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ON UNIT STABLE LENGTHS OF TRANSLATIONS OF POINT-PUSHING PSEUDO-ANOSOV MAPS ON CURVE COMPLEXES

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Abstract

Let $S_{p,1}$ be a hyperbolic Riemann surface of genus $p > 1$ with one puncture x. In this paper, we consider the subgroup $\mathscr{F}$ of the mapping class group of $S_{p,1}$ that consists of point-pushing mapping classes, and show that the minimum $L_{\mathcal{C}}(\mathscr{F})$ of stable translation lengths for the actions of all pseudo-Anosov elements of $\mathscr{F}$ on the curve complex $\mathcal{C}(S_{p,1})$ is one. It is well known that every pseudo-Anosov element $f \in \mathscr{F}$ determines an oriented filling closed geodesic γ on $S_{p,1} \cup \{x\}$. We further show that $L_{\mathcal{C}}(\mathscr{F})$ can be achieved by those pseudo-Anosov elements f so that γ intersect some simple closed geodesics only once. As consequences, we prove that the set of the stable translation lengths for the actions of all pseudo-Anosov elements of $\mathscr{F}$ is unbounded. We also give a sufficient condition for a pseudo-Anosov element $f \in \mathscr{F}$ to have invariant bi-infinite geodesics in $\mathcal{C}(S_{p,1})$.

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1. Introduction and Main Results

Let $S_{p,n}$ be a hyperbolic Riemann surface of genus $p \geq 0$ with $n \geq 0$ punctures. Let *x* be a puncture if $n \geq 1$. Assume that $S_{p,n-1} = S_{p,n} \cup \{x\}$ is also hyperbolic. Let $\mathscr{F}$ be the subgroup of the mapping class group $Mod(S_{p,n})$ consisting of mapping classes projecting to the trivial mapping class on $S_{p,n-1}$.

It is well-known (Kra [7]) that there are infinitely many pseudo-Anosov mapping classes in $\mathscr{F}$, each of which contains a homeomorphism f : $S_{p,n} \to S_{p,n}$ that keeps invariant a pair $(\mathcal{F}_+, \mathcal{F}_-)$ of transverse measured foliations on $S_{p,n}$ with the property that there is a real number $\lambda > 1$ such that

$$f(\mathcal{F}_+) = \lambda \mathcal{F}_+ \text{ and } f(\mathcal{F}_-) = (1/\lambda)\mathcal{F}_-.$$

λ is called the *dilatation* of *f*. Thurston [10] showed that λ is an algebraic number. It is important to note that *f* is irreducible, by which we mean that for every simple closed geodesic *u* on $S_{p,n}$ and any positive integer *i*, $f^i(u)$ is not homotopic to *u*. Here and throughout the paper, we denote by $f^i(u)$ the geodesic homotopic to the image curve of *u* under the map f^i.

We can thereby consider the f^i-iterations of *u* and obtain an infinite orbit

$$\mathscr{S} = \{u,\ f(u),\ f^2(u), \ldots\}.$$

Geodesics in $\mathscr{S}$ are distinct and can be viewed as vertices on the curve complex $\mathcal{C}(S_{p,n})$ (see Harvey [5] for the definition of the curve complex). Denote by $\mathcal{C}_0(S_{p,n})$ the set of vertices of $\mathcal{C}(S_{p,n})$. $\mathcal{C}(S_{p,n})$ is equipped with the path metric $d_{\mathcal{C}}$ defined as follows. For any two vertices $u, v \in \mathcal{C}_0(S_{p,n})$, we declare $d_{\mathcal{C}}(u, v) = 1$ if and only if *u* and *v* are disjoint;

otherwise, $d_{\mathcal{C}}(u, v)$ is one more than the minimum number of geodesics $v_1, \ldots, v_s$ that lie in between u and v and satisfy the conditions

$$d_{\mathcal{C}}(u, v_1) = 1,\ d_{\mathcal{C}}(v_s, v) = 1, \text{ and } d_{\mathcal{C}}(v_j, v_{j+1}) = 1 \text{ for } j = 1, \ldots, s-1.$$

It is obvious that

$$d_{\mathcal{C}}(u, f^m(u)) \geq m \tag{1.1}$$

for $m = 0, 1$. From Proposition 4.6 of Masur-Minsky [8], $d_{\mathcal{C}}(u, f^m(u)) \geq 3$ for all large integers m. In [15, 16], we showed that (1.1) is true for $3 \leq m \leq 11$ for surfaces $S_{p,1}$.

