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GENERALIZED MAXIMUM PRINCIPLES FOR SOME DIFFERENTIAL INEQUALITIES WITH APPLICATIONS

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Abstract

We introduce maximum principles for some elliptic partial differential inequalities, and give some of its applications.

0. Introduction

We consider several types of differential equations and discuss the maximum principle for them. In general, the maximum principle tells us that the maximum value of the function, which is a solution of a differential equation, is attained at the boundary of the region. In this paper, we deal with elliptic equations. The most important and easy equation is the Laplace equation. The homogeneous version of Laplace's equation is

$$\Delta u = 0.$$

It is often written with minus sign since the (delta-operator) with this sign becomes strict monotone operator in the operator theory, which means that it has a unique solution. The non-homogeneous version of Laplace's equation

$$\Delta u = f$$

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is called *Poisson's equation*. It is convenient to include a minus sign here because Δ is a negative definite operator.

The Laplace and Poisson equations, and their generalizations, arise in many different contexts:

(1) Potential theory, e.g., in the Newtonian theory of gravity, electrostatics, heat flow, and potential flows in fluid mechanics.

(2) Riemannian geometry, e.g., the Laplace-Beltrami operator.

(3) Stochastic processes, e.g., the stationary Kolmogorov equation for Brownian motion.

(4) Complex analysis, e.g., the real and imaginary parts of an analytic function of a single complex variable are harmonic.

The classical Dirichlet problem for Poisson's equation: If D is a bounded domain in R^n for $n > 1$, then it is to find a function u such that

$$u : D \to R, \quad u \in C^2(D) \cap C(\overline{D})$$

and

$$\Delta = f \text{ in } D,$$

$$u = g \text{ on } \partial D.$$

In Section 1, we consider ordinary differential equations (one-dimensional) and study the maximum principle in various differential inequalities. In Sections 2-4, we discuss the maximum principle for multidimensional equations.

1. The Maximum Principle

A function $u(x)$ that is continuous on the closed interval $[a, b]$ takes on its maximum at a point on this interval. If $u(x)$ has a continuous second derivative, and if u has a relative maximum at some point c between a and b, then we know from elementary calculus that

$$u'(c) = 0 \quad \text{and} \quad u''(c) \leq 0. \tag{1.1}$$

Suppose that in an open interval (a, b), u is known to satisfy a differential inequality

$$L[u] \equiv u'' + g(x)u' > 0, \tag{1.2}$$

where $g(x)$ is any bounded function. Then it is clear that relation (1.1) cannot be satisfied at any point c in (a, b). Consequently, whenever (1.2) holds, the maximum of u in the interval cannot be attained anywhere except at the endpoints a or b. We have here the simplest case of a maximum principle.

An essential feature of the above argument is the requirement that the inequality (1.2) be strict; that is, we assume that $u'' + g(x)u'$ is never zero. In the study of differential equations and in many applications, such a requirement is overly restrictive, and it is important that we remove it if possible. We note, however, that for the nonstrict inequality $u'' + g(x)u' \geq 0$, the solution $u =$ constant is admitted. For a constant solution, the maximum is attained at every point. We shall prove that this exception is the only one possibility.

2. Maximum Principles for Elliptic Equations

2.1. The Laplace operator

Let $u(x_1, x_2, ..., x_n)$ be a twice continuously differentiable function defined in a domain D in n-dimensional Euclidean space. The Laplace operator or Laplacian Δ is defined as

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

If the equation $\Delta u = 0$ is satisfied at each point of a domain D, then we say that u is a *harmonic function*. Suppose that u has a local maximum at an interior point of D. Then we know that

$$\frac{\partial u}{\partial x_1} = 0,\ \frac{\partial u}{\partial x_2} = 0,\ \ldots,\ \frac{\partial u}{\partial x_n} = 0$$

and

$$\frac{\partial^2 u}{\partial x_1^2} \le 0,\ \frac{\partial^2 u}{\partial x_2^2} \le 0,\ \ldots,\ \frac{\partial^2 u}{\partial x_n^2} \le 0.$$

