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# STUDY OF GENERALIZED ZK-BBM EQUATION TO CONSTRUCT SOLITARY PATTERNS SOLUTIONS VIA VARIATIONAL ITERATION METHOD 

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#### Abstract

The variational iteration method (VIM) is used for the solution of generalized Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZK-BBM) equation subject to the appropriate initial condition. The numerical solution of the generalized ZK-BBM equation shows a solitary pattern solution. The solution obtained by variational iteration method is compared with the exact solution which shows that variational iterate solution gives almost exact solution. The convergence analysis of VIM solution of generalized ZK-BBM equation shows that the solution is convergent. The modified VIM iterate formula is also proposed for the solution of generalized ZK-BBM equation. It has been shown that VIM performs extremely well in terms of accuracy, efficiency, simplicity and can be used for the solution of many non-linear evolution equations in mathematical physics.


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## 1. Introduction

The mathematical modeling of many physical systems leads to the nonlinear evolution equations. The dispersive, dissipative and diffusion effects in the physical problem makes it more complex. The non-linear evolution equations arise in physical science like fluid dynamics, mathematical biology, plasma physics, chemical reaction and optical fibers etc. Many nonlinear evolution equations have solitary pattern solution which has attracted the attention of several authors to investigate them in details. Zabusky and Kruskal [22] in 1965 coined the term soliton in study of a non-linear anharmonic oscillator from the continuum point of view. They showed that these solutions preserve their shape and velocities after two of them collide, interact and then spread apart again. Many authors have reported the solitary pattern solution of Benjamin-Bona-Mahony (BBM), Modified Benjamin-Bona-Mahony (MBBM), Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZKBBM), Kadomtsev-Petvishvili-Benjamin-Bona-Mahony (KPBBM), Caudrey-Dodd-Gibbon (CDG) and Korteweg-de Vries (KdV) equations getting motivation from application in real life problems.

The Benjamin-Bona-Mahony (BBM) equation [4]

$$
\begin{equation*}
u_{t}+a u_{x}+b u u_{x}-c u_{x x t}=0, \tag{1}
\end{equation*}
$$

where $a, b$ and $c$ are real constants, $u(x, t)$ is non-periodic functions defined on $-\infty<x<\infty, t \geq 0$, was proposed in 1972 to incorporate the non-linear dispersion effect in study of long waves of small but finite amplitude. Zakharov and Kuznetsov [23], in 1974 proposed a generalization of KdV equation to describe the behavior of weakly non-linear ion-sound oscillation in a low pressure magnetized plasma. The ZK equation was given by

$$
\begin{equation*}
u_{t}+a u u_{x}+\left(u_{x x}+u_{y y}+u_{z z}\right)_{x}=0, \tag{2}
\end{equation*}
$$

where the $u$ is non-dimensional ion velocity in the problem of ion sound wave propagation along the magnetic field and $a$ is real constant. Wazwaz have combined the BBM equation and ZK equation in sense of ZK-model and solved the resulting generalized form of $(2+1)$ dimensional ZK-BBM
equations

$$
\begin{equation*}
u_{t}+u_{x}+a\left(u^{n}\right)_{x}+b\left(u_{x t}+u_{y y}\right)_{x}=0, \quad n>1 \tag{3}
\end{equation*}
$$

in 2005 by tanh method and sin-cosine method [17] and

$$
\begin{equation*}
u_{t}+u_{x}-a\left(u^{n}\right)_{x}-\left(b\left(u^{n}\right)_{x t}+k\left(u^{n}\right)_{y t}\right)_{x}=0, \quad n>1 . \tag{4}
\end{equation*}
$$

in 2008 by extended tanh method [18]. Khan et al. [10] reported the solitary wave solutions to the generalized ZK-BBM equation by modified simple equation method. Guner et al. [5] found the bright and dark soliton solution of generalized ZK-BBM equation.

An effective method to analyze the mathematical model conforming a physical reality is always required. Many analytic methods linearize the governing equations or make assumption of insignificant non-linearity. It is desirable to get an exact solution of non-linear problems but it is not always possible. In recent years, many powerful mathematical methods such as inverse scattering [1], tanh-sech method [11, 19], extended tanh method [20], sine-cosine method [21], variational iteration method (VIM) [8, 10, 16] and many others have been proposed for exact and approximate solutions of nonlinear problems. He [6, 7] proposed variational iteration method in 1998 and applied systematically in 1999 [8]. This method was applied for solution of Helmholtz equation [12], non-linear differential equation of fractional order [13], non-linear fluid mechanics problem [3], non-linear thermoelasticity [15] successfully. Jafari et al. [9] proposed modified variational iteration method in 2013 for the solution of Riccati differential equation and to enlarge the convergence region of iterative approximate solution. Variational iteration method, neither requires linearization nor perturbation for the solution of the problems. The VIM gives approximate analytic or closed form analytic solution of the non-linear evolution equations.