For surfaces $S_{p,n}$ with $3p - 4 + n > 0$ and $n > 0$, it was shown in [13, 14] that (1.1) remains true for $m = 3, 4$.

The stable (or asymptotic) translation length $\tau_{\mathcal{C}}(f)$ for the action of f on $\mathcal{C}(S_{p,n})$ is defined as

$$\tau_{\mathcal{C}}(f) = \liminf_{m \to \infty} \frac{d_{\mathcal{C}}(u, f^m(u))}{m}$$

for a vertex $u \in \mathcal{C}_0(S_{p,n})$. It is easy to show that $\tau_{\mathcal{C}}(f)$ does not depend on the choice of u. So $\tau_{\mathcal{C}}(f)$ is well defined. By the same result of [8], as mentioned earlier, there is a positive constant $c_{p,n}$, depending only on p and n, such that for all pseudo-Anosov elements $f \in \mathscr{F}$, we have $\tau_{\mathcal{C}}(f) \geq c_{p,n}$, which means that

$$L_{\mathcal{C}}(\mathscr{F}) = \inf\{\tau_{\mathcal{C}}(f); \text{ for any pseudo-Anosov mapping class } f \in \mathscr{F}\}$$

has a positive lower bound c_p. In [15, 16], we showed that $c_p \geq 0.8$ for surfaces $S_{p,1}$ with $p > 1$.

An upper bound for $L_{\mathcal{C}}(\mathscr{F})$ can be easily obtained from the triangle inequality. Observe that every pseudo-Anosov element $f \in \mathscr{F}$ determines (via an isotopy) an oriented closed filling closed geodesic γ on $S_{p,n-1}$. That is, γ intersects every simple closed geodesic on $S_{p,n-1}$. Let $\gamma \subset S_{p,n-1}$ be such a filling geodesic that intersects some simple geodesics $\tilde{u}$ only once. Let u be the vertex in $\mathcal{C}_0(S_{p,n})$ obtained from $\tilde{u}$ by removing a point $x \in \gamma$. Let f be a pseudo-Anosov mapping class constructed from pushing x along γ in a full cycle. Then $f \in \mathscr{F}$ (Theorem 2 of [7]) and u is disjoint from $f(u)$, and so we have $d_{\mathcal{C}}(u, f(u)) = 1$. By the triangle inequality and the fact that f acts on $\mathcal{C}(S_{p,n})$ as an isometry with respect to the metric $d_{\mathcal{C}}$, we get $d_{\mathcal{C}}(u, f^m(u)) \leq m$ for all $m \geq 1$. It follows that $\tau_{\mathcal{C}}(f) \leq 1$ and thus $L_{\mathcal{C}}(\mathscr{F}) \leq 1$.

The main purpose of this paper is to fill in the gap between the lower and upper bounds of $L_{\mathcal{C}}(\mathscr{F})$ mentioned above. We will prove the following result:

Theorem 1.1. *For any Riemann surface* $S_{p,1}$ *with* $p > 1$, *we have* $L_{\mathcal{C}}(\mathscr{F}) = 1$, *which can be achieved by those* $\tau_{\mathcal{C}}(f)$ *for which f determines filling geodesics that intersect some simple closed geodesics only once.*

Well-known results. For any subgroup H of $Mod(S_{p,n})$, let $L_{\mathcal{C}}(H) = \inf\{\tau_{\mathcal{C}}(f);$ for any pseudo-Anosov mapping class $f \in H\}$. From Proposition 4.6 of Masur-Minsky [8], there is a positive lower bound for $L_{\mathcal{C}}(Mod(S_{p,n}))$. Bowditch [2] proved that $\tau_{\mathcal{C}}(f)$ is a rational number with bounded denominator for every pseudo-Anosov element $f \in Mod(S_{p,n})$. For a closed Riemann surface $S_{p,0}$ of genus $p > 1$, an upper bound for $L_{\mathcal{C}}(Mod(S_{p,0}))$ is given by [3], where Farb-Leininger-Margalit proved that

$$L_{\mathcal{C}}(Mod(S_{p,0})) < \frac{4\log(2+\sqrt{3})}{p\log\left(p - \frac{1}{2}\right)}.$$

