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SEQUENTIAL ESTIMATION OF RATIO OF SCALE PARAMETERS IN EXPONENTIAL DISTRIBUTIONS

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Abstract

Fixed-width confidence intervals for the ratio of scale parameters in exponential distributions are considered. The first-order asymptotic results of the sequential procedure are established. An application to estimating certain reliability is provided.

1. The Formulation of the Problem

Let $(X_i, Y_i), i = 1, ..., n$ be a random sample from the bivariate density $f(x, y) = f_1(x)f_2(y)$, where

$$f_1(x) = (\mu\theta)^{-1} \exp(-x/\mu\theta)$$

and

$$f_2(y) = \mu^{-1} \exp(-y/\mu).$$

We are interested in estimating θ by $\hat{\theta}_n$ such that

$$P(\theta \in (\hat{\theta}_n \pm d)) \geq 1 - \alpha, \tag{1.1}$$

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where d and α are the specified constants. One can easily obtain the likelihood equations for θ and μ to be

$$\mu\theta = \overline{X}_n$$

and

$$2\mu = \frac{\overline{X}_n}{\theta} + \overline{Y}_n, \tag{1.2}$$

where $\overline{X}_n$ and $\overline{Y}_n$ denote the sample means of the X and Y samples. Thus,

$$\hat{\mu} = \overline{Y}_n \quad \text{and} \quad \hat{\theta} = \overline{X}_n / \overline{Y}_n. \tag{1.3}$$

Further, the information matrix is given by

$$I(\theta, \mu) = n\begin{pmatrix} \theta^{-2} & (\theta\mu)^{-1} \\ (\theta\mu)^{-1} & 2\mu^{-2} \end{pmatrix}$$

from which we obtain the asymptotic variance of $\hat{\theta}$, namely $\sigma^2_{\hat{\theta}}$ as

$$\sigma^2_{\hat{\theta}} = 2\theta^2/n. \tag{1.4}$$

From the large-sample properties of the maximum likelihood estimates, we have

$$(\hat{\theta} - \theta)/\sigma_{\hat{\theta}} \overset{d}{\approx} \text{normal } (0, 1). \tag{1.5}$$

For sufficiently large n, (1.1) implies that

$$\frac{d^2}{\sigma^2_{\hat{\theta}}} \geq z^2_{\alpha/2}$$

or

$$n \geq 2\theta^2 z^2/d^2 = n^*, \text{ (say)}$$

where $z = z_{\alpha/2}$ denotes the upper $100(1-\alpha/2)$th percentile of the standard normal distribution. However, n^* involves the unknown parameter θ. Hence, we resort to the following adaptive sequential rule: sample pairs

(X_i, Y_i), $i = 1, \ldots$ sequentially and stop at N, where

$$N = \inf\left\{n \geq m : n \geq \frac{2z^2}{d^2}\hat{\theta}_n^2\right\}, \tag{1.6}$$

and m denotes the initial sample size.

2. Properties of the Sequential Procedure

Property 1. The sequential procedure has a finite termination with probability one.

Proof. Let $b = 2z^2/d^2$ and consider

$$P(N = \infty) = \lim_{n\to\infty} P(N > n) \leq \lim_{n\to\infty} P(n < b\hat{\theta}_n^2) = 0$$

since $\hat{\theta}_n$ converges to θ in probability.

Property 2.

$$N \to \infty \text{ a.s., } EN \to \infty \text{ as } d \to 0 \text{ and } \lim_{d\to 0}(N/n^*) = 1 \text{ a.s.} \tag{2.1}$$

Let $Z_n = (\hat{\theta}_n/\theta)^2$, $f(n) = n$ and $t = b\theta^2$. Then we can rewrite the stopping time as

$$N = N(t) = \inf\{n \geq m : Z_n \leq f(n)/t\}. \tag{2.2}$$

Thus Z_n is a sequence of random variables such that $Z_n > 0$ and $\lim_{d\to 0} Z_n = 1$ a.s. due to the fact that $\hat{\theta}_n/\theta \to 1$ as $n \to \infty$. N is well-defined and non-decreasing as a function of t and one can easily verify that

$$N \to \infty \text{ a.s. and } EN \to \infty \text{ as } t \to \infty.$$

Next, for $N > 1$, (proceedings as in Chow and Robbins [2]), we have

$$N \geq b\hat{\theta}_N^2 \text{ and } N - 1 < b\hat{\theta}_{N-1}^2$$

from which we obtain

$$b\hat{\theta}_N^2 \leq N \leq 1 + b\hat{\theta}_{N-1}^2$$

$$\frac{\hat{\theta}_N^2}{\theta^2} \leq \frac{N}{n^*} \leq \frac{1}{n^*} + \frac{\hat{\theta}_{N-1}^2}{\theta^2}.$$

By taking limits on both sides of the preceding inequality as $d \to 0$ (and hence, $b \to \infty$), we establish the last part of (2.1).