Therefore, at a local maximum, the inequality

$$\Delta u \le 0$$

must hold. If a function satisfies the strict inequality $\Delta u > 0,$ at each point of a domain D, then u cannot attain its maximum at any interior point of D. Suppose $b_1(x_1, x_2, \ldots, x_n), b_2(x_1, x_2, \ldots, x_n), \ldots, b_n(x_1, x_2, \ldots, x_n)$ are any bounded functions defined in D. Without any change in the argument above, we conclude that if u satisfies the strict inequality

$$\Delta u + b_1 \frac{\partial u}{\partial x_1} + b_2 \frac{\partial u}{\partial x_2} + \cdots + b_n \frac{\partial u}{\partial x_n} > 0$$

in D, then u cannot attain its maximum at an interior point.

2.2. Second-order elliptic operator transformations

We shall be concerned with second-order differential operators of the form $\sum_{i,\, j=1}^{n} \alpha_{ij}(x_1, x_2, \ldots, x_n).$ Since $\partial^2/\partial x_i \partial x_j \equiv \partial^2/\partial x_j \partial x_i,$ we may define

$$a_{ij} = \frac{1}{2}(\alpha_{ij} + \alpha_{ji})$$

and write the above differential expression as

$$\mathcal{L} \equiv \sum_{i,\, j=1}^{n} \alpha_{ij}(x_1, x_2, \ldots, x_n) \frac{\partial^2}{\partial x_i \partial x_j}, \quad a_{ij} = a_{ji}, \quad i, j = 1, 2, \ldots, n.$$

In other words, there is no loss of generality in supposing that the coefficients of the second-order operator $\mathcal{L}$ are symmetric.

The operator $\mathcal{L} \equiv \sum_{i,\, j=1}^{n} \alpha_{ij}(x_1, x_2, ..., x_n) \frac{\partial^2}{\partial x_i \partial x_j}$, $a_{ij} = a_{ji}$, $i, j = 1, 2, ..., n$ is called *elliptic* at a point $X = (x_1, x_2, ..., x_n)$ if and only if there is a positive quantity $\mu(X)$ such that

$$\sum_{i,\, j=1}^{n} a_{ij}(X)\xi_i \xi_j \geq \mu(X) \sum_{i=1}^{n} \xi_i^2$$

for all *n*-tuples of real numbers $(\xi_1, \xi_2, ..., \xi_n)$. The operator *L* is said to be *elliptic* in a domain *D* if it is elliptic at each point of *D*. It is uniformly elliptic in *D* if $\left(\sum_{i,\, j=1}^{n} a_{ij}(X)\xi_i \xi_j \geq \mu(X) \sum_{i=1}^{n} \xi_i^2\right)$ holds for each point of *D* and if there is a positive constant μ_0 such that $\mu(X) \geq \mu_0$ for all *X* in *D*.

2.3. The maximum principle of Hopf [2]

Consider the strict differential inequality

$$L[u] \equiv \sum_{i,\, j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} > 0$$

in a domain *D*, and assume that *L* is elliptic in *D*. If *u* has a relative maximum at a point $\overline{X} = (\overline{x}_1, \overline{x}_2, ..., \overline{x}_n)$, then we know from the calculus of several variables that at $\overline{X}$,

$$\frac{\partial u}{\partial z_k} = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial z_k^2} \leq 0, \quad k = 1, 2, ..., n$$

for any coordinates $z_1, z_2, ..., z_n$ obtained from the coordinates $x_1, x_2, ..., x_n$ by a linear transformation. In particular, if $\mathcal{L}$, the principal part of *L*, is the Laplace operator in *z*-coordinates, then

$$L[u] \equiv \sum_{i,\, j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} > 0$$

cannot hold at $\overline{X}$. Whenever *L* is elliptic, we can find a linear transformation of coordinates so that at $\overline{X}$, the operator $\mathcal{L}$ becomes the Laplace operator. We conclude that if *L* is elliptic, then a function *u* which satisfies $L[u] > 0$ in a domain *D* cannot have a maximum in *D*. As in the one-dimensional case, we shall extend the maximum principle to include the possibility that $L[u]$ satisfies an inequality which may not be strict.