Later, Gadre-Tsai [4] improved their results by showing that

$$\frac{1}{162(2p-2)^2+30(2p-2)} < L_{\mathcal{C}}(Mod(S_{p,0})) \leq \frac{4}{p^2+p-4}. \tag{1.2}$$

For real valued functions $F(t)$ and $G(t)$, we write $F(t) \asymp G(t)$ if there is a constant C such that $1/C < F(t)/G(t) < C$ for all $t \in \mathbf{R}$. Using this notation, we can write (1.2) as $L_{\mathcal{C}}(Mod(S_{p,0})) \asymp 1/p^2$ as $p \to +\infty$. Valdivia [9] showed that for all sufficiently large integers n with $p \geq 2$ fixed, $L_{\mathcal{C}}(Mod(S_{p,n})) \asymp 1/n$. He also showed that $L_{\mathcal{C}}(Mod(S_{0,n})) \asymp 1/n^2$ and $L_{\mathcal{C}}(Mod(S_{1,2n})) \asymp 1/n^2$. Recently, Kin-Shin [6] proved that $L_{\mathcal{C}}(Mod(S_{1,n})) \asymp 1/n^2$.

Quantitative estimations of $L_{\mathcal{C}}(H)$ for certain subgroups H of a mapping class group were also obtained in [3] and [6]. Let Γ_0 be the fundamental group of $S_{p,0}$. For any $k \geq 1$, let Γ_k be the kth term of the lower central series for Γ_0. Denote by $\mathcal{N}_k$ the kernel of the natural homomorphism of $Mod(S_{p,0})$ onto $Out(\Gamma/\Gamma_k)$. Then for the sequence of the subgroups $\mathcal{N}_k$, Theorem 6.1 of [3] asserts that for all $k \geq 1$, we have $L_{\mathcal{C}}(\mathcal{N}_k) \to 0$ as $p \to +\infty$.

Let $\mathcal{H}, \mathcal{H}' < Mod(S_{p,0})$ denote the handlebody and hyperelliptic subgroups, respectively. It was shown in [6] that $L_{\mathcal{C}}(\mathcal{H}) \asymp 1/p^2$, $L_{\mathcal{C}}(\mathcal{H}') \asymp 1/p^2$, and $L_{\mathcal{C}}(\mathcal{H} \cap \mathcal{H}') \asymp 1/p^2$. Additionally, let D_n denote a closed disk with n points $x_1, x_2, ..., x_n$ removed. There is a natural homomorphism $\iota : Mod(D_n) \to Mod(S_{0,n+1})$ defined by collapsing the disk D_n to the $(n+1)$st puncture x_{n+1} on $S_{0,n+1} = \mathbf{S}^2 \setminus \{x_1, ..., x_n, x_{n+1}\}$. Kin-Shin [6] also proved that $L_{\mathcal{C}}(\iota(Mod(D_n))) \asymp 1/n^2$.

Theorem 1.1 follows from the following result:

Theorem 1.2. *Let* $S_{p,1}$ *be a Riemann surface of genus* $p > 1$ *with one puncture. Let* $f \in \mathscr{F}$ *be any pseudo-Anosov element. Then there exists* $u \in \mathcal{C}_0(S_{p,1})$ *such that* (1.1) *holds for any nonnegative integer* m.

Outline of proof of Theorem 1.2. Throughout we fix $S = S_{p,1}$ and let $\widetilde{S} = S \bigcup \{x\}$. We use the same notations and assumptions as in [15, 16]. Let $f \in \mathscr{F}$ be a pseudo-Anosov element. From Theorem 2 of [7], we know that f can be identified with an essential hyperbolic Möbius transformation g on a hyperbolic plane $\mathbf{H}$ which has two distinct fixed points on $\mathbf{S}^1$. Denote by $\{\mathcal{L}, \mathcal{R}\} = \mathbf{S}^1 \backslash \{\text{fixed points of } g\}$. Points on $\mathcal{L}$ or $\mathcal{R}$ are naturally ordered. Thus, it makes sense to write $U \le U'$ or $U > U'$ for points $U, U' \in \mathcal{L}$ or $U, U' \in \mathcal{R}$.