Property 3. The coverage probability of the fixed-width interval tends to the nominal level, $1 - \alpha$ as $d \to 0$.

Proof. Consider

$$\hat{\theta}_N - \theta = \frac{\overline{X}_N}{\overline{Y}_N} - \frac{\lambda}{\mu} = \frac{\overline{X}_N - \lambda}{\overline{Y}_N} - \theta\frac{(\overline{Y}_N - \mu)}{\overline{Y}_N}. \tag{2.3}$$

Because Anscombe's [1] theorem holds for sum of i.i.d. variables, we infer that

$$n^{*1/2}(\overline{X}_N - \lambda) \overset{d}{\approx} \text{normal } (0, 1)$$

and

$$n^{*1/2}(\overline{Y}_N - \mu) \overset{d}{\approx} \text{normal } (0, 1),$$

where $\lambda = \theta\mu$. Thus, $\overline{Y}_N$ converges to μ in probability as $d \to 0$. Hence, we can rewrite (2.3) (after using Slutsky's theorem) as

$$n^{*1/2}(\hat{\theta}_N - \theta) \approx \frac{n^{*1/2}}{\mu}\{(\overline{X}_N - \lambda) - \theta(\overline{Y}_N - \mu)\}$$

$$\approx n^{*-1/2}\sum_1^N W_i, \tag{2.4}$$

where $\{W_i = X_i - \lambda - \theta(Y_i - \mu)\}$ form an i.i.d. sequence. Thus applying again Anscombe's [1] theorem, we infer that

$$n^{*1/2}(\hat{\theta}_N - \theta) \approx \text{normal } (0, \operatorname{var} W_1 = 2\theta^2). \tag{2.5}$$

Hence,

$$P(|\hat{\theta}_N - \theta| \le d) \approx 2\Phi\left(\frac{d}{\theta}\sqrt{\frac{n^*}{2}}\right) - 1 = 2\Phi(z) - 1 = 1 - \alpha. \tag{2.6}$$

Finally, in order to establish the asymptotic efficiency of the sequential procedure, namely, that EN/n^* tends to 1 as d tends to zero, we need the following lemma.

Lemma 2.1 [3]. *Let* $\{Z_k, k \geq 1\}$ *be a sequence of positive random variables and* $\{m_k, k \geq 1\}$ *be a sequence of positive real numbers such that* m_k *increases with* k *and* $Z_k/m_k \to 1$ *a.s. as* $k \to \infty$. *Also, for any* $b > 0$, *let*

$$T(b) = \inf\{k \geq 1 : Z_k \geq b\},\ t(b) = \inf\{k \geq 1 : m_k \geq b\} \tag{2.7}$$

and assume that

$$\lim_{\rho \to 1} \lim_{b \to \infty} [t(b\rho)/t(b)] = 1. \tag{2.8}$$

If for some $\delta > 0$,

$$\sum_{k=1}^{\infty} P\{Z_k < \delta m_k\} < \infty, \tag{2.9}$$

then, as $b \to \infty$,

$$E\{T(b)/t(b)\} \to 1. \tag{2.10}$$

Property 4. We have

$$\lim_{d \to 0} (EN/n^*) = 1. \tag{2.11}$$

Proof. In Lemma 2.1, set $Z_n = \hat{\theta}_n^2, b = 2z^2/d^2$ and it suffices to verify (2.9).

Let $g(\lambda, \mu) = \lambda/\mu$ and expand $g(\overline{X}_n, \overline{Y}_n)$ as

$$g(\overline{X}_n, \overline{Y}_n) = g(\lambda, \mu) + (\overline{X}_n - \lambda)\frac{\partial g}{\partial \lambda} + (\overline{Y}_n - \mu)\frac{\partial g}{\partial \mu} + R, \tag{2.12}$$

where

$$\frac{\partial g}{\partial \lambda} = \frac{1}{\mu},\ \frac{\partial g}{\partial \mu} = \frac{-\lambda}{\mu^2} = \frac{-\theta}{\mu}$$

and the remainder term involves higher powers of $(\overline{X}_n - \lambda)$ and $(\overline{Y}_n - \mu)$. Consider, for some $0 < \delta < 1$,