Theorem 2.1 [5]**.** *Let* $(x_1, x_2, ..., x_n)$ *satisfy the differential inequality*

$$L[u] \equiv \sum_{i,\, j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} \geq 0$$

in a domain D, where L is uniformly elliptic. Suppose the coefficients a_{ij} *and* b_i *are uniformly bounded. If u attains maximum at a point of D, then* $u(x) \equiv M$ *in D.*

Theorem 2.2 [3]**.** *Let u satisfy the differential inequality*

$$(L + h)[u] \geq 0$$

with $h \leq 0$, *L uniformly elliptic in D, and the coefficients of L and h bounded. If u attains a nonnegative maximum M at an interior point of D, then* $u \equiv M$.

Remark 2.1. The restriction $h \leq 0$ is essential, as a counterexample for if $h > 0$.

Example 2.1 [3]**.** The function $u = e^{-r^2}$ has an absolute maximum at $r = 0$ and is a solution of the equation $\Delta u + (2n - 4r^2)u = 0$ in *n* dimensions.

Let u satisfy $L[u] \geq 0$ in a domain D with a smooth boundary ∂D. We know that if u takes on a maximum at all, then it must do so at a boundary point. We shall now suppose that u is continuous and bounded on $D \cup \partial D$ and that there is a point P on ∂D at which u takes on its maximum value. If D is bounded, then such a point P will always exist. First of all, we observe that the directional derivative of u at P taken in any direction on the boundary that points outward from cannot be negative. If it were, the function u would start increasing as we enter the domain D at P, and so the maximum could not occur at P. Let $n = (\eta_1, \eta_2, ..., \eta_n)$ be the unit normal vector in an outward direction at a point P on the boundary of D. We say that the vector $v = (v_1, v_2, ..., v_n)$ points *outward* from D at the boundary point P if

$$v \cdot n < 0.$$

We define the directional derivative of u at the boundary point P in the direction v as

$$\frac{\partial u}{\partial v} \equiv \lim_{x \to P} [v \cdot grad\, u(x)] = \lim_{x \to P} \left(v_1 \frac{\partial u}{\partial x_1} + \cdots + v_n \frac{\partial u}{\partial x_n} \right),$$

if it exists. The directional derivative is said to be *outward* if v points outward from D. Then, if u has a maximum at P, we have $\partial u / \partial v \geq 0$ at P. We shall now show that unless u is a constant, the strict inequality $\partial u / \partial v \geq 0$ holds at P.

Theorem 2.3 [5]**.** *Let u satisfy the inequality*

$$L[u] \equiv \sum_{i,\, j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} \geq 0$$

in a domain D in which L is uniformly elliptic. Suppose that $u < M$ *in D and that* $u = M$ *at a boundary point P. Assume that P lies on the boundary of a ball* ∂K_1 *in D. If u is continuous in* $D \cup P$ *and an outward directional derivative* $\partial u / \partial v$ *exists at P, then*

$$\frac{\partial u}{\partial v} > 0 \ \ at\ P$$

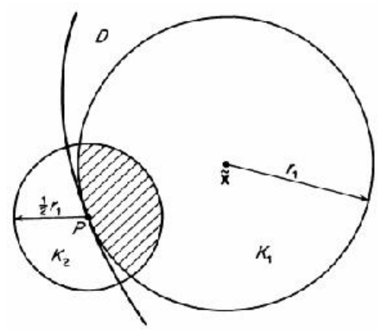
unless $u \equiv M$.

