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REPRESENTATION OF THE DERIVATIVE FOR SPLIT-QUATERNIONIC FUNCTIONS

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Abstract

In the split-quaternion domain, functions have standard differentiability conditions and results which are not applied by calculations of differential operators, directly. We introduce the unusual representation of the derivative, called the *SR* calculus, which is useful to calculate derivatives of such functions. We show the derivatives and their examples by using the simple calculation process on split-quaternionic functions.

1. Introduction

Split-quaternions were introduced by Cockle [2] in 1849. Split-quaternions are elements of a 4-dimensional associative but not commutative algebra over $\mathbb{R}$ for multiplication. Unlike the quaternion algebra introduced by Hamilton in 1843, the split-quaternions have zero divisors, nilpotent elements and nontrivial idempotents (see [11]). In differential geometry and some algebraic properties of Hamilton operators of split-quaternions, there

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are many studies. Kula and Yayli [9] showed that the algebraic structure of split-quaternions has a product of rotations in semi-Euclidean and Minkowski 3-space. Jafari and Yayli [4] studied De-Moivre's and Euler's formulae for matrices associated with split-quaternions. Kim and Shon [6, 7] proposed a split-hyperholomorphic function and a split-harmonic function with values in split-quaternions, which are expressed polar coordinate forms for split quaternions, and obtained properties of split-hyperholomorphic mappings.

An involution is an inverse linear mapping itself. Many examples of non-trivial involutions contain the complementation in set theory and complex conjugations. Knus et al. [8] recalled basic properties of central simple algebras and studied involutions into symplectic and unitary types. Bekar and Yayli [1] expressed geometric interpretations of involutions and anti-involutions of real quaternions, biquaternions and split-quaternions.

The *HR* calculus has been used to calculate formal derivatives of both analytic and non-analytic functions of quaternion variables. The *HR* derivative can be proposed the left-hand and right-hand versions of quaternionic derivatives, based on a general orthogonal system. Actually, the *HR* calculus gives the simple way to deal with the chain rule, the mean-valued theorem and Taylor's theorem. Jahanchahi et al. [5] introduced the *HR* calculus to provide information within four-dimensional quaternion valued signals, for the calculation of the derivatives of analytic quaternion valued functions. Mandic et al. [10] gave the *HR* calculus which conforms with the maximum change of the gradient and the direction of the conjugate gradient, based on the isomorphism with quaternion involutions.

In this paper, we consider *HR* calculus and *HR* derivative on split-quaternions. We introduce the *SR* calculus which is useful to represent derivatives of split-quaternion valued functions. From the *SR* calculus, we show the standard differentiability conditions and calculations of differential operators in the split-quaternion domain. Also, we give some examples to show convenience in use of the *SR* calculus.

2. Preliminaries

A set consisting of split-quaternions is defined as

$$\mathbb{H}_{\mathcal{S}} = \{p = x_0 + ix_1 + jx_2 + kx_3 \,|\, x_r \in \mathbb{R},\ r = 0, 1, 2, 3\},$$

where the imaginary units i and the unit elements j and $k = ij$ as components of a basis for $\mathbb{H}_{\mathcal{S}}$ satisfy

$$i^2 = -1 \quad \text{and} \quad j^2 = k^2 = ijk = 1.$$

The product for split-quaternions is non-commutative, that is, $pq \neq qp$ and

$$ij = k = -ji, \quad jk = -i = -kj \quad \text{and} \quad ki = j = -ik.$$

For $p = x_0 + ix_1 + jx_2 + kx_3$ and $q = y_0 + iy_1 + jy_2 + ky_3$ in $\mathbb{H}_{\mathcal{S}}$, the corresponding dot product, denoted by $\langle p, q\rangle_{(s)}$, is given by

$$\langle p, q\rangle_{(s)} = x_0y_0 + x_1y_1 - x_2y_2 - x_3y_3.$$

The split-quaternion conjugate of $p \in \mathbb{H}_{\mathcal{S}}$ is defined as

$$p^* = x_0 - ix_1 - jx_2 - kx_3 = S(p) - V(p),$$

and the modulus is written by

$$\langle p, q\rangle_{(s)} = pp^* = p^*p = x_0^2 + x_1^2 - x_2^2 - x_3^2 \in \mathbb{R}$$

and thus the inverse element of each split-quaternion in $\mathbb{H}_{\mathcal{S}}^{\circ}$ is given by

