$, and so $at = as_1t'$. Since eS satisfies Condition (GP') and $aS \cong eS$, thus there exist $u, v \in S$ and $n \in \mathbb{N}$ such that $a = au, as_1 = a' = av$ and $ut^n = vt'^n$, that is, A_S satisfies Condition (GP'), as required. □

Theorem 3.13. *Every regular cyclic right S-act satisfies Condition (GP').*

Proof. This follows from definition. □

As we saw in Section 2, weak flatness of acts does not imply Condition (GP'). Now it is natural to ask for monoids S over which weak flatness implies Condition (GP').

An element s of S is called *left e-cancellable*, for an idempotent $e \in S$, if $s = se$ and $\ker \lambda_s \leq \ker \lambda_e$. The monoid S is called *right PP* if every principal right ideal of S is projective, as a right S-act. This is equivalent to saying that every element $s \in S$ is left e-cancellable for some idempotent $e \in S$ (see [1]). The monoid S is called *left PSF* if every principal left ideal of S is strongly flat, as a left S-act. This is equivalent to saying that S is right semi-cancellative, that is, whenever $su = s'u$, for $s, s', u \in S$, there exists $r \in S$ such that $u = ru$ and $sr = s'r$ (see [12]).

Theorem 3.14. *The following statements are equivalent*:

(1) *S is right cancellative*;

(2) *S is left PSF and all torsion free right S-acts satisfy Condition* (GP');

(3) *S is left PSF and all principally weakly flat right S-acts satisfy Condition* (GP');

(4) *S is left PSF and all weakly flat right S-acts satisfy Condition* (GP');

(5) *S is left PSF and all flat right S-acts satisfy Condition* (GP').

Proof. (1) $\Rightarrow$ (2). It is obvious that every right cancellative monoid is left PSF. Suppose A_S is a torsion free right *S*-act, $at = a't'$ and $tz = t'z$, for $a, a' \in A_S$ and $t, t', z \in S$. Then by assumption $t = t'$, and so $at = a't$. Now since A_S is torsion free, thus $a = a'$. Therefore, A_S satisfies Condition (GP'), as required.

Implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.

(4) $\Rightarrow$ (1). If *S* is not right cancellative, then $I = \{s \in S \mid s$ is not right cancellable$\}$ is a proper right ideal of *S*, and so $A(I)$ is flat, by [15, Lemma 2.11]. Thus, by the assumption that $A(I)$ satisfies Condition (GP'), which is a contradiction, by Lemma 3.1. □

Recall from [10] that an element *s* of *S* is called *left almost regular* if there exist elements $r, r_1, ..., r_m, s_1, s_2, ..., s_m \in S$ and right cancellable elements $c_1, c_2, ..., c_m \in S$, such that

$$\begin{aligned} s_1c_1 &= sr_1 \\ s_2c_2 &= s_1r_2 \\ &\cdots \\ s_mc_m &= s_{m-1}r_m \\ s &= s_mrs. \end{aligned}$$

A monoid S is called *left almost regular* if all its elements are left almost regular.

Notice that the above theorem is also valid when left *PSF* is substituted by left *PP* or left almost regular.

By a similar argument as in the proof of [8, IV, Proposition 9.3], we can show the following theorem.

Theorem 3.15. *If all finitely generated flat right S-acts satisfy Condition* (GP'), *then* $E(S) = \{1\}$.

By a similar argument as in the proof of [14, Theorem 2.9], and Theorems 3.14 and 3.15, we can show the following theorem.

Theorem 3.16. *The following statements are equivalent*:

(1) *S is right cancellative*;

(2) *there exists a regular left S-act and* $|E(S)| = 1$;

(3) *there exists a regular left S-act and all principally weakly flat right S-acts satisfy Condition* (GP');

(4) *there exists a regular left S-act and all weakly flat right S-acts satisfy Condition* (GP');

(5) *there exists a regular left S-act and all flat right S-acts satisfy Condition* (GP').