Every vertex $u \in \mathcal{C}_0(S)$ is homotopic to a vertex $\widetilde{u} \in \mathcal{C}_0(\widetilde{S})$ as the puncture x is filled in. (2.3) tells us that u can be mapped to a convex and unbounded region Ω_u as shown in Figure 1:

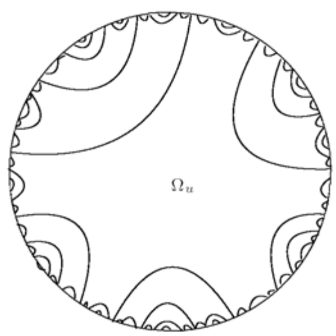
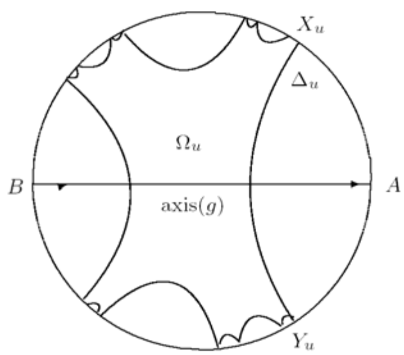


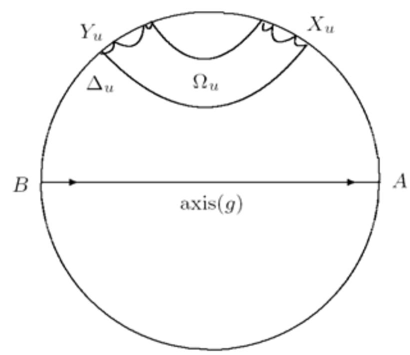
Figure 1

The complement $\mathbf{H} \backslash \overline{\Omega}_u$ is a disjoint union of half-planes each of which contains infinitely many geodesics projecting to $\widetilde{u}$ under the universal covering map $\varrho : \mathbf{H} \to \widetilde{S}$. In particular, every component of $\partial\Omega_u$ projects to $\widetilde{u}$ under ϱ.

All such regions Ω_u can be classified as type (I) or type (II) regions with respect to g as drawn in Figures 2(a) and 2(b), where $\{X_u, Y_u\} = \mathbf{S}^1 \cap \partial\Delta_u$ and Δ_u is the half-plane in $\mathbf{H}\backslash\overline{\Omega}_u$ covering the attracting fixed point of g.



(a) type (II) region



(b) type (I) region

Figure 2

Let $u, v \in \mathcal{C}_0(S)$ be mapped to Ω_u and Ω_v, respectively. Note that $d_{\mathcal{C}}(u, v) = 1$ implies that either $d_{\mathcal{C}}(\tilde{u}, \tilde{v}) = 1$ or $d_{\mathcal{C}}(\tilde{u}, \tilde{v}) = 0$ (i.e., $\tilde{u} = \tilde{v}$). By Lemma 2.1, Lemma 2.2 of [15] and Lemma 4 of [12], $d_{\mathcal{C}}(u, v) = 1$ with $d_{\mathcal{C}}(\tilde{u}, \tilde{v}) = 1$ if and only if $\partial\Omega_u \cap \partial\Omega_v = \varnothing$ and $\Omega_u \cap \Omega_v \neq \varnothing$; and $d_{\mathcal{C}}(u, v) = 1$ with $d_{\mathcal{C}}(\tilde{u}, \tilde{v}) = 0$ if and only if Ω_u and Ω_v are adjacent components of $\mathbf{H}\backslash\{\varrho^{-1}(\tilde{u})\}$ in the sense that $\overline{\Omega}_u \cap \overline{\Omega}_v$ is a geodesic in $\{\varrho^{-1}(\tilde{u})\}$.