Proof. By shrinking K_1 slightly if necessary, we may assume that ∂K_1 lies entirely in $D \cup P$. Construct a ball K_2 with center at *P* and radius $\frac{1}{2} r_1$, where r_1 is the radius of K_1. We illustrate this figure below. We define the function *z* again as

$$z(x) = e^{-\alpha \sum_{i=1}^{n} (x_i - \tilde{x}_i)^2} - e^{-\alpha r_1^2}$$

selecting α so large that $L[z] > 0$ in K_2. The function $w = u + \varepsilon z$ is now formed. According to Theorem 2.1, if $u \neq M$, then $u < M$ in K_1, and on its boundary except at the point *P*. We recall that $z = 0$ on the boundary of K_1. We select $\varepsilon > 0$ so small that $w < M$ on the portion of the boundary of K_2, lying in K_1. Then $w < M$ on the entire boundary of the shaded region shown in the figure above. Because $L[w] > 0$ in this region, the maximum of *w* occurs at *P* and $w(P) = M$. Therefore, at *P*,

$$\frac{\partial w}{\partial v} = \frac{\partial u}{\partial v} + \varepsilon \frac{\partial z}{\partial v} \geq 0.$$



We shall now show that $\partial z/\partial v < 0$ at *P*, so that $\partial u/\partial v > 0$ at *P*. Selecting $\tilde{x}$ as the origin of our coordinate system and letting *r* represent

Euclidean distance from $\tilde{x}$, we have

$$z = e^{-\alpha r^2} - e^{-\alpha r_1^2}.$$

Then

$$\frac{\partial z}{\partial x_i} = -2\alpha x_i e^{-\alpha r^2}$$

and

$$\eta_i = \frac{x_i}{r}.$$

Hence

$$\frac{\partial z}{\partial x_i} = -2\alpha r e^{-\alpha r^2} \sum_{i=0}^{n} v_i \eta_i < 0.$$

Therefore, $\partial u / \partial v > 0$, establishing the conclusion of the theorem.

Theorem 2.4. *Let u satisfy the inequality*

$$(L + h)[u] \equiv \sum_{i,\, j=1}^{n} a_{i,\, j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + h(x)u \geq 0,$$

and $h \leq 0$ in D. Suppose that $u \leq M$ in D, that $u = M$ at a boundary point P, and that $M \geq 0$. Assume that P lies on the boundary of a ball in D. If u is continuous in $D \cup P$, then any outward directional derivative of u at P is positive unless $u \equiv M$ in D.

Proof. The proof of this theorem follows exactly the same lines as that of Theorem 2.3 when $h \leq 0$ and $M \geq 0$.

This time choose α so large that $(L + h)[z] > 0$ in K_2 (this was shown to be possible in the previous theorem).

Proceed as before to define *w* and ε suitably and conclude again that at *P*,

$$\frac{\partial w}{\partial v} = \frac{\partial u}{\partial v} + \varepsilon \frac{\partial z}{\partial v} \geq 0.$$

Defining $\tilde{x}$ and r^2 as before, we again conclude that

$$\frac{\partial z}{\partial x_i} = -2\alpha x_i e^{-\alpha r^2}.$$

Finally, we again note that $\eta_i = \frac{x_i}{r}$, and conclude that

$$\frac{\partial z}{\partial v} = \sum_{i=1}^{n} \left(v_i \frac{\partial z}{\partial x_i} \right) = -2\alpha r e^{-\alpha r^2} \sum_{i=1}^{n} v_i \eta_i < 0$$

as before. Again, this shows that $\frac{\partial u}{\partial v} > 0$, which proves the theorem.

Example 2.2 [5]**.** A solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u^2 = 0$$

in a domain $D \subset \mathbb{R}^2$ cannot attain a maximum in D unless $u \equiv 0$, because

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u^2 = 0$$

reduces to

$$\Delta u = u^2.$$

Now suppose that u attains a non-zero maximum in D. By basic calculus, $\Delta u \leq 0$ at a local maximum. But u non-zero gives $u^2 > 0$, and the above equation then forces $\Delta u > 0$, which is a contradiction. Now suppose that u attains a maximum at some point $u(d) = 0$. Then $\Delta u = 0$. Hence u is a constant, and since $u(d) = 0$, we have $u \equiv 0$.