Theorem 3.17. *The following statements are equivalent*:

(1) *S is right cancellative*;

(2) *S is left PSF and all GP-flat right S-acts satisfy Condition* (GP');

(3) *there exists a regular left S-act and all GP-flat right S-acts satisfy Condition* (GP');

(4) *left diagonal act* ${}_S(S \times S)$ *is regular and all GP-flat right S-acts satisfy Condition* (GP').

Proof. (1) $\Rightarrow$ (4). Left diagonal act ${}_S(S \times S)$ is strongly faithful, by duality of [13, Corollary 4.8], and so ${}_S(S \times S)$ is regular, by [8, III, Proposition 19.13]. By using a similar argument as in the proof of (1) $\Rightarrow$ (2) of Theorem 3.14, we have all *GP*-flat right *S*-acts satisfy Condition (GP').

(4) $\Rightarrow$ (3). This is clear.

(3) $\Rightarrow$ (1). Since every principally right *S*-act is *GP*-flat, thus by the assumption that every principally weakly flat right *S*-act satisfies Condition (GP'), and so *S* is right cancellative, by Theorem 3.16.

(1) $\Rightarrow$ (2). This is similar to the proof of (1) $\Rightarrow$ (2) of Theorem 3.14.

(2) $\Rightarrow$ (1). This is similar to the proof of (4) $\Rightarrow$ (1) of Theorem 3.14. □

Notice that the above theorem is also valid when left *PSF* is substituted by left *PP* or left almost regular.

For fixed elements $u, v \in S$ define the relation $P_{u,v}$ on *S* as

$$(x, y) \in P_{u,v} \Leftrightarrow ux = vy \ (x, y \in S).$$

For $s, t \in S$, let $\mu_{s,t} = \ker \lambda_s \vee \ker \lambda_t$, and for any right ideal *I* of *S* let ρ_I denote the right Rees congruence, i.e., for $x, y \in S$,

$$(x, y) \in \rho_I \Leftrightarrow x = y \text{ or } x, y \in I.$$

The what follows ρ_{sS} is the right Rees congruence on *S* by the principal right ideal *sS*.

Lemma 3.18. *For any* $s, t, u, v \in S, n \in \mathbb{N}$, *the following statements are equivalent*:

(1) $us^n = vt^n$, $P_{u,v} \subseteq P_{1,s} \circ \mu_{s,t} \circ P_{t,1}$;

(2) $P_{1,s^n} \circ \mu_{s,t} \circ P_{t^n,1} \subseteq P_{u,v} \subseteq P_{1,s} \circ \mu_{s,t} \circ P_{t,1}$;

(3) $P_{1,s^n} \circ \mu_{s^n,t^n} \circ P_{t^n,1} \subseteq P_{u,v} \subseteq P_{1,s} \circ \mu_{s,t} \circ P_{t,1}$.

Proof. (1) $\Rightarrow$ (2). Let $p_1, p_2 \in S$, and $(p_1, p_2) \in P_{1, s^n} \circ \mu_{s,t} \circ P_{t^n, 1}$. Thus, there exist $y_1, y_2 \in S$ such that $(p_1, y_1) \in P_{1, s^n}, (y_1, y_2) \in \mu_{s,t}$, $(y_2, p_2) \in P_{t^n, 1}$. So $p_1 = s^n y_1, p_2 = t^n y_2, (y_1, y_2) \in \mu_{s,t} = \ker \lambda_s \vee \ker \lambda_t$. Then by [8, I, Lemma 4.37], there exist $z_1, ..., z_m \in S$ such that

$$\begin{array}{ccccc} sy_1 = sz_1 & sz_2 = sz_3 & \cdots & sz_{m-1} = sz_m & \\ & tz_1 = tz_2 & \cdots & & tz_m = ty_2. \end{array}$$

Consequently,

$$up_1 = us^n y_1 = us^n z_1 = vt^n z_1 = \cdots = vt^n z_m = vt^n y_2 = vp_2,$$

and so $(p_1, p_2) \in P_{u,v}$, as required.