This paper uses the variational iteration method for solution of generalized ZK-BBM equation under an initial condition. The variational iteration formula for solution of generalized ZK-BBM equation is derived in Section 4. In Section 5, we have obtained the numerical solution of generalized ZK-BBM equation by variational iteration formula and
compared with the exact solution obtained in [5]. The convergence analysis and modification of variational iteration method for the solution of generalized ZK-BBM equation are discussed in Subsections 5.3 and 5.4, respectively. The result and discussion given in Section 7 justify that variational iteration method can be used to study the solitary wave solution of generalized ZK-BBM equation.

## 2. The Generalized ZK-BBM Equation and Condition

For the study of long waves of small but finite amplitude in sense of ZKmodel, the generalized ZK-BBM equation [5] is taken as

$$
\begin{equation*}
u_{t}+u_{x}+\alpha\left(u^{n}\right)_{x}+\beta\left(u_{x t}+u_{y y}\right)_{x}=0, \quad n>1, \tag{5}
\end{equation*}
$$

where parameters $\alpha, \beta$ are real constants. This is a typical non-linear evolution equation due to the non-linear term $\left(u^{n}\right)_{x}$. Since this equation has odd order derivatives $u_{x x t}$ and $u_{y y x}$, therefore it incorporate the dispersion effect. The interaction of dispersion effect and non-linear effect has to be investigated. We study generalized ZK-BBM equation (5) by variational iteration method for $n=2$, under the initial condition

$$
\begin{equation*}
u(x, y, 0)=\gamma \operatorname{sech}^{2}(a x+b y), \tag{6}
\end{equation*}
$$

where $a, b$ are real constants and $\gamma=6 \beta\left(b^{2}-a^{2}\right) / \alpha\left(1+4 a^{2} \beta\right)$ is the amplitude of the soliton and $\alpha\left(1+4 a^{2} \beta\right) \neq 0$, see Guner et al. [5]. However, for $n=3$, the initial condition is taken as

$$
\begin{equation*}
u(x, y, 0)=\gamma \operatorname{sech}(a x+b y), \tag{7}
\end{equation*}
$$

where $a, b$ are real constants and $\gamma= \pm \sqrt{2 \beta\left(b^{2}-a^{2}\right) / \alpha\left(1+a^{2} \beta\right)}$ is the amplitude of the soliton, see Guner et al. [5]. The main aim of this paper is to obtain solitary wave solutions of generalized ZK-BBM equation (5) and to verify the application of variational iteration method.

## 3. Basic Idea of He's Variational Iteration Method (VIM)

To clarify the basic ideas of variational iteration method, we consider the following differential equation:

$$
\begin{equation*}
L u(x, y, t)+N u(x, y, t)=g(x, y, t) \tag{8}
\end{equation*}
$$

where $L$ is a linear operator, $N$ is a non-linear operator, and $g$ is an inhomogeneous term. According to the VIM, one can write down a correctional functional as follows:

$$
\begin{align*}
& u_{k+1}(x, y, t) \\
= & u_{k}(x, y, t)+\int_{0}^{t} \lambda\left\{L u_{k}(x, y, \xi)+N \tilde{u}_{k}(x, y, \xi)-g(x, y, \xi)\right\} d \xi \tag{9}
\end{align*}
$$

where $\lambda$ is a general Lagrangian multiplier which can be identified optimally via the variational theory. The subscript $k$ indicates the $k$ th approximation, and $\tilde{u}_{k}$ is considered as a restricted variation, i.e., $\delta \tilde{u}_{k}=0$.

We determine the Lagrangian multiplier $\lambda$ optimally via integration by part of integral using restricted variation. The successive approximation $u_{k}(t), k \geq 0$ of the solution $u(x, y, t)$ will be calculated by choosing the initial approximation $u_{0}$ to satisfy the initial and boundary conditions of the problem. Consequently, the exact solution of the problem, if exist, may be obtained as

$$
\begin{equation*}
u(x, y, t)=\lim _{k \rightarrow \infty} u_{k}(x, y, t) \tag{10}
\end{equation*}
$$

The solution of the problem by VIM and its convergence depends on the initial approximation $u_{0}$. Some modifications of VIM to improve the convergence speed and to increase interval of convergence for VIM series solution are suggested in $[2,16,14]$. For approximate solution of non-linear problem (8), we use the $k$ th order iteration $u_{k}(x, y, t)$. The existence of the limit function $u(x, y, t)$ in equation (10) is discussed in Subsection 5.3.