$$p^{-1} = \frac{p^*}{pp^*} = \frac{p^*}{\langle p, p\rangle_{(s)}},$$

where $\mathbb{H}_{\mathcal{S}}^{\circ} = \mathbb{H}_{\mathcal{S}} \backslash D$ and $D = \{p = x_0 + ix_1 + jx_2 + kx_3 \,|\, x_0^2 + x_1^2 = x_2^2 + x_3^2\}$, for example,

$$i^{-1} = -i, \quad j^{-1} = j \quad \text{and} \quad k^{-1} = k.$$

Consider a similarity relation '~' between p and q in $\mathbb{H}_{\mathcal{S}}$ such that

$$p \sim q \text{ if and only if } p = \mu q \mu^{-1},$$

where $\mu \in \mathbb{H}_{\mathcal{S}}^{\circ}$ is non-trivial. Then we obtain the following properties:

Proposition 2.1. *For p and q in* $\mathbb{H}_{\mathcal{S}}$, *if* $p \sim q$, *then we have*

$$\langle p, p\rangle_{(s)} = \langle q, q\rangle_{(s)}. \tag{1}$$

Proof. From the assumption $p \sim q$, there is $\mu \in \mathbb{H}_{\mathcal{S}}^{\circ}$ such that $p = \mu q \mu^{-1}$. Since the modulus on $\mathbb{H}_{\mathcal{S}}$ satisfies $\langle \cdot \rangle_{(s)} \in \mathbb{R}$ and a split-quaternion and its conjugate are commutative for product, we can calculate as follows:

$$\langle p, p\rangle_{(s)} = pp^* = (\mu q \mu^{-1})(\mu q \mu^{-1})^*$$

$$= (\mu q \mu^{-1})((\mu^{-1})^* q^* \mu^*) = qq^* = \langle q, q\rangle_{(s)}.$$

Thus, we obtain equation (1). □

Example 2.2. Since i, j and k satisfy the following equations:

$$i(i)i^{-1} = -i(i)i = i, \; j(j)j^{-1} = j(j)j = j \text{ and } k(k)k^{-1} = k(k)k = k,$$

the three units satisfy the similarity relation each other, that is, $i \sim j \sim k$.

Remark 2.3. An involution is denoted by the mapping $x \mapsto I(x)$, which satisfies the following axioms [12]:

Axiom 1. An involution is its own inverse, that is, $I(I(x)) = x$.

Axiom 2. An involution is linear, that is,

$$I(\alpha x + \beta y) = \alpha I(x) + \beta I(y),$$

where α and β are real constants.

Axiom 3. An involution satisfies $I(xy) = I(x)I(y)$.

Let the equivalence relations be involutions of split-quaternions by referring [3]:

$$\begin{cases} p^i = ipi^{-1} = -ipi = x_0 + ix_1 - jx_2 - kx_3, \\ p^j = jpj^{-1} = jpj = x_0 - ix_1 + jx_2 - kx_3, \\ p^k = kpk^{-1} = kpk = x_0 - ix_1 - jx_2 + kx_3. \end{cases} \tag{2}$$

Also, the conjugate of a split-quaternion is also an involution and satisfies $(p^*)^* = p$. Based on the above involutions in (2), the four real components of a split-quaternion can be expressed as

$$x_0 = \frac{1}{4}(p + p^i + p^j + p^k), \quad x_1 = -\frac{1}{4}i(p + p^i - p^j - p^k),$$

$$x_2 = \frac{1}{4}j(p - p^i + p^j - p^k), \quad x_3 = \frac{1}{4}k(p - p^i - p^j + p^k). \tag{3}$$

By using the above equations in (2) and (3), any split-quaternion-valued function of the four real variables (x_0, x_1, x_2, x_3) can be written as a function of the split-quaternion variable p and its involutions (p^i, p^j, p^k).

3. Representation of the *SR* Derivative

Consider the derivatives of a split-quaternion-valued function and a corresponding composite function of the four real variables. Let $f : \mathbb{H}_{\mathcal{S}} \to \mathbb{H}_{\mathcal{S}}$ be a function such that

$$f(p) = f(x_0 + ix_1 + jx_2 + kx_3) = u_0 + iu_1 + ju_2 + ku_3,$$

where $u_r = u_r(x_0, x_1, x_2, x_3) \in \mathbb{R}$ $(r = 0, 1, 2, 3)$ are real-valued functions.