(2) $\Rightarrow$ (3). Let $p_1, p_2 \in S$, and $(p_1, p_2) \in P_{1, s^n} \circ \mu_{s^n, t^n} \circ P_{t^n, 1}$. Thus, there exist $y_1, y_2 \in S$ such that $(p_1, y_1) \in P_{1, s^n}$, $(y_1, y_2) \in \mu_{s^n, t^n}$, $(y_2, p_2) \in P_{t^n, 1}$. So $p_1 = s^n y_1, p_2 = t^n y_2, (y_1, y_2) \in \mu_{s^n, t^n} = \ker \lambda_{s^n} \vee \ker \lambda_{t^n}$. Then there exist $z_1, ..., z_m \in S$ such that

$$\begin{array}{ccccc} s^n y_1 = s^n z_1 & s^n z_2 = s^n z_3 & \cdots & s^n z_{m-1} = s^n z_m & \\ & t^n z_1 = t^n z_2 & \cdots & & t^n z_m = t^n y_2. \end{array}$$

Obviously, we have $(s^n, 1) \in P_{1, s^n}, (1, 1) \in \mu_{s,t}, (1, t^n) \in P_{t^n, 1}$, and so $(s^n, t^n) \in P_{1, s^n} \circ \mu_{s,t} \circ P_{t^n, 1} \subseteq P_{u,v}$, which implies $us^n = vt^n$. Consequently,

$$up_1 = us^n y_1 = us^n z_1 = vt^n z_1 = \cdots = vt^n z_m = vt^n y_2 = vp_2,$$

and so $(p_1, p_2) \in P_{u,v}$, as required.

(3) $\Rightarrow$ (1). Obviously, we have $(s^n, 1) \in P_{1, s^n}$, $(1, 1) \in \mu_{s^n, t^n}$, $(1, t^n) \in P_{t^n, 1}$, and so $(s^n, t^n) \in P_{1, s^n} \circ \mu_{s^n, t^n} \circ P_{t^n, 1} \subseteq P_{u, v}$, which implies $us^n = vt^n$. □

Theorem 3.19. *The following statements are equivalent*:

(1) *All fg-weakly injective right S-acts satisfy Condition* (GP');

(2) *all weakly injective right S-acts satisfy Condition* (GP');

(3) *all injective right S-acts satisfy Condition* (GP');

(4) *all cofree right S-acts satisfy Condition* (GP');

(5) *for any* $s, t, z \in S$, $sz = tz$ *implies that there exist* $u, v \in S$ *and* $n \in \mathbb{N}$ *such that* $(s^n, t^n) \in P_{u, v}$ *and the following conditions hold*:

(i) $P_{u, v} \subseteq P_{1, s} \circ \mu_{s, t} \circ P_{t, 1}$;

(ii) $\ker \lambda_u \subseteq \rho_{sS}$;

(iii) $\ker \lambda_v \subseteq \rho_{tS}$;

(iv) $\ker \lambda_{us} \bigcup \ker \lambda_{vt} \subseteq \mu_{s, t}$;

(6) *for any* $s, t, z \in S$, $sz = tz$ *implies that there exist* $u, v \in S$ *and* $n \in \mathbb{N}$ *such that the following conditions hold*:

(i) $P_{1, s^n} \circ \mu_{s, t} \circ P_{t^n, 1} \subseteq P_{u, v} \subseteq P_{1, s} \circ \mu_{s, t} \circ P_{t, 1}$;

(ii) $\ker \lambda_u \subseteq \rho_{sS}$;

(iii) $\ker \lambda_v \subseteq \rho_{tS}$;

(iv) $\ker \lambda_{us} \bigcup \ker \lambda_{vt} \subseteq \mu_{s, t}$;

