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GRAPHICAL REPRESENTATION OF MAXIMAL GROUPS OF GENUS TWO AND THEIR HYPERELLIPTIC CURVES

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Abstract

In 2007, Cardona and Quer studied curves of genus 2 whose automorphism groups are isomorphic to two non-maximal groups D_8 and D_{12} of orders 16 and 24, see [1]. They did not, however, refer to maximal automorphism groups of genus 2. In this paper, we investigate distinct types of maximal groups of genus two in complete detail. We prove that there are exactly four types of maximal groups of genus two and give their presentations as finitely presented, transitive permutation representations of certain degrees. Furthermore,

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their character tables, matrix representations, and their primary invariants and their Cayley color graphs and/or their Shreier's coset graphs are found. We also compute the equations of the hyperelliptic curves covered by these four types of groups, which happen to be the groups obtained from the maximal automorphism bound for the soluble automorphism groups $(|G| = 48(g-1))$, supersoluble automorphism groups $|G| = 24$, nilpotent automorphism groups $(16(g-1))$ and the quaternion group $(4, 4, 4)$-group of order 8 for genus $g = 2$.

1. Generalization of Distinct Types of Groups of Genus Two

We proceed by considering the well-known Riemann-Hurwitz relation [1, 4, 6, 7] for the case $g = 2$. We know that this relation is an equation involving the positive integers $N = |G| \geq 2$, r, $2 \leq m_1 \leq m_2 \leq \cdots \leq m_r$, i.e., the order of the finite group G, the minimum number of generators of finite order and the periodic generators of the finite groups. The equation in the case of $g = 2$ will take the simpler form of as follows:

$$r - 2 = \frac{2}{N} + \sum_{j=1}^{r} \frac{1}{m_j}. \tag{1.1}$$

Clearly, as it has been shown in several papers, the only admissible value for r will be three. Thus, we have the following equation:

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{2}{N} = 1. \tag{1.2}$$

First of all, no two of the m_i's could be equal to two, this rules out all the dihedral groups. Thus, first suppose only $m_1 = 2$, to be the smallest generator. Then we must have $m_2 \geq 3$, $m_3 \geq 3$ and $N \geq 8$. Next, it can easily be checked that the only possible values for m_2 is either 3 or 4 which give the following equations for m_3:

$$\text{if } m_2 = 3, \text{ then } m_3 = \frac{6N}{N-12} \text{ and if } m_2 = 4, \text{ then } m_3 = \frac{4N}{N-8}. \tag{1.3}$$

However, the fractions on the right hand side of the above equations can only be an integer as a value for m_3 and give admissible desired triangle group, if and only if $N = 48,$ in the first case of (1.3) and $N = 24,$ or $N = 16$ in the second case of (1.3).

These values will give $m_3 = 8, 6, 8.$ These will give the minimal soluble bound $(2, 3, 8)$ group of order $48(g - 1)$ for $g = 2,$ see [6], the particular supersoluble bound $(2, 4, 6)$-group of order 24, see [7, 8] and the smallest maximal nilpotent automorphism $(2, 4, 8)$ triangle group of order $16(g - 1)$ for $g = 2,$ see [6], which are incidentally the minimum values of the bounds for the soluble, supersoluble and nilpotent automorphism groups of Riemann curves of genus $g \geq 2.$ Their orders of course will match the values of the bounds of $48(g - 1),$ $24(g - 1)$ and $16(g - 1)$ which are all attained for $g = 2.$ Next, we assume that the minimum value of the generators in (1.2) is greater than two. There, of course, will be only two possible values of $m_1 = 3$ and $m_1 = 4.$ If $m_1 = 3,$ then it is easy to show that $m_2 = 3$ is not admissible. But, if we let $m_2 = 4,$ we have the following equation for m_3:

$$m_3 = \frac{12N}{5N - 24}. \tag{1.4}$$

However, by elementary number theory, it is easy to show that the fraction on the right hand side is an integer and gives an admissible group, if and only if, $N = 12.$ This will of course give the unique $(3, 4, 4)$-group of order 12 which can actually be shown that, it is a semi-direct product of the cyclic groups Z_3 and $Z_4.$ Finally, the only other admissible value for the minimum value of the generators is four. If $m_1 = 4$ and if we let $m_2 = 4$ as well, then we will obtain the following equation for m_3:

$$m_3 = \frac{2N}{N - 4}. \tag{1.5}$$

Once again the right hand side of the above equation is an integer and gives an admissible group if and only if $N = 8$. This will give the $(4, 4, 4)$-group of order 8, which turns out to be isomorphic to the quaternion group. We have therefore obtained the totality of the integral solutions of the Diophantine equation (1.2).

2. Maximal Automorphism Groups of the Genus 2 Hyperelliptic Curves

In this section, we give the presentations as finitely presented, permutation representation, and matrix representation for all four groups obtained in Section 1.

Proposition 1. *As we noted in the first section, the first group is a* $(2, 3, 8)$ *soluble group of order* 48 *and has the following known finitely presented presentation*:

$$G_1 \cong \langle u, v \mid u^2 = v^8 = (uv)^3 = 1; uv^4 = v^4u \rangle. \tag{2.1}$$

The above group is determined by the observation that it is of genus two and admits an action on the cyclic group $\mathbb{Z}_8$. Its permutation representation as a subgroup of the symmetric group $\mathbb{S}_{16}$, has the following generators of orders two and eight with the commutator of order 6:

$$\begin{cases} U = (1, 2)(3, 5)(4, 7)(6, 10)(8, 13)(9, 15)(11, 14)(12, 16), \\ V = (1, 3, 6, 11, 14, 8, 4, 2)(5, 7, 12, 16, 13, 10, 15, 9), \\ UVUV^{-1} = (1, 4, 12, 14, 6, 15)(2, 9, 10, 11, 16, 7)(3, 8)(5, 13), \end{cases} \tag{2.2}$$

which also satisfy the same relations of (2.1) of course in $\mathbb{S}_{16}$.

The group G_1 is the rotation group of a map of 6 octagons coming together three at a vertex on a surface of genus two. Its Cayley color graph shown below has 56 edges and 48 vertices.

The Shreier's coset graph for this group is shown in Figure 1 where we have used two octagons for the generator of order 8 and two hexagons plus two double-sided edges for its commutator of order six.

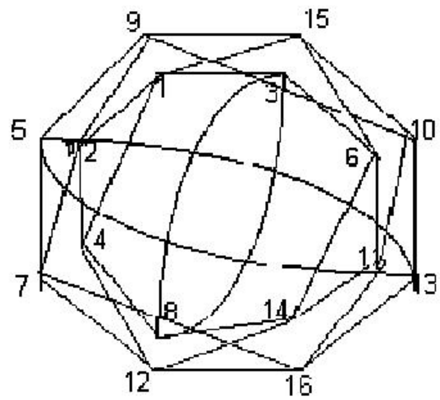


Figure 1

Moreover, G_1 is the smallest soluble automorphism group obtained from the maximal bound $48(g-1)$ for $g = 2$ and it is isomorphic to $GL(2, 3)$ (the general linear group over the Galois field $GF(3)$) and has been shown by Kulkarni [3] to be the full automorphism group of the Lefschetz curve with affine equation: $y^8 = x(x-1)^3(x+1)^4$. This equation with a change of variables of course can easily be transform to the following equation:

$$y^8 = (x-1)^3(x+1). \qquad (2.3)$$

Its character table is given in the following table:

Size	1	1	12	8	6	8	6	6
Order	1	2	2	3	4	6	8	8
X_3	2	2	0	–1	2	–1	0	0

Using the above table, we compute one of its representations of degree 2 over the cyclotomic field of degree 8. Following simple calculations, we obtain the following matrices for the generators of its representation of degree two as a subgroup of the general linear group $GL(2, F\langle\omega\rangle)$:

$$\left\langle \begin{bmatrix} \frac{1}{2}(\omega^3 - \omega) & \frac{1}{2}(\omega^3 + \omega) \\ -\frac{1}{2}(\omega^3 + \omega) & \frac{1}{2}(-\omega^3 + \omega) \end{bmatrix}, \begin{bmatrix} \frac{1}{2}(\omega^2 - 1) & \frac{1}{2}(\omega^2 + 1) \\ \frac{1}{2}(\omega^2 - 1) & -\frac{1}{2}(\omega^2 + 1) \end{bmatrix} \right\rangle, \qquad (2.4)$$

where ω is an 8th root of unity.

Next, using invariant theory, we can show that the primary invariants for this group are given by the two polynomials:

$$\langle I_1 = x^5y - xy^5,\ I_2 = x^8 + 14x^4y^4 + y^8\rangle \tag{2.5}$$

which have degrees six and eight. Since $6 \times 8 = 48$, all the invariants of this group can then be obtained using the above two polynomials. Any curve of genus two with equation involving $\langle I_1, I_2\rangle$ alone leaves all the birational transformations of the group (2.4) fixed. Over the Galois field of degree three, *magma* gives a curve of genus two with equation obtained by simply equating the above two invariants in (2.5). It is also interesting to note that the curve with equation $x^8 + 14x^4y^4 + y^8 = x^5y - xy^5$ has genus 6, when taken over the field $GF(p)$, $\forall p \equiv 1 \pmod 8$ and still gives the same group (2.1) for a full automorphism group.

We will show that the curve covered by the group (2.1) of order 48 is actually hyperelliptic. We can find an equation of the form $W^2 = F(Z)$, where $F(Z)$ is a polynomial of degree $2g + 1 = 5$.

To obtain an equation for the covering group (2.1) in the hyperelliptic form, we performed the following change of variables:

$$\left\langle x = \frac{-W^2 - 2Z}{W^2},\ y = \frac{\omega^2 Z\sqrt{2}}{W}\right\rangle. \tag{2.6}$$

Transform (2.2) to the following:

$$\boxed{W^2 = Z(Z^4 - 1)}. \tag{2.7}$$

We will now show that the hyperelliptic curve (2.7) of genus two actually admits our group of order 48 for a full automorphism group as follows.

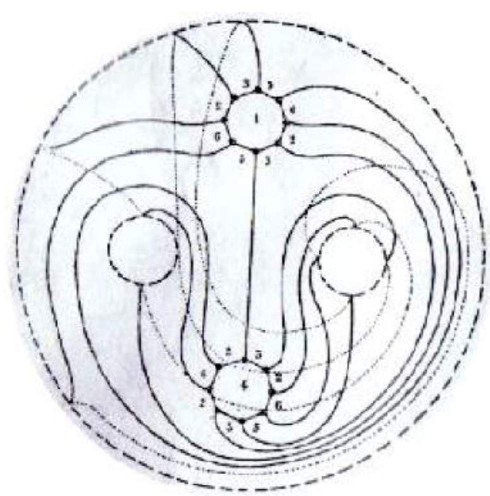
Suppose ω is as above a primitive 8th root of unity. First of all the above curve clearly admits the transformation: $v(x, y) = (\omega^2 x, \omega y)$, which is of order 8, and may be used for the generator v in the presentation (2.1). Now consider the birational transformation: $x' = \dfrac{x - \omega^2}{\omega^2 x - 1}$ and use equation (2.7) to compute y' as follows:

$$y'^2 = x'(x'^4 - 1) = \frac{8x(x^4 - 1)}{(1 - \omega^2 x)^6}.$$

Thus, we have the following two transformations:

$$\left\langle u(x,\ y) = \left(\frac{-x + \omega^2}{1 - \omega^2 x},\ \frac{-2y\sqrt{2}}{(1 - \omega^2 x)^3}\right),\ v(x,\ y) = (\omega^2 x,\ \omega y)\right\rangle. \tag{2.8}$$

It is not hard to note that the above birational transformations of orders two and eight transform the curve into itself and it is easily verified that the two transformations $\langle u(x,\ y),\ v(x,\ y)\rangle$ satisfy all the relations in (2.1). This proves our conjecture.