## 4. Solution of Generalized ZK-BBM Equation by Variational Iteration Method

For the solution of generalized ZK-BBM equation (5) by the variational iteration method, we write the correction functional for equation (5) as

$$
\begin{align*}
& u_{k+1}(x, y, t) \\
&= u_{k}(x, y, t)+\int_{0}^{t} \lambda\{ \\
&+\beta\left(\frac{\partial u_{k}(x, y, \xi)}{\partial \xi}+\frac{\partial \tilde{u}_{k}(x, y, \xi)}{\partial x}+n \alpha \tilde{u}_{k}^{n-1} \frac{\partial \tilde{u}_{k}(x, y, \xi)}{\partial x}\right.  \tag{11}\\
&=(11
\end{align*}
$$

where $\delta \tilde{u}_{k}$ is considered as a restricted variation. Taking the variation of the above correctional functional, we get

$$
\begin{aligned}
& \delta u_{k+1}(x, y, t) \\
= & \delta u_{k}(x, y, t)+\delta \int_{0}^{t} \lambda(\xi)\left\{\frac{\partial u_{k}(x, y, \xi)}{\partial \xi}+\frac{\partial \tilde{u}_{k}(x, y, \xi)}{\partial x}\right. \\
& \left.+n \alpha \tilde{u}_{k}^{n-1} \frac{\partial \tilde{u}_{k}(x, y, \xi)}{\partial x}+\beta\left(\frac{\partial^{3} \tilde{u}_{k}(x, y, \xi)}{\partial x^{2} \partial \xi}+\frac{\partial^{3} \tilde{u}_{k}(x, y, \xi)}{\partial y^{2} \partial x}\right)\right\} d \xi .
\end{aligned}
$$

Applying the optimal variational theory and using the restricted variation $\delta \tilde{u}_{k}=0$, we obtain

$$
\delta u_{k+1}(x, y, t)=\delta u_{k}(x, y, t)+\delta \int_{0}^{t} \lambda(\xi)\left(\frac{\partial u_{k}(x, y, \xi)}{\partial \xi}\right) d \xi .
$$

Now on integration by parts, we obtain

$$
\delta u_{k+1}(x, y, t)=\left(1+\left.\lambda(\xi)\right|_{\xi=t}\right) \delta u_{k}(x, y, t)-\int_{0}^{t} \frac{d \lambda}{d \xi} \delta u_{k}(x, y, \xi) d \xi .
$$

For the stationary conditions, we must have $\delta u_{k+1}=0$, for arbitrary $\delta u_{k}$.
This gives the stationary conditions $1+\left.\lambda(\xi)\right|_{\xi=t}=0$ and $\frac{d \lambda}{d \xi}=0$, which in
turn implies that

$$
\begin{equation*}
\lambda(\xi)=-1 \tag{12}
\end{equation*}
$$

Therefore, we get the following correction variational functional in $x, y$ and $t$, as

$$
\begin{aligned}
& u_{k+1}(x, y, t) \\
=u_{k}(x, y, t)-\int_{0}^{t}\{ & \left\{\frac{\partial u_{k}(x, y, \xi)}{\partial \xi}+\frac{\partial u_{k}(x, y, \xi)}{\partial x}+n \alpha u_{k}^{n-1} \frac{\partial u_{k}(x, y, \xi)}{\partial x}\right. \\
& \left.+\beta\left(\frac{\partial^{3} u_{k}(x, y, \xi)}{\partial x^{2} \partial \xi}+\frac{\partial^{3} u_{k}(x, y, \xi)}{\partial y^{2} \partial x}\right)\right\} d \xi, \quad k \geq 0
\end{aligned}
$$

for the solution of the generalized ZK-BBM equation (5).