Example 2.3. We show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u^3$$

in the domain $D : x^2 + y^2 < 1$ with

$$u = 0 \text{ for } x^2 + y^2 = 1$$

has no solution other than $u \equiv 0$.

Solution. Suppose that $u > 0$ at some point in D. Then $u^3 > 0$ and hence $\Delta u = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) > 0$ since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u^3$. Since u is continuous, each point at which $u > 0$ lies in an open ball contained entirely in D, and such that $u(x) > 0$ at each point x in B. Hence for each $u > 0$ in D, the above equations hold in an open connected subset of D (a domain). We consider the largest possible such domain in each case. Then by Theorem 2.1, since $u = 0$ on ∂D and by continuity on the boundary of each of the domains described above, $u \leq 0$ on each of these domains. Hence there are no points at which u is strictly positive.

Suppose now that $u < 0$. Then $u^3 < 0$ and hence $\Delta u < 0$ for the same reasons as before. We again construct the relevant domains (this time the domains in which $u < 0$). Applying Theorem 2.1 to $(-u)$, we see as above that $(-u) \leq 0$ on each of the described domains. Hence there are no points at which u is strictly negative.

3. The Generalized Maximum Principle

The condition $h(x) \leq 0$ in Theorem 2.4 cannot be removed entirely. As in Section 1, the methods used to prove a maximum principle with $h \leq 0$ can be extended to establish a generalized maximum principle.

Theorem 3.1 [5]**.** *Let* $u(X)$ *satisfy the differential inequality*

$$(L+h)[u] \equiv \sum_{i,\, j=1}^{n} a_{i,\, j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + h(x)u \geq 0$$

in a domain D, where L is uniformly elliptic. If there exists a function $w(X)$ *such that*

$$w(X) > 0 \text{ on } D \cup \partial D,$$

$$(L+h)[w] \leq 0 \text{ in } D,$$

then $u(X)/w(X)$ *cannot attain a nonnegative maximum in D unless it is a constant. If* $u(X)/w(X)$ *attains its nonnegative maximum at a point P on* ∂D *which lies on the boundary of a ball in D and if* u/w *is not constant, then*

$$\frac{\partial}{\partial v}\left(\frac{u}{w}\right) > 0 \text{ at } P,$$

where $\partial/\partial v$ *is any outward directional derivative.*

Proof. We now give a specific method for determining a function $w(X)$ having properties $w > 0$ on $D \cup \partial D$ and $(L+h)[w] \leq 0$ in D, provided the domain D is contained in a sufficiently narrow slab bounded by two parallel hyperplanes. Suppose that the bounded domain D is contained in a slab $a < x_1 < b,$ where x_1 is the first coordinate of $X = (x_1, x_2, \ldots, x_n);$ we set

$$w(X) = 1 - \beta e^{\alpha(x_1 - a)},$$

where α and β are to be determined. A computation shows that

$$(L+h)[w] = -\beta[\alpha^2 a_{11}(X) + \alpha b_1(X) + h(X)]e^{\alpha(x_1 - a)} + h(X).$$

By the uniform ellipticity hypothesis, $a_{11} \geq \mu_0.$

We suppose that $h(X)$ is bounded and that $b_1(X)$ is bounded from below; that is,

$$-m \le h(X) \le M,$$

$$-m \le b_1(X),$$

where m and M are nonnegative. We choose α so large that

$$\alpha^2\mu_0 - (\alpha + 1)m > 0.$$

Then we select

$$\beta = \frac{M}{\alpha^2\mu_0 - (\alpha + 1)m}.$$

Under these circumstances,

$$(L + h)[w] \le 0 \text{ on } D \cup \partial D.$$

However, to insure that $w > 0$ on $D \cup \partial D,$ we must have

$$\beta e^{\alpha(b-a)} < 1.$$

That is, the inequality

$$M < [\alpha^2\mu_0 - (\alpha + 1)m]e^{-\alpha(b-a)}$$

must be satisfied. We are still free to increase the size of α if we wish. We may choose α so that the right side of the last inequality is a maximum. Notice that the right side becomes larger as $b - a$ becomes smaller. Also, last inequality becomes less restrictive as M, the maximum of $h(x)$, becomes smaller.