(7) *for any* $s, t, z \in S$, $sz = tz$ *implies that there exist* $u, v \in S$ *and* $n \in \mathbb{N}$ *such that the following conditions hold*:

(i) $P_{1, s^n} \circ \mu_{s^n, t^n} \circ P_{t^n, 1} \subseteq P_{u, v} \subseteq P_{1, s} \circ \mu_{s, t} \circ P_{t, 1}$;

(ii) $\ker \lambda_u \subseteq \rho_{sS}$;

(iii) $\ker \lambda_v \subseteq \rho_{tS}$;

(iv) $\ker \lambda_{us} \cup \ker \lambda_{vt} \subseteq \mu_{s, t}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious, and implications (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) are follows, from Lemma 3.18.

(4) $\Rightarrow$ (5). Let there exist $s, t, z \in S$ such that $sz = tz$, and for any $u, v \in S$ and $n \in \mathbb{N}$ either $us^n \neq vt^n$ or one of the Conditions (i)-(iv) does not hold. Let S_1 and S_2 be sets such that $|S_1| = |S_2| = |S|$, and let $\alpha : S \to S_1$ and $\beta : S \to S_2$ be bijections. Let $X = (S/\mu_{s,t}) \coprod S_1 \coprod S_2$, and define mappings $f, g : S \to X$ as follows:

$$f(x) = \begin{cases} [y]_{\mu_{s,t}}, & \exists y \in S;\ x = sy, \\ \alpha(x), & x \in S \backslash sS, \end{cases}$$

$$g(x) = \begin{cases} [y]_{\mu_{s,t}}, & \exists y \in S;\ x = ty, \\ \beta(x), & x \in S \backslash tS. \end{cases}$$

If there exist $y_1, y_2 \in S$ such that $sy_1 = sy_2$ or $ty_1 = ty_2$, then $(y_1, y_2) \in \mu_{s,t}$, and respectively, $f(sy_1) = [y_1]_{\mu_{s,t}} = [y_2]_{\mu_{s,t}} = f(sy_2)$ or $g(ty_1) = [y_1]_{\mu_{s,t}} = [y_2]_{\mu_{s,t}} = g(ty_2)$, that is, f and g are well defined. According to the definition, $fs = gt$. Since X^S is cofree, by [8, II, Proposition 4.8], thus by the assumption that X^S satisfies Condition (GP'), and so there exist $h : S \to X, u, v \in S$ and $n \in \mathbb{N}$ such that $f = hu$, $g = hv$, and $us^n = vt^n$. Now there are four possibilities that can arise:

If one of conditions (i)-(iii) does not hold, then a similar argument as [9, Proposition 2.2], leads to a contradiction. So we suppose that condition (iv) does not hold. Then there exist $p_1, p_2 \in S$ such that $(p_1, p_2) \in \ker \lambda_{us} \cup \ker \lambda_{vt} \backslash \mu_{s,t}$. That is,

$$(\exists p_1, p_2 \in S)((usp_1 = usp_2) \vee (vtp_1 = vtp_2)) \wedge (p_1, p_2) \notin \mu_{s,t}.$$

If $usp_1 = usp_2$, then $f(sp_1) = hu(sp_1) = h(usp_1) = h(usp_2) = hu(sp_2) = f(sp_2)$. By definition of f and g, we have $[p_1]_{\mu_{s,t}} = [p_2]_{\mu_{s,t}}$, and so $(p_1, p_2) \in \mu_{s,t}$, which is a contradiction. If $vtp_1 = vtp_2$, then analogously it leads to a contradiction.