Since $\mathbb{H}_{\mathcal{S}}$ and $\mathbb{R}^4$ are isomorphic, we let $g : \mathbb{R}^4 \to \mathbb{R}^4$ be a corresponding composite function satisfying

$$g(x_0, x_1, x_2, x_3) = (u_0, u_1, u_2, u_3),$$

where $g(x_0, x_1, x_2, x_3) \cong f(p)$ and $u_r = u_r(x_0, x_1, x_2, x_3)(r = 0, 1, 2, 3)$.

By the chain rule for the function of the four real variables, we have the differential of the function g of the four real variables as follows:

$$dg = \frac{\partial g}{\partial x_0} dx_0 + \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 + \frac{\partial g}{\partial x_3} dx_3$$

$$= \frac{\partial f(p)}{\partial x_0} dx_0 + i \frac{\partial f(p)}{\partial x_1} dx_1 + j \frac{\partial f(p)}{\partial x_2} dx_2 + k \frac{\partial f(p)}{\partial x_3} dx_3.$$

Since each of real variables x_0, x_1, x_2 and x_3 can be written by using p^i, p^j and p^k (see (2) and (3)), we have a function $h : \mathbb{H}_{\mathcal{S}}^4 \to \mathbb{H}_{\mathcal{S}}$ which is a corresponding composite function such that

$$(p, p^i, p^j, p^k) \mapsto h(p, p^i, p^j, p^k) \in \mathbb{H}_{\mathcal{S}}.$$

Example 3.1. For $p = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}_{\mathcal{S}}$, a function $f(p) = p^2$ is

$$f(p) = f(x_0 + ix_1 + jx_2 + kx_3) = (x_0 + ix_1 + jx_2 + kx_3)^2$$

$$= (x_0^2 - x_1^2 + x_2^2 + x_3^2) + i2x_0x_1 + j2x_0x_2 + k2x_0x_3$$

$$= \frac{1}{8}\{10p^2 + 2(p^i)^2 + 2(p^j)^2 + 2(p^k)^2 + 6pp^i + 4pp^j$$

$$+ 4pp^k - 4p^ip^j - 4p^ip^k - 4p^jp^k - 2pp^k\}$$

$$= h(p, p^i, p^j, p^k).$$

From equations (2) and (3), we also have the derivatives of the function h and the components of a split-quaternion as follows:

$$dx_0 = \frac{1}{4}(dp + dp^i + dp^j + dp^k), \quad dx_1 = -\frac{1}{4} i(dp + dp^i - dp^j - dp^k),$$

$$dx_2 = \frac{1}{4} j(dp - dp^i + dp^j - dp^k), \quad dx_3 = \frac{1}{4} k(dp - dp^i - dp^j + dp^k), \quad (4)$$

and

$$dh = \frac{\partial h}{\partial p}dp + \frac{\partial h}{\partial p^i}dp^i + \frac{\partial h}{\partial p^j}dp^j + \frac{\partial h}{\partial p^k}dp^k. \quad (5)$$

Theorem 3.2. *For* $p \in \mathbb{H}_{\mathcal{S}}$, *let* $h : \mathbb{H}_{\mathcal{S}}^4 \to \mathbb{H}_{\mathcal{S}}$ *be a differentiable function in* $\mathbb{H}_{\mathcal{S}}$. *Then the complete set of the SR derivatives is*

$$\left(\frac{\partial h}{\partial p}\frac{\partial h}{\partial p^i}\frac{\partial h}{\partial p^j}\frac{\partial h}{\partial p^k}\right)^t = \frac{1}{4}\begin{pmatrix} 1 & -i & j & k \\ 1 & -i & -j & -k \\ 1 & i & j & -k \\ 1 & i & -j & k \end{pmatrix}\left(\frac{\partial h}{\partial x_0}\frac{\partial h}{\partial x_1}\frac{\partial h}{\partial x_2}\frac{\partial h}{\partial x_3}\right)^t, \quad (6)$$

where $h = h(p, p^i, p^j, p^k)$ *and* $(\ )^t$ *is a transposed matrix of* $(\ )$.

