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SOLVING FUZZY NONLINEAR EQUATION VIA LEVENBERG-MARQUARDT METHOD

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Abstract

This paper presents the Levenberg-Marquardt (LV) approach for solving fuzzy nonlinear problems, in which the LV steps are computed at every iteration. We begin by transforming the fuzzy quantities into its equivalent parametric form. Numerical experiments with encouraging results are presented to illustrate the efficiency of the proposed method.

1. Introduction

The development of methods for numerical evaluations was a response to the continuous demand of numerical computation, mainly in systems

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nonlinear equations. This is due to its role in solving problems arising from areas such as engineering, medicine, social sciences. However, most parameters of these systems are usually represented by fuzzy numbers rather than crisp numbers. Therefore, the outcome depends on the roots of fuzzy equations (Abbasbandy and Asady [9]). Standard analytical techniques such as Buckley and Qu [7, 8], and Muzzioli and Reynaerts [12] are not suitable for solving nonlinear equations such as:

(I) $ax^4 + bx^3 + cx^2 + dx - e = f,$

(II) $ae^x + b = d,$

(III) $x - \sin(x) = g,$

where x, a, b, c, d, e, f, g are fuzzy numbers. Thus, there is a need to explore numerical methods for solving fuzzy nonlinear equations. New ideas on numerical methods spread quickly across the globe. Recently, Sulaiman et al. [14] applied the Regula Falsi method to solve fuzzy nonlinear equation. Abbasbandy and Asady [9] parameterized some fuzzy quantities and applied Newton's approach to solve the equivalent fuzzy nonlinear equations. Waziri and Moyi [6] employed the Chord's Newton method to solve dual fuzzy nonlinear equations. Also, Amirah et al. [10] introduced the Broyden's method to obtain the solutions of a fuzzy nonlinear equation. Broyden and Chord's Newton's methods are variants of Newton's method and possess most properties of Newton's method. The convergence of Newton's method is straightforward particularly when the function is quadratic [4]. However, the method may be undefined if the Jacobian is singular. In a major revolution in numerical practice and to overcome some of these drawbacks, we suggested the Levenberg-Marquardt method for the solution of fuzzy nonlinear equation. This method introduces a parameter μ_k to Newton's algorithm and also possesses quadratic rate of convergence of Newton's method if the Jacobi at the solution point x^* is nonsingular and if μ_k is adequately chosen at each iteration.

This paper is structured as follows: Section 2 discusses the brief overview and some fundamental results of fuzzy numbers. In Section 3, we present the description of the Levenberg-Marquardt method. In Section 4, we propose Levenberg-Marquardt algorithm for the solution of fuzzy nonlinear equation. Numerical examples on well-known benchmark problems are illustrated in Section 5. Finally, we present the conclusion in Section 6.

2. Preliminaries

This section presents some useful definitions of fuzzy numbers.

Definition 1 [11]**.** A fuzzy number is a set like $u : R \to I = [0, 1]$ which satisfies the following:

(1) u is upper semi-continuous,

(2) $u(x) = 0$ outside some interval $[c, d]$,

(3) there are real numbers a, b such that $c \le a \le b \le d$ and,

(3.1) $u(x)$ is monotonic increasing on $[c, a]$,

(3.2) $u(x)$ is monotonic decreasing on $[b, d]$,

(3.3) $u(x) = 1, a \le x \le b.$

The set of all these fuzzy numbers is denoted by E. An equivalent parametric is as also given in [13].

Definition 2 [11]**.** Fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of function $\underline{u}(\alpha), \overline{u}(\alpha), 0 \le \alpha \le 1$ which satisfies the following requirement:

(1) $\underline{u}(\alpha)$ is a bounded monotonic increasing left continuous function,

(2) $\overline{u}(\alpha)$ is a bounded monotonic decreasing left continuous function,

(3) $\underline{u}(\alpha) \le \overline{u}(\alpha), 0 \le \alpha \le 1.$

A popular fuzzy number is the trapezoidal fuzzy number $u = (x_0, y_0, \sigma, \beta)$ with interval defuzzifier $[x_0, y_0]$ and left fuzziness σ and right fuzziness β, where the membership function is

$$u(x) = \begin{cases} \frac{1}{\sigma}(x - x_0 + \sigma), & x_0 - \sigma \leq x \leq x_0, \\ & x \in [x_0, y_0], \\ \frac{1}{\beta}(y_0 - x + \beta), & y_0 \leq x \leq y_0 + \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Its parametric form is

$$\underline{u}(r) = x_0 - \sigma + \sigma r, \quad \overline{u}(r) = y_0 + \beta - \beta r.$$

Let $TF(\mathbb{R})$ be the set of all trapezoidal fuzzy numbers. The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented as follows [9].