(5) $\Rightarrow$ (1). Suppose A_S is an fg-weakly injective right S-act. Let $as = a't$ and $sz = tz$, for $a, a' \in A_S$ and $s, t, z \in S$. By assumption, there exist $u, v \in S$ and $n \in \mathbb{N}$ such that $us^n = vt^n$ and conditions (i)-(iv) hold. Define the mapping $\varphi : (uS \cup vS)_S \to A_S$ as follows:

$$\varphi(x) = \begin{cases} ap, & \exists p \in S;\ x = up, \\ a'q, & \exists q \in S;\ x = vq. \end{cases}$$

If there exist $p, q \in S$ such that $up = vq$, then by (i) there exist $y_1, y_2 \in S$ such that $(p, y_1) \in P_{1,s}$, $(y_1, y_2) \in \mu_{s,t}$ and $(y_2, q) \in P_{t,1}$. That is $p = sy_1, q = ty_2$ and $(y_1, y_2) \in (\ker \lambda_s \vee \ker \lambda_t)$, i.e., there exist $z_1, \ldots, z_m \in S$ such that

$$\begin{array}{cccc} sy_1 = sz_1 & sz_2 = sz_3 & \cdots & sz_{m-1} = sz_m \\ tz_1 = tz_2 & & \cdots & tz_m = ty_2. \end{array}$$

Consequently,

$$ap = asy_1 = asz_1 = a'tz_1 = a'tz_2 = \cdots = a'tz_m = a'ty_2 = a'q.$$

Suppose that there exist $p_1, p_2 \in S$ such that $up_1 = up_2$. If $p_1 = p_2$, then $ap_1 = ap_2$. If $p_1 \neq p_2$, then by condition (ii), there exist $y_1', y_2' \in S$ such that $p_1 = sy_1'$ and $p_2 = sy_2'$. Then $usy_1' = up_1 = up_2 = usy_2'$ and so $(y_1', y_2') \in \ker \lambda_{us}$, which by condition (iv) implies that $(y_1', y_2') \in \mu_{s,t} = (\ker \lambda_s \vee \ker \lambda_t)$, and so

$$\begin{array}{ccccc} sy_1' = sz_1 & sz_2 = sz_3 & \cdots & sz_{m-1} = sz_m & \\ & tz_1 = tz_2 & \cdots & & tz_m = ty_2'. \end{array}$$

Hence,

$$ap_1 = asy_1' = asz_1 = a'tz_1 = a'tz_2 = \cdots = a'tz_m = a'ty_2' = asy_2' = ap_2.$$

If there exist $q_1, q_2 \in S$ such that $vq_1 = vq_2$, then by using the conditions (iii) and (iv), we get analogously that $a'q_1 = a'q_2$. Then φ is a well defined homomorphism. By *fg*-weak injectivity of A_S there exists a homomorphism $\psi : S \to A_S$ which extends φ. Let $a'' = \psi(1)$. Then $a = \varphi(u) = \psi(u) = \psi(1)u = a''u$, and $a' = \varphi(v) = \psi(v) = \psi(1)v = a''v$. Thus, A_S satisfies Condition (GP'), as required. □

Lemma 3.20. *For any monoid S, and for any* $s \in S$:

$$P_{1,s} \circ \ker \lambda_s \circ P_{s,1} = (sS \times sS) \cap \Delta_S.$$

Proof. Let $p_1, p_2 \in S$, and $(p_1, p_2) \in P_{1,s} \circ \ker \lambda_s \circ P_{s,1}$. Then there exist $y_1, y_2 \in S$ such that $(p_1, y_1) \in P_{1,s}, (y_1, y_2) \in \ker \lambda_s, (y_2, p_2) \in P_{s,1}$, and so $p_1 = sy_1, sy_1 = sy_2, p_2 = sy_2$. That is, $(p_1, p_2) \in (sS \times sS) \cap \Delta_S$, as required. It is easy to see the reverse inclusion. □

Corollary 3.21. *Let S be a monoid such that the set of all left cancellable elements is commutative. Then all cofree right S-acts satisfy Condition (GP') if and only if S is a group.*