Let $u_0 \in \mathcal{C}_0(S)$. Write $u_m = f^m(u_0)$ and consider a geodesic

$$\mathscr{G} = [u_0, v_1, v_2, ..., v_s, u_m]$$

joining from u_0 to u_m. These vertices are mapped to regions $\Omega_0', \Omega_1, ..., \Omega_s, \Omega_m'$ in $\mathbf{H}$ that all look like the region depicted in Figure 1. $\{\Omega_0', \Omega_1, ..., \Omega_s, \Omega_m'\}$ satisfies the conditions:

(A1) $\Omega_0' \cap \Omega_1 \neq \varnothing$, $\Omega_s \cap \Omega_m' \neq \varnothing$, $\Omega_i \cap \Omega_{i+1} \neq \varnothing$ for $i = 1, \ldots, s-1$, and

(A2) $\partial\Omega_0' \cap \partial\Omega_1 = \varnothing$, $\partial\Omega_s \cap \partial\Omega_m' = \varnothing$, $\partial\Omega_i \cap \partial\Omega_{i+1} = \varnothing$ for $i = 1, \ldots, s-1$.

Notice that each Ω_i is either a type (I) or a type (II) region with respect to g. One may assume that Ω_0' is of type (II) so that $\Omega_0' \subset \mathbf{H} \backslash \{\overline{\Delta}_0, \overline{\Delta}_0'\}$ (refer to Figure 3). Then all $\Omega_i' = g^i(\Omega_0')$, $i \geq 0$, are also type (II) regions.

We must compare the geodesic $\mathscr{G}$ with the quasi-geodesic

$$\mathscr{QG} = [u_0,\ f(u_0),\ f^2(u_0), \ldots, u_m]$$

through their vertices. $\mathscr{QG}$ determines a sequence $\Delta_0' \subset \Delta_1' \subset \cdots \subset \Delta_m'$ of nested half-planes in $\mathbf{H}$ for $\Delta_i' = g^i(\Delta_0')$, as well as those labeled points $\{P_i\}$ and $\{Q_i\}$. See Figure 3 also:

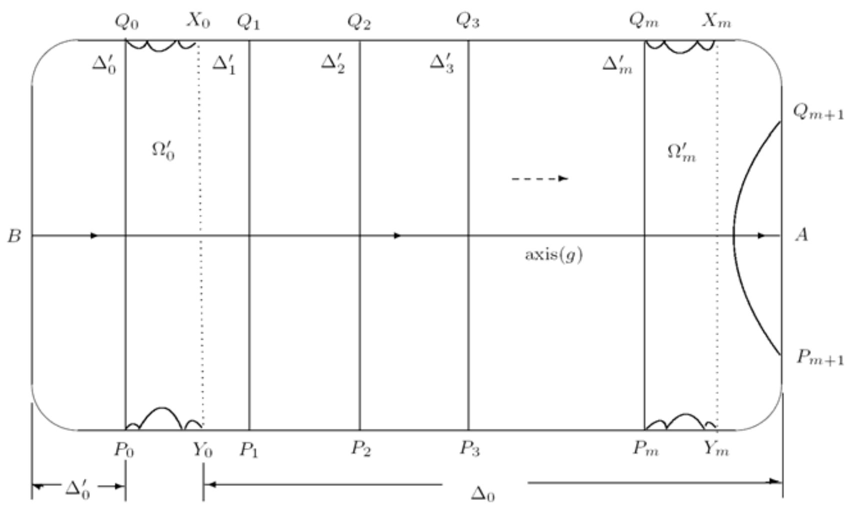


Figure 3

We see that $\Omega_0' \subset \Delta_1' \backslash \overline{\Delta_0'}$ and for every $i \geq 0$, $\Omega_i' \subset \Delta_{i+1}' \backslash \overline{\Delta_i'}$. Unfortunately, Ω_i may not sit in $\Delta_{i+1}' \backslash \overline{\Delta_i'}$. In any event, however, the conditions $\Omega'_m \subset \mathbf{H}\backslash\Delta'_m$ and $d_{\mathcal{C}}(v_s, u_m) = 1$ imply that $\Omega_s \cap \Omega'_m \neq \emptyset$, which tells us that the sequence $\{\Omega_i\}$ moves to catch up Ω'_m. So necessarily we have $P_m \leq Y_s$, $Q_m \leq X_s$ if Ω_s is of type (II); $Q_m < Y_s < X_s$ if Ω_s is of type (I) and is supported on $\mathcal{L}$; and $P_m < Y_s < X_s$ if Ω_s is of type (I) and is supported on $\mathcal{R}$.