For arbitrary $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$, and $k > 0$, the addition $(u + v)$ and multiplication by scalar k are defined as

$$(\underline{u + v})(\alpha) = \underline{u}(\alpha) + \underline{v}(\alpha), \quad (\overline{u + v})(\alpha) = \overline{u}(\alpha) + \overline{v}(\alpha),$$

$$(\underline{ku})(\alpha) = k\underline{u}(\alpha), \quad (\overline{ku})(\alpha) = k\overline{u}(\alpha).$$

3. Levenberg-Marquardt Modification (LVM)

Consider a system of nonlinear equations

$$F(x_k) = 0. \tag{3.1}$$

The Levenberg-Marquardt modification is classical and one of the most popular solution methods for solving (3.1) [2]. The technique introduces a parameter μ_k to Newton's method to ensure that the search direction is a descent direction even when $J(x^k)$ is not positive definite. Given the Newton algorithm as

$$x_{k+1} = x_k - J(x_k)^{-1}F(x_k), \tag{3.2}$$

or

$$d = x_{k+1} - x_k = -J(x_k)^{-1}F(x_k). \tag{3.3}$$

Levenberg-Marquardt modification introduces μ_k to (3.2) and hence, replaces

$$J(x_k)^{-1} = (J(x_k) + \mu_k I)^{-1}, \tag{3.4}$$

where $\mu_k \geq 0$ is a parameter being updated from iteration to iteration, and I an identity matrix. The parameter μ_k plays an important role in the convergence of the method. It prevents the search direction from being too large when $J(x_k)$ is nearly singular [3]. During the iteration process, we start with a small value for the parameter μ_k, and then increase the value slowly during the iteration process until the descent condition, that is $f(x_{k+1}) < f(x_k)$ is achieved [4]. However, it has always been difficult choosing the initial value. Numerous studies have been done that focus on choosing the value of the parameter μ_k. Yamashita and Fukushima [2] suggested the choice of $\mu_k = \| F(x_k) \|^2$. However, Fan and Yuan [3] pointed out that when the sequence $\{x_n\}$ is close to the solution set, $\mu_k = \| F(x_k) \|^2$ may be very small and hence loose its role. Also, when the sequence is far from the solution set, $\mu_k = \| F(x_k) \|^2$ may be too large which makes the search direction to be too small, hence slow the rate of convergence of the sequence. Thus, they suggested the choice of $\mu_k = \| F(x_k) \|$ and proved that the quadratic convergence of the Levenberg-Marquardt method still holds using this parameter. Moreover, when $\mu_k \to 0$, the Levenberg-Marquardt modification approaches the behavior of simple Newton's method, and also when $\mu_k \to \infty$, approaches the pure gradient method [4].

Definition 3.1 [2]**.** Let N be a subset of R^n such that $N \cap X^* \neq \emptyset$. We say that $\| F(x) \|$ provides a local error bound on N for system (3.1), if there exists a positive constant $c > 0$ such that $\| F(x) \| \geq c\, dist(x, X^*), \forall x \in N$.

To study the convergence of the method, we consider the following:

Assumption 3.1 [3]**.** (a) Suppose $F(x)$ is continuously differentiable, and the Jacobi $J(x)$ is Lipschitz continuous on some neighborhood of $x^* \in X^*$, that is, there exist positive constants L_1 and $b_1 < 1$ such that

$$\| J(y) - J(x) \| \leq L \| y - x \|, \forall x, y \in N(x^*, b_1) = \{x / \| x - x^* \| \leq b_1\}. \tag{3.5}$$

(b) $\| F(x) \|$ provides a local error bound of $N(x^*, b_1)$ for (3.1), there exists a constant say $c_1 > 0$ such that

$$\| F(x) \| \geq c_1\, dist(x, X^*), \forall x, y \in N(x^*, b_1). \tag{3.6}$$

Assumption 3.2 [3]**.** Suppose $\mu_k = \| F(x_k) \|, \forall k$. Let $\bar{x}_k$ denote the vector in X^* satisfying the condition

$$\| x - \bar{x}_k \| = dist(x_k, X^*).$$

Theorem 3.1 [3]**.** *Suppose* x_0 *is chosen sufficiently close to* X^*, *if the conditions of Assumptions* 3.1 *and* 3.2 *hold, then we say equation* (3.2) *converges superlinearly.*