## 5. Numerical Solution of the Generalized ZK-BBM Equation

For, numerical solution of the generalized ZK-BBM equation (5) under the initial conditions (6) and (7), we choose the zeroth order approximate $u_{0}(x, y, t)$ equal to the initial conditions and compute the higher order approximate (or exact) solution from equation (13) in following subsections:

### 5.1. Case $n=2$

For the VIM solution of the generalized ZK-BBM equation under the initial condition (6), we obtained the following iterative approximations:

$$
\begin{aligned}
u_{0}(x, y, t)= & \gamma \operatorname{sech}^{2}(a x+b y), \\
u_{1}(x, y, t)= & \gamma \operatorname{sech}^{2}(a x+b y)+2 \gamma a \operatorname{sech}^{2}(a x+b y) \tanh (a x+b y) t \\
& +4 \gamma^{2} \alpha a \operatorname{sech}^{4}(a x+b y) \\
& \times \tanh (a x+b y) t+24 \gamma \beta a b^{2} \tanh ^{3}(a x+b y) \operatorname{sech}^{2}(a x+b y) t \\
& -16 \gamma \beta a b^{2} \operatorname{sech}^{2}(a x+b y) \tanh (a x+b y) t .
\end{aligned}
$$

Similarly, we have computed the second and third approximate solution $u_{2}(x, y, t)$ and $u_{3}(x, y, t)$, respectively, using the software package Mathematica. The higher approximation can also be computed on similar way. From Guner et al. [5], the solution of equation (5) exists for the inequality

$$
\begin{equation*}
\alpha \beta\left(b^{2}-a^{2}\right)\left(1+4 a^{2} \beta\right)>0 . \tag{14}
\end{equation*}
$$

Therefore, for numerical computation of the solution by VIM, we have chosen the constant parameters $\alpha, \beta, a, b \in \mathbb{R} \backslash\{0\}$, so that the inequality (14) must be satisfied. The solution of generalized ZK-BBM equation (5) for $n=2$ under the initial condition (6) have been computed numerically for third iteration $u_{3}(x, y, t)$ and is presented in form of Figures 1-2 and Tables $1-2$ for $\alpha>0, \beta>0$ and $\alpha>0, \beta<0$ and is compared with the exact solution [5]. The numerical solution for the other values of $\alpha$ and $\beta$ has the similar solitary pattern solution.

Table 1. Comparison of VIM solution $u_{3}(x, y, t)$ with exact solution $u(x, y, t)$ for the fixed values of $\alpha=1, \beta=1, a=0.01, b=1, x=0$, $t=0.5$

| $y$ | Exact $u(0, y, 0.5)$ | VIM $u_{3}(0, y, 0.5)$ | Absolute error |
| :---: | :---: | :---: | :---: |
| 0 | 5.99326 | 5.99325 | $5.87052 \times 10^{-6}$ |
| 1 | 2.61561 | 2.61561 | $1.03359 \times 10^{-6}$ |
| 2 | 0.444587 | 0.444588 | $1.45633 \times 10^{-7}$ |
| 3 | 0.062183 | 0.062183 | $7.23523 \times 10^{-9}$ |
| 4 | 0.008453 | 0.008453 | $1.78315 \times 10^{-9}$ |
| 5 | 0.001144 | 0.001144 | $2.56982 \times 10^{-10}$ |

Table 2. Comparison of VIM solution $u_{3}(x, y, t)$ with exact solution $u(x, y, t)$ for the fixed values of $\alpha=0.2, \quad \beta=-0.2, \quad a=0.2, \quad b=1 / 6$, $y=0, t=0.5$

| $x$ | Exact $u(x, 0,0.5)$ | VIM $u_{3}(x, 0,0.5)$ | Absolute error |
| :---: | :---: | :---: | :---: |
| 0 | 0.074939 | 0.075007 | 0.000017 |
| 1 | 0.07502 | 0.075015 | $4.84439 \times 10^{-6}$ |
| 2 | 0.069369 | 0.069347 | 0.000022 |
| 3 | 0.059635 | 0.059617 | 0.000017 |
| 4 | 0.048145 | 0.048141 | $3.64064 \times 10^{-6}$ |
| 5 | 0.036941 | 0.036946 | $5.03996 \times 10^{-6}$ |


(a)

(b)

(c)

Figure 1. Comparison of the exact solution (a) with the third approximate solution (b) for the fixed values of $\alpha=1, \beta=1, a=0.01, b=1, x=0$; a 2-D representation (c) of exact solution $u$, first approximate $u[1]$, second approximate $u[2]$ and third approximate $u[3]$ solution for the fixed values $\alpha=1, \beta=1, a=0.01, b=1, x=0, t=0.5$.