Let w be positive on $D \cup \partial D,$ and define

$$v(x) = \frac{u(x)}{w(x)}.$$

Then computing $(L+h)[u]$, we see that

$$(L+h)[u] \equiv \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2(vw)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial(vw)}{\partial x_i} + h(x)(vw)$$

$$= \sum_{i,j=1}^{n} a_{ij}(x)\left(\frac{\partial^2 w}{\partial x_i \partial x_j} + 2\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j} + w\frac{\partial^2 v}{\partial x_i \partial x_j}\right)$$

$$+ \sum_{i=1}^{n} b_i(x)\left(w\frac{\partial v}{\partial x_i} + v\frac{\partial w}{\partial x_i}\right) + h(x)(vw)$$

$$= w\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\left\{2\sum_{j=1}^{n} a_{ij}(x)\frac{\partial w}{\partial x_j} + b_i(x)w\right\}\frac{\partial v}{\partial x_i}$$

$$+ \left(\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial w}{\partial x_i} + h(x)w\right)v$$

$$= w\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\left\{2\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 w}{\partial x_j} + b_i(x)w\right\}\frac{\partial v}{\partial x_i}$$

$$+ (L+h)[w]v \geq 0.$$

Dividing through by $\frac{1}{w}$ gives

$$\frac{1}{w}(L+h)[w] = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\left\{\frac{2}{w}\sum_{j=1}^{n} a_{ij}\frac{\partial w}{\partial x_j} + b_i\right\}\frac{\partial v}{\partial x_i}$$

$$+ \frac{1}{w}(L+h)[w]v \geq 0.$$

According to the properties of w, we conclude that the maximum principle as in the previous theorems holds for $v(x)$.

4. Applications of the Maximum Principle

4.1. The *P*-method [1]

We now deal with a possible application of the maximum principle, namely the P function method. The method consists of determining a function

$$P = P(x, u, \nabla u, ...)$$

satisfying a maximum principle, i.e.,

$$\max_{x \in \overline{\Omega}} P = \max_{x \in \partial\Omega} P,$$

where u is a solution of the studied equation (boundary value problem).

We start with a simple example. We consider the one-dimensional equation

$$u'' + 2 = 0 \text{ in } D = (0, \alpha)$$

and multiply it by $2u'$ and then integrate. We get

$$(u')^2 + 4u \equiv \text{const. in } \Omega.$$

Hence, we can define the function

$$P_1 = (u')^2 + 4u$$

satisfying a maximum principle, i.e., the function P_1 takes its maximum value either at a critical point of u or at some point on the boundary, unless it is a constant. This function P_1 is the one-dimensional version of

$$P_1 = (|\nabla u|)^2 + 4u.$$

This function is related to the torsion problem (the St.-Venant problem):

$$\begin{cases} \Delta u = -2 & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases}$$

The proof follows from the differential inequality

$$\Delta P_1 + \frac{1}{|\nabla u|^2}\left\{4\nabla P_1 \cdot \nabla u + \frac{1}{2}|\nabla P_1|^2\right\} \geq 0 \ \text{ in } D,$$

and the maximum principle.

Similarly the function

$$P_2 = (|\nabla u|)^2$$

attains its maximum on the boundary. We can actually prove the following result: The function

$$P_3 = (|\nabla u|)^2 + \frac{4}{n}u$$

takes its maximum value at some point on the boundary, unless P_3 is a constant. Moreover, P_3 is an identically constant in if and only if is an n dimensional ball.

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