Proof. Necessity. Let $s \in S$. There exist $u, v \in S$, $n \in \mathbb{N}$, such that

$$P_{1, s^n} \circ \mu_{s^n, s^n} \circ P_{s^n, 1} \subseteq P_{u, v} \subseteq P_{1, s} \circ \mu_{s, s} \circ P_{s, 1},$$

and so

$$P_{1, s^n} \circ \ker \lambda_{s^n} \circ P_{s^n, 1} \subseteq P_{u, v} \subseteq P_{1, s} \circ \ker \lambda_s \circ P_{s, 1},$$

by Theorem 3.19, and so $(s^n S \times s^n S) \cap \Delta_S \subseteq P_{u, v} \subseteq (sS \times sS) \cap \Delta_S$, by Lemma 3.20. Let $\ker \lambda_u \neq \Delta_S$. Then there exists $(p_1, p_2) \in \ker \lambda_u$ such that $p_1 \neq p_2$. By Theorem 3.19, $\ker \lambda_u \subseteq \rho_{sS}$, and so there exist $y_1, y_2 \in S$, such that $p_1 = sy_1$ and $p_2 = sy_2$. Since $up_1 = up_2$, thus $usy_1 = usy_2$, and so $(y_1, y_2) \in \ker \lambda_{us}$. Again by Theorem 3.19, $\ker \lambda_{us} \subseteq \mu_{s, s} = \ker \lambda_s$, and so $p_1 = p_2$, which is a contradiction. That is, $\ker \lambda_u = \Delta_S$. By a similar argument we can show that $\ker \lambda_v = \Delta_S$. Hence, u, v are cancellable elements, and so by assumption they commute. Now,

$$uv = vu \Rightarrow (v, u) \in P_{u, v} \subseteq (sS \times sS) \cap \Delta_S \Rightarrow u = v,$$

$$(1, 1) \in \ker \lambda_u = P_{u, u} = P_{u, v} \subseteq (sS \times sS) \cap \Delta_S \Rightarrow \exists x \in S;\ sx = 1,$$

and so $sS = S$, for every $s \in S$, that is, S is a group.

Sufficiency. If S is a group, then all right S-acts satisfy Condition (GP'), by Theorem 3.2. □

Corollary 3.22. *Let S be a commutative monoid. Then all cofree right S-acts satisfy Condition (GP') if and only if S is a group.*

Corollary 3.23. *Let S be a finite monoid. Then all cofree right S-acts satisfy Condition (GP') if and only if S is a group.*

Proof. Necessity. Let $s \in S$. By a similar argument as in Corollary 3.21, there exist $u, v \in S$ and $n \in \mathbb{N}$ such that $\ker \lambda_u = \ker \lambda_v = \Delta_S$ and $(s^n S \times s^n S) \bigcap \Delta_S \subseteq P_{u,v} \subseteq (sS \times sS) \bigcap \Delta_S$. So, $uS \cong S/\ker \lambda_u = S/\Delta_S \cong S$, and so $|uS| = |S|$. Since S is finite and $uS \subseteq S$, the last equality implies that $uS = S$. Thus, there exists $x \in S$ such that $ux = v$, and $(x, 1) \in P_{u,v}$ $\subseteq (sS \times sS) \bigcap \Delta_S$, hence $x = 1$ and so $u = v$. Now by a similar argument as in the proof of Corollary 3.21, S is a group.

Sufficiency. If S is a group, then all right S-acts satisfy Condition (GP'), by Theorem 3.2. □

Corollary 3.24. *Let S be a monoid such that every left cancellable element of S has a right inverse. Then all cofree right S-acts satisfy Condition (GP') if and only if S is a group.*

Proof. By assumption, for any $u \in C_l(S)$, $uS = S$, and so a similar argument as in the proof of Corollary 3.23, gives the result. □

Corollary 3.25. *Let S be an idempotent monoid. Then all cofree right S-acts satisfy Condition (GP') if and only if $S = \{1\}$.*