Theorem 3.2 [3]**.** *Suppose the sequence* $\{x_k\}$ *is generated by the Levenberg-Marquardt modification without line search, with the initial point* x_0 *sufficiently close to the solution point* x^*. *If conditions of Assumptions* 3.1 *and* 3.2 *hold true, then we say* $\{x_k\}$ *converges to the solution* x^* *quadratically.*

4. Levenberg-Marquardt Modification for Solving Fuzzy Nonlinear Equation

Consider a fuzzy nonlinear equation

$$F(x) = c. \tag{4.1}$$

And its parametric form is defined as

$$\begin{cases} \underline{F}(\underline{x}, \overline{x}, r) = \underline{c}(r), \\ \overline{F}(\underline{x}, \overline{x}, r) = \overline{c}(r), \end{cases} \quad \forall r \in [0, 1]. \tag{4.2}$$

To obtain the solution of the above equation, we start with a given initial point x_0, employing the Levenberg-Marquardt method, we generate a sequence of points $\{x_n\}$ that converge to the solution x^*. We describe the method by the following algorithm:

Algorithm 1 (Levenberg-Marquardt algorithm)

Step 1. Given a fuzzy nonlinear equation, transform it into parametric form.

Step 2. Solve the parametric form for $r = 0$ and $r = 1$ to obtain the initial guess x_0.

Step 3. Evaluate the function $F(\underline{x}, \overline{x}, r)$ and compute the parameter

$$\mu_k(\underline{x}, \overline{x}, r) = \left\| \begin{matrix} \underline{F}(\underline{x}, \overline{x}, r) \\ \overline{F}(\underline{x}, \overline{x}, r) \end{matrix} \right\|.$$

Step 4. Compute the Jacobian matrix $J(\underline{x}, \overline{x}, r)$.

Step 5. Use Levenberg-Marquardt formula to update the new form of Jacobian as

$$J(x_k) = (J(x_k) + \mu_k I),$$

where $x_k = (\underline{x}, \overline{x}; r)$.

Step 6. Compute for the next iterative value

$$x_{k+1} = x_k - (J(x_k) + \mu_k I)^{-1} F(x_k), \quad k = 1, 2, 3, \ldots.$$

Step 7. Check tolerance if $\varepsilon \leq 10^{-5}$, then stop. Otherwise.

Step 8. Repeat Steps 3 to 6 continuously until tolerance criteria are satisfied.

5. Numerical Results

In this section, we present the solutions of two examples using some numerical methods and compare their performance based on their number of iterations to illustrate the efficiency of Levenberg-Marquardt. Also, the solutions were plotted in Figure 1 and Figure 2, respectively. The considered problems are from [9, 10].

Example 1. Consider the fuzzy nonlinear equation

$$(3, 4, 5)x^2 + (1, 2, 3)x = (1, 2, 3).$$

Without loss of generality, let x be positive, and hence the parametric form of this equation is as follows:

$$(3 + r)\underline{x}^2(r) + (1 + r)\, \underline{x}(r) = (1 + r),$$

$$(5 - r)\overline{x}^2(r) + (3 - r)\overline{x}(r) = (3 - r),$$

and can be rewritten as

$$(3 + r)\underline{x}^2(r) + (1 + r)\underline{x}(r) - (1 + r) = 0,$$

$$(5 - r)\overline{x}^2(r) + (3 - r)\overline{x}(r) - (3 - r) = 0.$$

To obtain the initial value, we use the above system.

For $r = 0$,

$$3\underline{x}^2(0) + \underline{x}(0) = 1,$$

$$5\overline{x}^2(0) + 3\overline{x}(0) = 3.$$

For $r = 1$,

$$4^2(1) + 2\underline{x}(1) = 2,$$

$$4\overline{x}^2(1) + 2\overline{x}(1) = 2.$$

Hence, we have $\underline{x}(0) = 0.4343$, $\overline{x}(0) = 0.5306$, and $\underline{x}(1) = \overline{x}(1) = 0.5$, which is very close to the exact solution. We propose another initial value to illustrate the efficiency of the method. Let $x_0 = (0.4, 0.5, 0.6)$, applying various numerical methods; we obtained the solution after some number of iterations as presented in Table 1 with the maximum error less than 10^{-5}. See Figure 1 for detail of the solution.