Proof. By using variables p, p^i, p^j and p^k, we give an expression of the derivative of g through the process and result of the formation of h as follows:

$$dh = D_0 dq + D_1 dp^i + D_2 dp^j + D_3 dp^k. \quad (7)$$

To find the solution D_r $(r = 0, 1, 2, 3)$ in equation (7), we calculate equation (7), applying equations (4) and (5) as follows:

$$\frac{\partial h}{\partial p} = \frac{\partial x_0}{\partial p}\frac{\partial h}{\partial x_0} + \frac{\partial x_1}{\partial p}\frac{\partial h}{\partial x_1} + \frac{\partial x_2}{\partial p}\frac{\partial h}{\partial x_2} + \frac{\partial x_3}{\partial p}\frac{\partial h}{\partial x_3},$$

$$= \frac{1}{4}\frac{\partial h}{\partial x_0} + \frac{-i}{4}\frac{\partial h}{\partial x_1} + \frac{j}{4}\frac{\partial h}{\partial x_2} + \frac{k}{4}\frac{\partial h}{\partial x_3},$$

$$\frac{\partial h}{\partial p^i} = \frac{\partial x_0}{\partial p^i}\frac{\partial h}{\partial x_0} + \frac{\partial x_1}{\partial p^i}\frac{\partial h}{\partial x_1} + \frac{\partial x_2}{\partial p^i}\frac{\partial h}{\partial x_2} + \frac{\partial x_3}{\partial p^i}\frac{\partial h}{\partial x_3}$$

$$= \frac{1}{4}\frac{\partial h}{\partial x_0} + \frac{-i}{4}\frac{\partial h}{\partial x_1} + \frac{-j}{4}\frac{\partial h}{\partial x_2} + \frac{-k}{4}\frac{\partial h}{\partial x_3},$$

$$\frac{\partial h}{\partial p^j} = \frac{\partial x_0}{\partial p^j}\frac{\partial h}{\partial x_0} + \frac{\partial x_1}{\partial p^j}\frac{\partial h}{\partial x_1} + \frac{\partial x_2}{\partial p^j}\frac{\partial h}{\partial x_2} + \frac{\partial x_3}{\partial p^j}\frac{\partial h}{\partial x_3}$$

$$= \frac{1}{4}\frac{\partial h}{\partial x_0} + \frac{i}{4}\frac{\partial h}{\partial x_1} + \frac{j}{4}\frac{\partial h}{\partial x_2} + \frac{-k}{4}\frac{\partial h}{\partial x_3},$$

$$\frac{\partial h}{\partial p^k} = \frac{\partial x_0}{\partial p^k}\frac{\partial h}{\partial x_0} + \frac{\partial x_1}{\partial p^k}\frac{\partial h}{\partial x_1} + \frac{\partial x_2}{\partial p^k}\frac{\partial h}{\partial x_2} + \frac{\partial x_3}{\partial p^k}\frac{\partial h}{\partial x_3}$$

$$= \frac{1}{4}\frac{\partial h}{\partial x_0} + \frac{i}{4}\frac{\partial h}{\partial x_1} + \frac{-j}{4}\frac{\partial h}{\partial x_2} + \frac{k}{4}\frac{\partial h}{\partial x_3}.$$

By calculating the above equations, we can find each D_r $(r = 0, 1, 2, 3)$. Therefore, we can consider the form by means of the following matrix:

$$\frac{1}{4}\begin{pmatrix} 1 & -i & j & k \\ 1 & -i & -j & -k \\ 1 & i & j & -k \\ 1 & i & -j & k \end{pmatrix}$$

and their products, and then we obtain equation (6). □

With this, for a corresponding composite function h if we want to obtain a derivative with the components of a split-quaternion such as p, p^i, p^j and p^k, like equation (6), then it can be induced by a derivative with four real variables.

Example 3.3. For $p \in \mathbb{H}_{\mathcal{S}}$, let $h(p) = p^*$. Using the *SR* derivatives in equation (6) for $h(p)$, we have the result such that

$$\left(\frac{\partial h}{\partial p}\frac{\partial h}{\partial p^i}\frac{\partial h}{\partial p^j}\frac{\partial h}{\partial p^k}\right)^t = \frac{1}{4}\begin{pmatrix} 1 & -i & j & k \\ 1 & -i & -j & -k \\ 1 & i & j & -k \\ 1 & i & -j & k \end{pmatrix}\begin{pmatrix} 1 \\ -i \\ -j \\ -k \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Furthermore, p^* can be written by

$$p^* = \frac{1}{2}(p^i + p^j + p^k - p).$$

Thus, we obtain $h'(p) = (p^*)' = -\frac{1}{2}$.

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