Figure 2. Comparison of the exact solution (a) with the third approximate solution (b) for the fixed values of $\alpha=0.2, \beta=-0.2, a=0.2, b=1 / 6$, $y=0$; a 2-D representation (c) of exact solution $u$, first approximate $u[1]$, second approximate $u[2]$ and third approximate $u[3]$ solutions for the fixed values $\alpha=0.2, \beta=-0.2, a=0.2, b=1 / 6, \quad y=0, t=0.5$.

### 5.2. Case $n=3$

The solution of the generalized ZK-BBM equation by variational iteration method for the case $n=3$ under initial condition (7) can be obtained by using iteration formula (13). The successive approximations are given below:

$$
\begin{aligned}
& u(x, y, 0)=\gamma \operatorname{sech}(a x+b y) \\
& u_{1}(x, y, t)=\gamma \operatorname{sech}(a x+b y)+\gamma a \operatorname{sech}(a x+b y) \tanh (a x+b y) t
\end{aligned}
$$

$$
\begin{aligned}
& +3 \gamma^{3} \alpha a \operatorname{sech}^{3}(a x+b y) \\
& \times \tanh (a x+b y) t+6 \gamma \beta a b^{2} \tanh ^{3}(a x+b y) \operatorname{sech}(a x+b y) t \\
& -5 \gamma \beta a b^{2} \operatorname{sech}(a x+b y) \tanh (a x+b y) t
\end{aligned}
$$

Similarly, we have computed $u_{2}(x, y, t)$ and higher order approximations can also be computed by using the software package Mathematica. By Guner et al. [5], the solution of equation (5) exists for the inequality

$$
\begin{equation*}
\alpha \beta\left(b^{2}-a^{2}\right)\left(1+a^{2} \beta\right)>0 . \tag{15}
\end{equation*}
$$

Therefore, for numerical computational of solution by VIM, we choose constant parameters $\alpha, \beta, a, b \in \mathbb{R} \backslash\{0\}$ so that the inequality (15) must satisfied. The second iterative solution $u_{2}(x, y, t)$ of generalized ZK-BBM equation for case $n=3$ by the variational iteration method has been obtained for $\alpha>0, \beta>0$ and $\alpha<0, \beta>0$ in form of Figures 3-4 and Tables 3-4 and compared with exact solution [5]. For the other cases of $\alpha$ and $\beta$ we get the similar result.

Table 3. Comparison of VIM solution $u_{2}(x, y, t)$ with exact solution $u(x, y, t)$ for the fixed values of $\alpha=1, \beta=1, a=0.1, b=1, x=0$, $t=0.5$

| $y$ | Exact $u(0, y, 0.5)$ | VIM $u_{2}(0, y, 0.5)$ | Absolute error |
| :---: | :---: | :---: | :---: |
| 0 | 1.39331 | 1.3927 | 0.000606 |
| 1 | 0.976318 | 0.9767 | 0.000381 |
| 2 | 0.409282 | 0.409247 | 0.000035 |
| 3 | 0.153464 | 0.153453 | 0.000011 |
| 4 | 0.0566038 | 0.056599 | $4.01358 \times 10^{-6}$ |
| 5 | 0.020830 | 0.020829 | $1.45985 \times 10^{-6}$ |

Table 4. Comparison of VIM solution $u_{2}(x, y, t)$ with exact solution $u(x, y, t)$ for the fixed values of $\alpha=-1 / 3, \beta=1 / 3, a=0.25, b=0.2$, $y=0, t=0.5$

| $x$ | Exact $u(x, 0,0.5)$ | VIM $u_{2}(x, 0,0.5)$ | Absolute error |
| :---: | :---: | :---: | :---: |
| 0 | 0.20835 | 0.208643 | 0.000292 |
| 1 | 0.208303 | 0.208443 | 0.000140 |
| 2 | 0.195947 | 0.19594 | $7.8068 \times 10^{-6}$ |
| 3 | 0.174619 | 0.174555 | 0.000063 |
| 4 | 0.149034 | 0.148974 | 0.000060 |
| 5 | 0.123235 | 0.123194 | 0.000041 |


(a)

(b)

(c)

Figure 3. Comparison of the exact solution (a) with the second approximate solution (b) for the fixed values of $\alpha=1, \beta=1, a=0.1, b=1, x=0$; a 2-D representation (c) of exact solution $u$, first approximate $u[1]$ and second approximate $u[2]$ solutions for the fixed values $\alpha=1, \beta=1, a=0.1$, $b=1, x=0, t=0.5$.