The color Cayley graph of G_1

Proposition 2. *Our second group of genus two is the smallest* $(2, 4, 8)$ *nilpotent automorphism group obtained from the bound* $16(g-1)$ *for nilpotent automorphism groups of Riemann surfaces given in* [6]. *It can be shown that this group is actually a subgroup of index* 3 *in the first group* G_1 *and can be generated by* $\langle a = v^6, b = v^2 u \rangle$. *The* 16 *vertices and* 24 *edges of the first group* G_1 *can be used as a Cayley diagram for this group*. *Its permutation representation as a subgroup of the symmetric group of degree* 8 *is given by the following permutations*:

$$A = (1, 2, 5, 4)(3, 7, 8, 6),$$

$$B = (1, 3, 7, 4, 5, 8, 6, 2).$$

These new set of generators satisfy the following relations:

$$G_2 = \langle a, b \,|\, a^4 = b^8 = (ab)^2 = I, ab^2a = b^2 \rangle. \tag{2.9}$$

Thus, the $(2, 4, 8)$-group of order 16, i.e., the smallest group obtained from the bound $16(g-1)$ for nilpotent automorphism groups in [6] also covers the same hyperelliptic curve given by (2.7). Moreover, the birational transformations are:

$$\langle a = v^6(x, y); b = v^2(x, y)u(x, y) \rangle. \tag{2.10}$$

Leave the curve (2.7) unchanged and also satisfy the relations (2.9).

Its permutation representation as a subgroup of the symmetric group of degree 16 is given by the permutations of orders 4 and 8 whose product has order 2:

$$a = (1, 9, 16, 5)(2, 13, 15, 3)(4, 12, 11, 14)(6, 8, 7, 10),$$

$$b = (1, 3, 8, 11, 16, 13, 10, 4)(2, 5, 12, 7, 15, 9, 14, 6).$$

The Cayley graph for this group is shown in Figure 2.

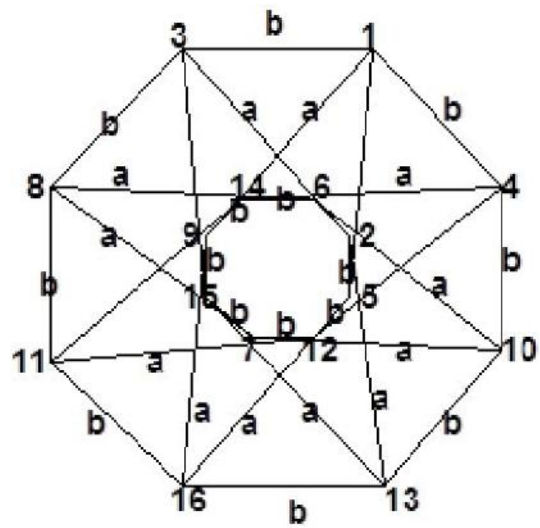


Figure 2

Proposition 3. *Our third genus two group is a semi-direct product of the cyclic groups* $\mathbb{Z}_3$ *and* $\mathbb{Z}_4$. *And it has the following presentation in generators and relations*:

$$\mathbb{Z}_3 \ltimes \mathbb{Z}_4 \cong \langle u, v \,|\, u^4 = v^4 = (uv)^3 = I;\, u^2 = v^2 = (uv^{-1})^3 \rangle. \qquad (2.11)$$