$$\begin{aligned} P(g(\overline{X}_n, \overline{Y}_n) < \delta g(\lambda, \mu)) &= P((\overline{X}_n - \lambda) - \theta(\overline{Y}_n - \mu) + \mu R < (\delta - 1)\lambda) \\ &= P(n^{1/2}\{\overline{X}_n - \lambda - \theta(\overline{Y}_n - \mu)\} + o_p(1) \leq (\delta - 1)n^{1/2}\lambda) \\ &\doteq P\left(n^{1/2}\sum_{1}^{n} W_i + o_p(1) \leq (\delta - 1)n^{1/2}\lambda\right), \end{aligned}$$

where $o_p(1)$ tends to zero in probability as $n \to \infty$. Consider

$$\sum_{1}^{\infty} P\left(\frac{\overline{X}_n}{\overline{Y}_n} > \delta\theta\right) = \sum_{1}^{m_o} + \sum_{m_o+1}^{\infty} P\left(n^{-1/2}\sum_{1}^{n} W_i/\sqrt{2}\lambda < \sqrt{n}(\delta-1)/\sqrt{2}\right),$$

where m_o is chosen large enough so that the central limit theorem holds for the sum of W_i random variables. Note that

$$EW_i = 0 \text{ and } \operatorname{var} W_i = 2\lambda^2.$$

Hence,

$$\sum_{1}^{\infty} P\left(\frac{\overline{X}_n}{\overline{Y}_n} > \delta\theta\right) = m_o + \sum_{m_o+1}^{\infty} \Phi\left(-(1-\delta)\sqrt{\frac{n}{2}}\right)$$

$$\leq m_o + \int_{m_o}^{\infty} \Phi\left(-(1-\delta)\sqrt{\frac{x}{2}}\right)dx$$

$$\leq m_o + \int_{m_o}^{\infty} \frac{\phi\left(\sqrt{\frac{x}{2}}(1-\delta)\right)}{\sqrt{\frac{x}{2}}(1-\delta)}dx$$

$$\leq m_o + \frac{1}{\sqrt{\frac{m_o}{2}}(1-\delta)} \frac{e^{-\frac{1}{4}(1-\delta)^2 m_o}}{\frac{1}{4}(1-\delta)^2\sqrt{2\pi}}$$

$$= m_o + \frac{4}{\sqrt{m_o\pi}}(1-\delta)^{-3}e^{-\frac{1}{4}(1-\delta)^2 m_o}$$

which is finite. This completes the proof of Property 4.

3. An Application

Suppose X and Y are independent having exponential distributions with scale parameters λ and μ respectively. Suppose we are interested in estimating

$$p = P(X < Y) = 1 - \lambda(\lambda + \mu)^{-1} = \mu(\lambda + \mu)^{-1} = (1 + \theta)^{-1}.$$

Estimating θ with a fixed-width confidence interval having width $2d$ (when d is small) is equivalent to estimating p with width $2p^2d$ with confidence at least $1 - \alpha$ because

$$\begin{aligned} 1 - \alpha &= P(\theta - d \le \hat{\theta}_N \le \theta + d) \\ &= P(1 + \theta - d \le 1 + \hat{\theta}_N \le 1 + \theta + d) \\ &= P\left(\frac{1}{p} - d \le \frac{1}{\hat{p}_N} \le \frac{1}{p} + d\right) \\ &= P\left(\frac{p}{1 + pd} \le \hat{p}_N \le \frac{p}{1 - pd}\right) \\ &\doteq P(p(1 - pd) \le \hat{p}_N \le p(1 + pd)) \\ &= P(|\hat{p}_N - p| \le p^2 d). \end{aligned}$$

Alternatively, if we are estimating p with a fixed width confidence interval having $2d$ is equivalent to estimating θ with width $2d(1 + \theta)^2$ with confidence at least $1 - \alpha$ for the following reason:

$$\begin{aligned} 1 - \alpha &= P(p - d \le \hat{p}_N \le p + d) \\ &= P\left(\frac{1}{p + d} \le \frac{1}{\hat{p}_N} \le \frac{1}{p - d}\right) \\ &= P\left(\frac{1}{p}\left(1 + \frac{d}{p}\right)^{-1} \le 1 + \hat{\theta}_N \le \frac{1}{p}\left(1 - \frac{d}{p}\right)^{-1}\right) \\ &\doteq P\left(\frac{1}{p}\left(1 - \frac{d}{p}\right) \le 1 + \hat{\theta}_N \le \frac{1}{p}\left(1 + \frac{d}{p}\right)\right) \\ &= P(1 + \theta - d(1 + \theta)^2 \le 1 + \hat{\theta}_N \le 1 + \theta + d(1 + \theta)^2) \\ &= P(|\hat{\theta}_N - \theta| \le d(1 + \theta)^2). \end{aligned}$$

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