Example 2. Consider the fuzzy nonlinear equation

$$(4, 6, 8)x^2 + (2, 3, 4)x - (8, 12, 16) = (5, 6, 7).$$

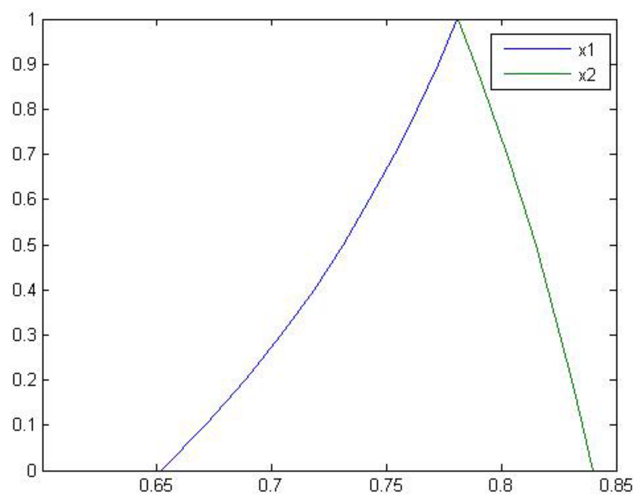

Figure 1. Solution of the Levenberg-Marquardt method for Example 1.

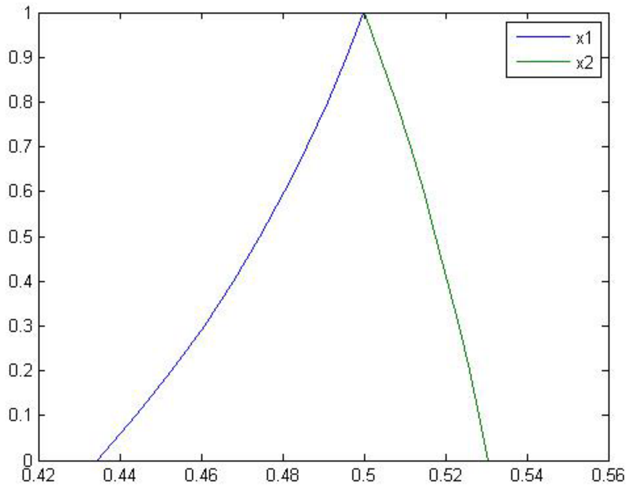

Figure 2. Solution of the Levenberg-Marquardt method for Example 2.

Without loss of generality, assume that x is positive, then the parametric form of this equation is as follows:

$$(4 + 2r)\underline{x}^2(r) + (2 + r)\underline{x}(r) - (8 + 4r) = (5 + r),$$

$$(8 - 2r)\overline{x}^2(r) + (4 - r)\overline{x}(r) - (16 - 4r) = (7 - r).$$

After solving the above parametric form for $r = 0$ and $r = 1$, we obtained the following initial guess $\underline{x}(0) = 0.6514$, $\overline{x}(0) = 0.8397$, and $\underline{x}(1) = \overline{x}(1) = 0.7808$, which is very close to the exact solution. Hence, to illustrate the efficiency of the method, we proposed a new initial guess $x_0 = (0.6, 0.8, 0.9)$. See Table 1 for number of iterations. It considers each method to obtain the solution using maximum error less than 10^{-5}, and Figure 2 for details of the solution.

Table 1. Number of iterations for solutions of Example 1 and Example 2 by various methods

Methods/Problems	Steepest descent method	Broyden's Method	Levenberg-Marquardt method	Newton's method
Example 1	15	6	2	2
Example 2	15	6	4	3

As can be observed from Table 1, Newton's method has the least number of iterations when applied to solve both problems with maximum error less than 10^{-5}. However, results obtained under the same stopping condition show that Levenberg-Marquardt method a variant of Newton method possesses lesser number of iterations compared to other numerical methods. It takes the same number of iterations with Newton method to obtain the solution for Example 1, and four iterations compared to Newton's three iterations for Example 2. This illustrates effectiveness of the proposed method and as such can be used as an alternative for solving fuzzy nonlinear equations.

6. Conclusions

In this paper, we investigated the implementations and applications of the Levenberg-Marquardt algorithm with other numerical methods, studying their effectiveness on some fuzzy nonlinear problems. A set of numerical experiments was presented to illustrate the practical performance of various algorithms. Our proposed method has shown efficiency as it performed better than steepest descent method by Abbasbandy and Jafarian [15] and slightly better than Broyden's algorithm by Amirah et al. [10] using maximum error less than 10^{-5}.

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