Our purpose is to determine the least number of regions $\{\Omega_i\}$ needed to satisfy (A1) and (A2) above, and to move across over all Δ'_i's so that $\{\Omega_i\}$ can get out of Δ'_m. There is a strong indication, due to (A1) and (A2), that the motion cannot be too rapid. Consider the subsequence $\{\Omega_{i_j}\}$ consisting of type (II) regions. We need to rule out the possibility that one endpoint $X_{i_j} = \partial\Delta_{i_j} \cap \mathcal{L}$ moves slowly towards the attracting fixed point A of g, while the other endpoint $Y_{i_j} = \partial\Delta_{i_j} \cap \mathcal{R}$ moves far down to A.

As a major step of the proof of Theorem 1.2, we show that the inclusion of type (I) regions in $\{\Omega_i\}$ will not increase the motion efficiency. That is to say, the least value s can be achieved by a sequence $\{\Omega_i\}$ whose members are all type (II) regions.

To carry this out, among other works, we let $[w_0, w_1, ..., w_{r+1}] \subset \mathscr{G}$ be a segment so that all Ω_{w_i} are type (I) regions. Then they stay on one side of $axis(g)$, which is the geodesic connecting the two fixed points of g. Denote $\sigma_{w_i} = \mathbf{H}\backslash\overline{\Delta}_{w_i}$. Note that σ_{w_i} is the half-plane containing Ω_{w_i} so that $\partial\sigma_{w_i} \in \{\partial\Omega_{w_i}\}$. Hence σ_{w_i} is disjoint from $axis(g)$. Suppose that $\bigcup \sigma_{w_i}$ is supported on $\mathcal{L}$ and covers an interval $[Q_j Q_{j+d-1}]$ for some integer $d \geq 2$. Then a sequence $\{\gamma_i\}_{0 \leq i \leq r+1}$ of geodesics can be found so that

(B1) $\gamma_i \in \{\varrho^{-1}(\varrho(\partial\sigma_{w_i}))\}$,

(B2) $\gamma_i \subset \Delta_{w_i}$ crosses $axis(g)$, and

(B3) γ_i intersects $[Q_j Q_{j+d-1}]$.

Note that for $0 \leq i \leq r$, either $\gamma_i = \gamma_{i+1}$, or γ_i and γ_{i+1} are disjoint. From (B1), γ_i and $\partial\sigma_{w_i}$ are also disjoint. Figure 4 demonstrates two special cases where $d = 2$. It is known that at least four type (I) regions are needed to cover an interval $[Q_j Q_{j+1}]$.

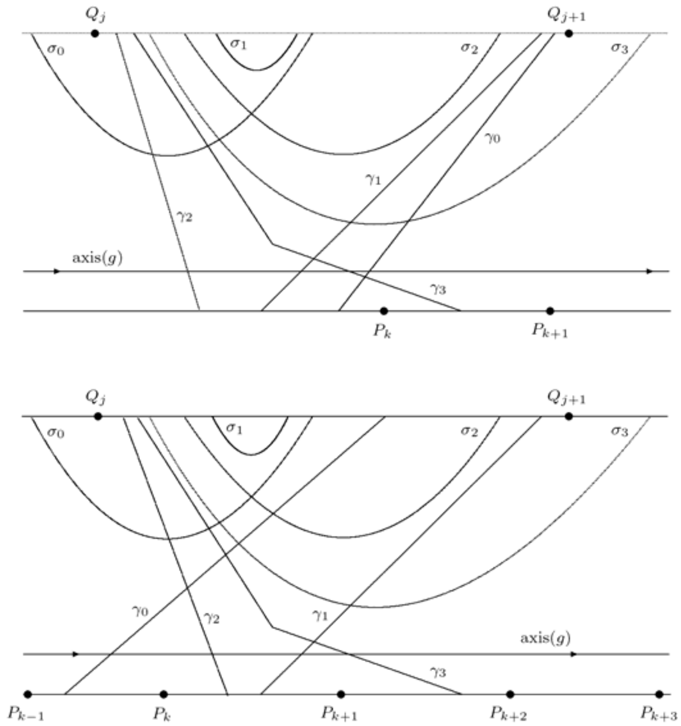


Figure 4

In each of the two figures, two finite sequences $\{\partial\sigma_0, \partial\sigma_1, \partial\sigma_2, \partial\sigma_3\}$ and $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ of geodesics are drawn that satisfy (B1)-(B3) as well as the property that $\bigcup \sigma_i$ covers $[Q_j Q_{j+1}]$. As we can see, in both examples, the sequence $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ is not properly ordered.