Using its permutation representation as a subgroup of the symmetric group of degree 12 namely the following two permutations:

$$\begin{cases} U = (1, 2, 6, 4)(3, 9, 5, 10)(7, 11, 8, 12), \\ V = (1, 3, 6, 5)(2, 7, 4, 8)(9, 12, 10, 11), \\ UV^{-1} = (1, 8, 9, 6, 7, 10)(2, 3, 11, 4, 5, 12). \end{cases} \qquad (2.12)$$

We find its character table as follows:

Size	1	1	2	3	3	2
Order	1	2	3	4	4	6
X_3	2	–2	–1	0	0	1

Then we compute its representation of degree 2 over the cyclotomic field F of degree 6. Following some elementary algebraic calculations, we obtain the following two matrices for the generators of its representation of degree two as a subgroup of $GL(2, F\langle\zeta\rangle)$, where ζ is a 6th root of unity:

$$\left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\zeta \\ -\zeta^2 & 0 \end{bmatrix} \right\rangle. \tag{2.13}$$

Once again, using invariant theory, it can be shown that the primary invariants for this group are the following two polynomials of degrees four and six:

$$\langle J_1 = x^2y^2,\ J_2 = x^6 + y^6 \rangle. \tag{2.14}$$

Trial and error leads us to the following 6th degree equation:

$$x^6 - x^2y^2 + y^6 = 0 \tag{2.15}$$

which leaves all the transformations of the group (2.11) fixed and it has genus two. Of course, the full automorphism group of the above curve turns out to be the $(2, 4, 6)$ supersoluble automorphism group of order 24 obtained in [7, p. 595] which contains our group (2.11) as its unique subgroup of index 2. Again we performed the following change of variables:

$$\left\langle x = \frac{Z}{W},\ y = \frac{Z^2}{W} \right\rangle. \tag{2.16}$$

On the algebraic curve with equation (2.15) and transform that equation into its hyperelliptic form, namely the equation:

$$\boxed{W^2 = Z^6 + 1}. \tag{2.17}$$

Next, we show that the above hyperelliptic curve of genus 2 has the supersoluble automorphism $(2, 4, 6)$-group obtained in [7], that is the group with the following presentation, for a full automorphism group:

$$\mathbb{Z}_3 \ltimes D_4 \cong \langle s, t \mid s^4 = t^6 = (st)^2 = (s^{-1}t)^2 = I \rangle. \tag{2.18}$$

First of all, the curve (2.17) clearly admits the transformation: $t(x,\ y) = (\zeta x,\ -y)$, which is of order 6, and may be used for the generator t in the above presentation.

Next, if we let $x' = \frac{\zeta}{x}$, then using the above equation, we obtain:

$$y'^2 = \frac{\zeta^6}{x^6} + 1 = \frac{x^6 + 1}{x^6} = \frac{y^2}{x^6}. \tag{2.19}$$

Thus, the transformation: $s(x,\ y) = \left(\frac{\zeta}{x},\ \frac{y}{x^3}\right)$ which is easy to prove is of order 4 leaves the curve unchanged and can serve as the generator s of order four in (2.18). Finally, the birational transformations $\langle s(x,\ y),\ t(x,\ y)\rangle$ do indeed satisfy the relations:

$$(st)^2 = (s^{-1}t)^2 = I$$

and leave the curve (2.17) unchanged.

The Cayley graph for this group is shown in Figure 3.

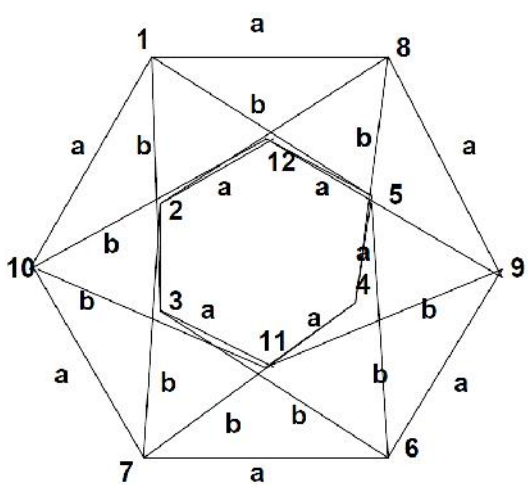


Figure 3

Proposition 4. *Finally, the last group of genus two, which we denote by* $\mathbb{Q}_8$, *is actually isomorphic to the quaternion group of order* 8.