Proof. Necessity. If $e \in S$, then by the proof of Corollary 3.21, there exist $u, v \in S$ and $n \in \mathbb{N}$ such that $\ker \lambda_u = \ker \lambda_v = \Delta_S$ and $P_{u,v} = (eS \times eS) \bigcap \Delta_S$. Thus, $(u, 1) \in \ker \lambda_u = \Delta_S$, and so $u = 1$, similarly $v = 1$. Therefore,

$$\Delta_S = \ker \lambda_1 = \ker \lambda_u = P_{u,u} = P_{u,v} = (eS \times eS) \bigcap \Delta_S \subseteq (eS \times eS),$$

since $(1, 1) \in \Delta_S \subseteq (eS \times eS)$, there exists $x \in S$ such that $ex = 1$, and so $e = 1$. Thus, $S = \{1\}$, as required.

Sufficiency. If $S = \{1\}$, then all right S-acts satisfy Condition (GP'), by Theorem 3.2. □

Theorem 3.26. *The following statements are equivalent*:

(1) *All right S-acts satisfying Condition* (GP') *are free*;

(2) *all right S-acts satisfying Condition* (GP') *are projective generators*;

(3) *all finitely generated right S-acts satisfying Condition* (GP') *are free*;

(4) *all finitely generated right S-acts satisfying Condition* (GP') *are projective generators*;

(5) *all cyclic right S-acts satisfying Condition* (GP') *are free*;

(6) *all cyclic right S-acts satisfying Condition* (GP') *are projective generators*;

(7) *all monocyclic right S-acts satisfying Condition* (GP') *are free*;

(8) *all monocyclic right S-acts satisfying Condition* (GP') *are projective generators*;

(9) $S = \{1\}$.

Proof. (1) $\Rightarrow$ (9). By the assumption that all right S-acts satisfying Condition (P) are free, and so by [8, IV, Theorem 12.8], $S = \{1\}$. By using a similar argument, it can be seen that other implications to (9), are also true.

(9) $\Rightarrow$ (1). This is clear. □

Recall from [8] that an element s of a semigroup S is called *aperiodic* if there exists $n \in \mathbb{N}$ such that $s^n = s^{n+1}$. A semigroup S is called *aperiodic* if every $s \in S$ is aperiodic.

Theorem 3.27. *Let S be a monoid such that every idempotent different from* 1 *is a right zero*. *Then the following statements are equivalent*:

(1) *All right S-acts satisfying Condition* (GP') *are regular*;

(2) *all finitely generated right S-acts satisfying Condition* (GP') *are regular*;

(3) *all cyclic right S-acts satisfying Condition* (GP') *are regular*;

(4) *every element of S different from* 1 *is right zero*.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). If all cyclic right *S*-acts satisfying Condition (GP') are regular, then all cyclic right *S*-acts satisfying Condition (*P*) are regular. Thus, all cyclic right *S*-acts satisfying Condition (*P*) are projective and so are strongly flat. Hence, *S* is aperiodic, by [8, IV, Theorem 10.2]. Thus, for every $s \in S$ there exists $n \in \mathbb{N}$ such that $s^{n+1} = s^n$. That is, s^n is an idempotent. If $s \neq 1$, then $s^n \neq 1$. Otherwise, $s^n = 1$ implies that $ss^n = s1$ or $s^{n+1} = s$. Since $s^{n+1} = s^n$, we have $s^n = s$, and so $s = 1$, which is a contradiction. Thus, $s^n \neq 1$, and so by the assumption that s^n is right zero. S_S, as a cyclic right *S*-act, satisfies Condition (GP'), by Theorem 2.2, and so by the assumption that S_S is regular, thus the principal right ideal *sS* is projective. Therefore, *s* is left *e*-cancellable, for some idempotent $e \in S$, by [8, III, Theorem 17.16], that is, $s = se$ and $ss^{n-1} = ss^n$ implies that $es^{n-1} = es^n$. Without loss of generality, we may assume that *n* is the smallest natural number such that $s^n = s^{n+1}$. Now, if $e = 1$, then $s^{n-1} = s^n$, which is a contradiction. Thus, $e \neq 1$, and so by the assumption that *e* is a right zero. Then $s = se = e$, that is, *s* is a right zero, as required.