This phenomenon is true, in general: for any $d \geq 2$, and any finite sequence $\{\sigma_i\}$ passing through $Q_j, \ldots, Q_{j+d-1}$, a sequence $\{\gamma_i\}$ of geodesics can be found so as to satisfy (B1)-(B3). Lemma 3.2 asserts that

$\{\gamma_i\}_{0 \le i \le r+1}$ is not properly ordered. Putting all these sequences together, we see that $\{\gamma_i\}$ overall moves towards the attracting fixed point of g as $m \to +\infty$, but the motion is not monotonic.

Let $\{L_i, R_i\} = \gamma_i \cap \mathbf{S}^1$ be the two endpoints of γ_i with $L_i \in \mathcal{L}$ and $R_i \in \mathcal{R}$. Lemma 3.6 asserts that

$$\max\{|R_0 R_{r+1}|, |L_0 L_{r+1}|\} \le r, \tag{1.3}$$

where and below $|UU'|$ denotes (for any U, U' in $\mathcal{L}$ or in $\mathcal{R}$) the number of the labeled points P_n or Q_n contained in the half-open interval $(UU']$.

We then investigate a segment $[u, \Gamma, v] \subset \mathscr{G}$, where Ω_u, Ω_v are of type (II) and $\Gamma = \{v_1, \ldots, v_k\}$ are all mapped to type (I) regions $\Omega_1, \ldots, \Omega_k$. Let Q_j be the first labeled point so that $X_u \le Q_j$. We can further divide Γ into three sub-sequences $\mathscr{A}$, $\mathscr{C}$ and $\mathscr{B}$, where $\mathscr{A}$ is a sub-sequence that lies prior to the first vertex in Γ whose corresponding (type (I)) region covers Q_j, and $\mathscr{B}$, if not empty, is the sub-sequence that lies after the first vertex in Γ whose corresponding (type (I)) region covers Q_{j+d-1}, where $d \ge 2$ and Q_{j+d-1} is the last labeled point covered by $\{\Omega_i\}_{1 \le i \le k}$. Thus, the vertices in the sub-sequence $\mathscr{C}$ are mapped to those half-planes σ_i so that $\bigcup \sigma_i$ covers $[Q_j Q_{j+d-1}]$.

It follows from Lemma 4.3 and (1.3) that

$$\max\{|X_u X_v|, |Y_u Y_v|\} \le k+1. \tag{1.4}$$

Notice that $\mathscr{G}$ is the concatenation of segments of forms $[u, \Gamma, v]$. By using (1.4) for each segment $[u, \Gamma, v]$, we conclude that the least number $s \ge m-1$ if $\{\Omega_i\}$ contains no type (I) regions; and $s \ge m$ if $\{\Omega_i\}$ contains some type (I) regions. Details can be found in Section 5.

2. Preliminary Background

Let $\mathbf{H}$ be a hyperbolic plane, and let $\varrho : \mathbf{H} \to \widetilde{S}$ be a universal holomorphic covering map with a covering group G, where $\widetilde{S} = S \cup \{x\}$ and G contains only hyperbolic Möbius transformations. For every element $h \in G$, there is an h-invariant geodesic in $\mathbf{H}$ joining the repelling fixed point to the attracting fixed point of h. This geodesic is called the axis of h and is denoted by $axis(h)$.

For any vertex $\widetilde{u} \in \mathcal{C}_0(\widetilde{S})$, let $\{\varrho^{-1}(\widetilde{u})\}$ be the collection of all (disjoint) geodesics in $\mathbf{H}$ projecting to $\widetilde{u}$ under ϱ. Denote by $\mathscr{R}_{\widetilde{u}}$ the set of components of $\mathbf{H} \backslash \{\varrho^{-1}(\widetilde{u})\}$ and by $\mathscr{N}$ the disjoint union of small crescent neighborhoods of geodesics in $\{\varrho^{-1}(\widetilde{u})\}$ so that $\varrho(\mathscr{N})$ is a thin cylinder with center geodesic $\widetilde{u}$. Fix $\Omega \in \mathscr{R}_{\widetilde{u}}$. See Figure 1.