Figure 4. Comparison of the exact solution (a) with the second approximate solution (b) for the fixed values of $\alpha=-1 / 3, \beta=1 / 3, a=0.25, b=0.2$, $y=0$; a 2-D representation (c) of exact solution $u$, first approximate $u[1]$ and second approximate $u[2]$ solutions for the fixed values $\alpha=-1 / 3$, $\beta=1 / 3, a=0.25, b=0.2, y=0, t=0.5$.

### 5.3. Convergence analysis

The convergence of solution of the generalized ZK-BBM equation by variational iteration method can be studied following [14]. We can rewrite iteration formula (13) in form of an operator $A\left(u_{k}\right)$ as

$$
\begin{equation*}
u_{k+1}(x, y, t)=u_{k}(x, y, t)+A\left(u_{k}\right), \quad k \geq 0, \tag{16}
\end{equation*}
$$

where the operator $A(u)$ is defined as

$$
\begin{aligned}
A(u)=-\int_{0}^{t}\{ & \left\{\frac{\partial u(x, y, \xi)}{\partial \xi}+\frac{\partial u(x, y, \xi)}{\partial x}+n \alpha u^{n-1} \frac{\partial u(x, y, \xi)}{\partial x}\right. \\
& \left.+\beta\left(\frac{\partial^{3} u(x, y, \xi)}{\partial x^{2} \partial \xi}+\frac{\partial^{3} u(x, y, \xi)}{\partial y^{2} \partial x}\right)\right\} d \xi .
\end{aligned}
$$

Now, we can write the solution of generalized ZK-BBM equation (5) in form

$$
\begin{equation*}
u(x, y, t)=\lim _{k \rightarrow \infty} u_{k}=\sum_{k=0}^{\infty} v_{k}=\sum_{k=0}^{\infty}\left(u_{k}-u_{k-1}\right), \tag{17}
\end{equation*}
$$

where the $u_{k}$ is $k$ th partial sum of the series $\sum_{k=0}^{\infty} v_{k}$. The component $v_{k}$ is given by

$$
v_{0}=u_{0}
$$

and

$$
v_{k}=A\left(v_{0}+v_{1}+v_{2}+\cdots+v_{k-1}\right), \quad(k \geq 1)
$$

If the series $\sum_{k=0}^{\infty} v_{k}$ is convergent and limit function $u$ satisfy equation (5), then the exact solution will exist and domain of existence of solution can be obtained. Following the Banach fixed point theorem and convergence of variational iteration method in [14], the series $\sum_{k=0}^{\infty} v_{k}$ converges to the an exact solution of generalized ZK-BBM equation provided that there exists $0<\gamma<1$ such that

$$
\left\|v_{k+1}\right\| \leq \gamma\left\|v_{k}\right\|, \quad \forall k \in \mathbb{N} \cup\{0\}
$$

In other words, above condition can be written as follows. The series $\sum_{k=0}^{\infty} v_{k}$ converges to the exact solution $u$, whenever there exists $\beta_{i}$ such that $0 \leq \beta_{i}<1$, for all $i \in \mathbb{N} \cup\{0\}$ and maximum absolute truncated error can be estimated as

$$
\begin{equation*}
\left\|u-\sum_{k=0}^{m} v_{k}\right\| \leq \frac{1}{1-\beta} \beta^{m+1}\left\|v_{0}\right\|, \quad \beta=\max \left\{\beta_{i}, i=0,1,2, \ldots, m\right\}, \tag{18}
\end{equation*}
$$

where the parameter $\beta_{i}, \forall i \in \mathbb{N} \bigcup\{0\}$ is given by

$$
\beta_{i}= \begin{cases}\frac{\left\|v_{i+1}\right\|}{\left\|v_{i}\right\|}, & \left\|v_{i}\right\| \neq 0  \tag{19}\\ 0, & \left\|v_{i}\right\|=0\end{cases}
$$

The convergence of VIM series solution of generalized ZK-BBM equation can be tested with respect to the time $t$ for the following two cases:

1. We have computed $v_{i}, i=0,1,2,3$, for $x=0, y=1, \alpha=1, \beta=1$, $a=0.01$, and $b=1$ as

$$
v_{0}=2.51859, v_{1}=0.191717 t
$$

$$
v_{2}=0.0000199756 t+0.00464803 t^{2}+0.000237926 t^{3}
$$

$$
v_{3}=-4.07524 \times 10^{-8} t+8.08251 \times 10^{-6} t^{2}-0.000319563 t^{3}-0.00027935 t^{4}
$$

$$
+0.61736 \times 10^{-8} t^{5}-4.32293 \times 10^{-9} t^{6}-1.65123 \times 10^{-11} t^{7}
$$

and therefore $\beta_{i}, i=0,1,2,3$ as

$$
\begin{aligned}
& \beta_{0}=\left\|v_{1}\right\| /\left\|v_{0}\right\|=0.076121<1, \quad \beta_{1}=\left\|v_{2}\right\| /\left\|v_{1}\right\|=0.0255894<1, \\
& \beta_{2}=\left\|v_{3}\right\| /\left\|v_{2}\right\|=0.0725024<1, \quad \beta_{3}=\left\|v_{4}\right\| /\left\|v_{3}\right\|=0.0801356<1 .
\end{aligned}
$$

2. Similarly, we have computed $\beta_{i}, i=0,1,2,3$ for $x=5, y=0$, $\alpha=0.2, \beta=-0.2, a=0.2$, and $b=1 / 6$ as above equal to

$$
\begin{aligned}
& \beta_{0}=\left\|v_{1}\right\| /\left\|v_{0}\right\|=0.310274<1, \quad \beta_{1}=\left\|v_{2}\right\| /\left\|v_{1}\right\|=0.118558<1, \\
& \beta_{2}=\left\|v_{3}\right\| /\left\|v_{2}\right\|=0.162657<1, \quad \beta_{3}=\left\|v_{4}\right\| /\left\|v_{3}\right\|=0.437901<1 .
\end{aligned}
$$

Therefore, $\beta_{i}<1, i=0,1,2,3$, into above two test cases so it is strongly expected that VIM series solution is convergent to the solution of generalized ZK-BBM equation. Similarly, we have computed the $\beta_{i}$,
$i=0,1,2,3$ for other higher values of $x$ and $y$ and constant parameters $\alpha$, $\beta, a$ and $b$, and observe that $\beta_{i}<1, i=0,1,2,3$. So we conclude that VIM series solution of generalized ZK-BBM equation converges to the exact solution of [5] for all $x$ and $y$. This is also confirmed by the error calculated from exact solution and shown in Tables 1-2. Similarly, one can perform convergence analysis of the VIM solution for the case $n=3$.

### 5.4. Modified variational iteration method

Following the work [9] on a modified variational iteration method we define a modified iteration formula corresponding to (16) as follows:

$$
\begin{equation*}
u_{k+1}(x, y, t)=u_{k}(x, y, t)+\eta A\left(u_{k}\right), \quad k \geq 0, \tag{20}
\end{equation*}
$$

where $\eta$ is an auxiliary parameter and is not equal to zero. It is used to control the convergence region of the solution by variational iteration method. From the convergence analysis in Subsection 5.3, we see that, the modified iteration formula (20) provide us the freedom to choose relatively small value of $|\eta|$ (generally less than one) to obtained a good approximate solution of generalized ZK-BBM equation. In other words, we can choose small value of $|\eta|$ to increase the region of convergence of the VIM solution.

## 6. Result and Discussion

The approximate solution $u_{3}(x, y, t)$ of generalized ZK-BBM equation (5) by the variational iteration method for the case $n=2$ has been computed for the case $\alpha>0, \beta>0$ and $\alpha>0, \beta<0$. The solutions have been presented and compared with the exact solution [5] in Figures 1-2 and Tables 1-2. The two dimensional representation of solution (see Figures 1(c)-2(c)) show that the exact solution, first order, second order and third order variational iterate solution of the generalized ZK-BBM equation overlap to each other and hence are almost same. These results show that there exists solitary pattern motion of the long waves of small but finite amplitude when we study them in sense of ZK-model. It is due to the
interaction of non-linear effect with the dispersion effect in the generalized ZK-BBM equation. The absolute error of approximate solution with the exact solution is also presented in the tables. The negligible absolute error and convergence analysis show that VIM solution of generalized ZK-BBM equation has very good agreement with the exact solution. For the case $n=3$, under the different subcases the approximate solution $u_{2}(x, y, t)$ of generalized ZK-BBM equation (5) is calculated and presented in Figures 3-4 and Tables 3-4 and it shows that there exists solitary wave solution of generalized ZK-BBM equation by VIM.