Oddly enough, this group like the second group is also a subgroup of the first group and it is generated by: $\langle a = v^6,\ b = uv^6u \rangle$.

These generators of course satisfy the following relations of the quaternion group:

$$\mathbb{Q}_8 \cong \langle a,\ b \,|\, a^4 = b^4 = (ab)^4 = I;\ a^2 = b^2 = (ab)^2 \rangle. \qquad (2.20)$$

This final group has the following well-known unique representation of degree two:

$$a \to \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}; \quad b \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \text{ where } i = \sqrt{-1}. \qquad (2.21)$$

But, to find two birational transformations for the generators of its representation of degree 2, which satisfy the relations of $\mathbb{Q}_8$ and leave the curve (2.7) unchanged, we use:

$$w(x,\ y) = v^6(x,\ y) = (-x,\ -\omega^2 y); \quad r(x,\ y) = u \circ w \circ u(x,\ y) = \left(\frac{-1}{x},\ \frac{-y}{x^3}\right). \qquad (2.22)$$

Its permutation representation as a subgroup of $\mathbb{S}_8$ has the following generators:

$$\begin{cases} a = (1,\ 2,\ 6,\ 7)(8,\ 3,\ 4,\ 5), \\ b = (1,\ 4,\ 6,\ 8)(2,\ 3,\ 7,\ 5). \end{cases} \qquad (2.23)$$

The Cayley graph for this group is shown in Figure 4.

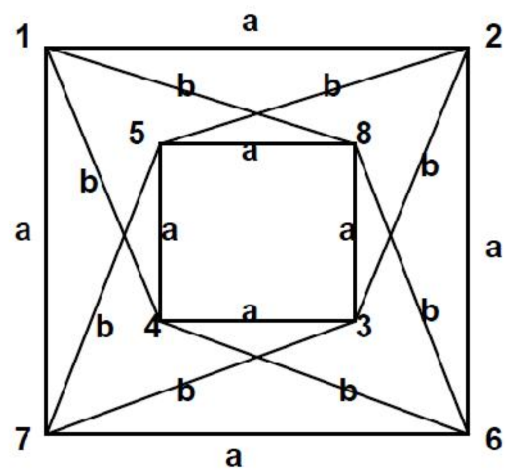


Figure 4

A view of one half of this surface of genus two divided into black and white triangles is given by the following figure:

This completes the discussion of all groups of genus two, their Cayley graph and their affine and hyperelliptic curves of genus 2 covered by these groups.

The following table summarizes our results:

No.	The presentation of genus two groups G	Curves of genus 2 covered by G	Order
1	$\langle u, v \mid u^2 = v^8 = (uv)^3 = 1; uv^4 = v^4 u \rangle$	$y^8 = (x-1)^3(x+1)$ or $W^2 = Z^5 - Z$	48
2	$\langle u, v \mid u^2 = v^8 = (uv)^4 = 1; uvu = v^3 \rangle$	$W^2 = Z^5 - Z$	16
3	$\langle u, v \mid u^4 = v^4 = (uv)^3 = I; u^2 = v^2 = (uv^{-1})^3 \rangle$	$x^6 + y^6 = x^2 y^2$ or $W^2 = Z^6 + 1$	12
4	$\langle u, v \mid u^4 = v^4 = (uv)^4 = I; u^2 = v^2 = (uv)^2 \rangle$	$W^2 = Z^5 - Z$	8

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