(4) $\Rightarrow$ (1). Suppose A_S satisfies Condition (GP'). We have to show that all cyclic subacts of A_S are projective. Let $a \in A_S$. Then either for all $1 \neq s \in S, as \neq a$ or there exists $s \in S$ such that $as = a$. If for all

$1 \neq s \in S, as \neq a$, then $\varphi : aS \to S$ defined by $\varphi(as) = s$ is an isomorphism, because if $as = at$, then since $ss = ts$, Condition (GP') implies that there exist $u, v \in S$, and $n \in \mathbb{N}$ such that $us^n = vt^n$. Since $1 \neq s, t$ are right zero, the last equality implies that $s = t$, and so φ well defined. Therefore, aS is projective. If there exists $1 \neq s \in S$ such that $as = a$, then by using a similar argument as above, there exists at most one such s. In this case $\varphi : aS \to sS$ defined by $\varphi(at) = st$ is an isomorphism. Since s is an idempotent aS is projective, by [8, III, Proposition 17.2(3)]. □

The monoid S is called *simple*, if S has no proper ideal (see [8]).

Theorem 3.28. *For any simple monoid S, the following statements are equivalent*:

(1) *All right S-acts satisfying Condition (GP') are regular*;

(2) *all finitely generated right S-acts satisfying Condition (GP') are regular*;

(3) *all cyclic right S-acts satisfying Condition (GP') are regular*;

(4) $S = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). By the assumption that all cyclic right S-acts satisfying Condition (P) are (being regular) projective and so every right reversible submonoid of S contains a left zero by [8, IV, Theorem 11.8]. Let $1 \neq e$ be an idempotent. Since S is simple, it has no proper right ideal, and so $eS = S$. Therefore, $e = 1$ which is a contradiction. Hence, the only idempotent of S is 1. Now let $s \in S$. As we showed in the proof of Theorem 3.27, S is aperiodic, and so there exists $n \in \mathbb{N}$ such that $s^n = s^{n+1}$. Since s^n is an idempotent, we have $s^n = 1$, and so $s^n = s^{n+1}$ implies that $s = 1$. Thus, $S = \{1\}$, as required.

(4) $\Rightarrow$ (1). If $S = \{1\}$, then all right S-acts are regular, by [8, IV, Theorem 14.4]. □

Theorem 3.29. *The following statements are equivalent*:

(1) *All right S-acts satisfying Condition (GP') are divisible*;

(2) *all finitely generated right S-acts satisfying Condition (GP') are divisible*;

(3) *all cyclic right S-acts satisfying Condition (GP') are divisible*;

(4) *all monocyclic right S-acts satisfying Condition (GP') are divisible*;

(5) *all left cancellable elements of S are left invertible*.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). By Theorem 2.2, S_S, as a right S-act, satisfies Condition (GP'). Since $S/\rho(s, s) = S/\Delta_S \cong S_S$, for $s \in S$, thus by the assumption that S_S is divisible, and so $Sc = S$, for any left cancellable element $c \in S$. Therefore, there exists $x \in S$, such that $xc = 1$.

(5) $\Rightarrow$ (1). This is obvious, by [8, III, Proposition 2.2]. □

We recall from [18] that an act A_S is called *strongly torsion free* if for any $a, b \in A_S$ and any $s \in S$, the equality $as = bs$ implies $a = b$.

Theorem 3.30. *The following statements are equivalent*:

(1) *All right S-acts satisfying Condition (GP') are strongly torsion free*;

(2) *all finitely generated right S-acts satisfying Condition (GP') are strongly torsion free*;

(3) *all cyclic right S-acts satisfying Condition (GP') are strongly torsion free*;

(4) *S is right cancellative*.

Proof. This follows from [18, Theorem 3.1]. □

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