## 7. Conclusion

In this paper, the variational iteration method has been applied successfully to find the solitary wave solution of generalized ZK-BBM equation. The solitary pattern solution is obtained due to interaction and balance between the non-linear and dispersion effect in the equation. The error obtained is negligible and solution is almost exact. The convergence analysis of VIM solution and the modified variational iteration formula also support it. These observation show that variational iteration method is quite efficient in finding the solitary pattern solution of generalized ZK-BBM equation. The variational iteration method can be used in further study of evolution problems and non-linear partial differential equations.

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## References

[1] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform, Cambridge University Press, Cambridge, 1990.
[2] T. A. Abassy, M. A. El-Tawil and H. El-Zoheiry, Solving nonlinear partial differential equations using the modified variational iteration Pad technique, J. Comput. Appl. Math. 207(1) (2007), 73-91.
[3] E. M. Abulwafa, M. A. Abdou and A. A. Mahmoud, Nonlinear fluid flows in pipe-like domain problem using variational-iteration method, Chaos Solitons Fractals 32(4) (2007), 1384-1397.
[4] R. T. Benjamin, J. L. Bona and J. J. Mahony, Model equations for long waves in nonlinear dispersive systems, Philos Trans. Roy. Soc. London Ser. A 272 (1972), 47-78.
[5] O. Guner, A. Bekir, L. Moraru and A. Biswas, Bright and dark soliton solutions of the generalized Zakharov-Kuznetsov-Benjamin-Bona-Mahony non-linear evolution equation, Proc. Rom. Acad. Ser. A 16(3) (2015) 422-429.
[6] J.-H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods Appl. Mech. Engrg. 167 (1998), 57-68.
[7] J.-H. He, Approximate solution of non-linear differential equations with convolution product non-linearities, Comput. Methods Appl. Mech. Engrg. 167 (1998), 69-73.
[8] J.-H. He, Variational iteration method a kind of non-linear analytical technique: some examples, Internat. J. Non-Linear Mech. 34 (1999), 699-708.
[9] H. Jafari, H. Tajadodi and D. Baleanu, A modified variational iteration method for solving fractional Riccati differential equation by Adomian polynomials, Fract. Calc. Appl. Anal. 16(1) (2013), 109-122.
[10] K. Khan, M. A. Akbar and N. H. M. Ali, The modified simple equation method for exact and solitary wave solutions of non-linear evolution equation: The gZK-BBM equation and right-handed noncommutative Burgers equations, ISRN Math. Phys. (2013), 146704.
[11] W. Malfliet, Solitary wave solutions of nonlinear wave equations, Amer. J. Phys. 60 (1992), 650-654.
[12] S. Momani and S. Abuasad, Application of He's variational iteration method to Helmholtz equation, Chaos Soliton Fractals 27 (2006), 1119-1123.
[13] S. Momani and Z. M. Odibat, Analytical approach to linear fractional partial differential equations arising in fluid mechanics, Phys. Lett. A 355(45) (2006), 271.
[14] Z. Odibat, A study on the convergence of variational iteration method, Math. Comput. Modelling 51 (2010), 1181-1192.
[15] N. H. Sweilam and M. M. Khader, Variational iteration method for one dimensional non-linear thermoelasticity, Chaos Solitons Fractals 32(1) (2007), 145-149.
[16] M. Tatari and M. Dehghan, Improvement of He's variational iteration method for solving systems of differential equations, Comput. Math. Appl. 58(11-12) (2009), 2160-2166.
[17] A. M. Wazwaz, Compact and noncompact physical structures for the ZK-BBM equation, Appl. Math. Comput. 169 (2005), 713-725.
[18] A. M. Wazwaz, The extended tanh method for new compact and noncompact solutions for the KP-BBM and the ZK-BBM equations, Chaos Solitons Fractals 38 (2008), 1505-1516.
[19] A. M. Wazwaz, The tanh method for travelling wave solutions of nonlinear equations, Appl. Math. Comput. 154 (2004), 713-723.
[20] A. M. Wazwaz, The extended tanh method for new soliton solutions for many forms of the fifth-order KdV equations, Appl. Math. Comput. 184 (2007), 1002-1014.
[21] A. M. Wazwaz, The tanh method and the sine-cosine method for solving the KP-MEW equation, Int. J. Comput. Math. 82 (2005), 235-246.
[22] N. J. Zabusky and M. D. Kruskal, Interaction of solitons in a collisionless plasma and the recurrence of initial states, Phys. Rev. Lett. 15 (1965), 240-243.
[23] V. E. Zakharov and E. A. Kuznetsov, On three-dimensional solitons, Soviet Phys. 39 (1974), 285-288.


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