Notice that every geodesic in $\{\varrho^{-1}(\widetilde{u})\}$ determines a half-plane which does not include Ω, and the set $\mathscr{U}$ of half-planes determined by $\{\varrho^{-1}(\widetilde{u})\}$ and Ω has an infinite tree structure and thus is of partially ordered defined by inclusions. Half-planes in $\mathscr{U}$ are arranged in different levels. All the components of $\mathbf{H} \backslash \overline{\Omega}$ are designated as level one half-planes in $\mathscr{U}$. A half-plane in $\mathscr{U}$ is a level two element if it is contained in a level one half-plane but is not contained in any other half-plane in $\mathscr{U}$, and so on. We can similarly define a half-plane in $\mathscr{U}$ in any level. There are infinitely many half-planes in $\mathscr{U}$ in any level.

Let $t_{\widetilde{u}}$ be the Dehn twist about $\widetilde{u}$, which is constructed from cutting $\widetilde{S}$ along $\widetilde{u}$, rotating one end 360° in counterclockwise direction, and then gluing back with the other end. It is obvious that $t_{\widetilde{u}}$ is a quasiconformal map whose Beltrami coefficient is supported on $\varrho(\mathscr{N})$ and can be lifted to an automorphism τ of $\mathbf{H}$ that keeps the identity on $\Omega \backslash \mathscr{N}$.

The lift τ can also be constructed as follows: let $\hat{u} \in \{\varrho^{-1}(\widetilde{u})\}$ be a boundary component of Ω, and D^* the component of $\mathbf{H}\backslash\{\hat{u}\}$ containing Ω. Set $D = \mathbf{H}\backslash\overline{D}^*$. Let $h_{\hat{u}} \in G$ be a primitive hyperbolic element such that $h_{\hat{u}}(D) = D$ (thus $h_{\hat{u}}(\hat{u}) = \hat{u}$ and $h_{\hat{u}}(D^*) = D^*$).

For any $h \in G$, if $h(D)$ does not include D, i.e., either $h(D)$ and D are disjoint, or $h(D) \subset D$, we define a map $\zeta_h : \mathbf{H} \to \mathbf{H}$ as

$$\zeta_h = \begin{cases} h h_{\hat{u}} h^{-1} & \text{on } h(D)\backslash\mathscr{N}, \\ \text{a q.c map making } \zeta_h \text{ continuous} & \text{on } h(D) \cap \mathscr{N}, \\ \text{id} & \text{on } \mathbf{H}\backslash h(\overline{D}); \end{cases}$$

and if $h(D) \supset D$, ζ_h is defined as

$$\zeta_h = \begin{cases} h h_{\hat{u}}^{-1} h^{-1} & \text{on } h(D^*)\backslash\mathscr{N}, \\ \text{a q.c map making } \zeta_h \text{ continuous} & \text{on } h(D^*) \cap \mathscr{N}, \\ \text{id} & \text{on } \mathbf{H}\backslash h(\overline{D}^*). \end{cases}$$

Remark. One of $\{h_{\hat{u}}, h_{\hat{u}}^{-1}\}$ is chosen as $h_{\hat{u}}$ so that the quasiconformal maps mentioned above are compatible with $t_{\widetilde{u}}$.

Let T_j be the product of all ζ_h's for which $h(D)$ or $h(D^*)$ are level j half-planes in $\mathscr{U}$. Then the map τ can be expressed as the product:

$$\tau = \prod_{j=1}^{\infty} T_j. \tag{2.1}$$

From the construction, we can verify that

$$\tau G \tau^{-1} = G \text{ and the restriction } \tau|_{\Omega\backslash\mathscr{N}} = \text{id}.$$

Also, τ does not depend on the choice of a boundary component of Ω, nor the order of the composition in (2.1); it only depends on the choice of

$\Omega \in \mathscr{R}_{\tilde{u}}$. Different choices of Ω in $\mathscr{R}_{\tilde{u}}$ give rise to different lifts τ of $t_{\tilde{u}}$. Note that τ naturally extends to $\mathbf{S}^1$ homeomorphically, as τ is quasiconformal.

Choose $\hat{x} \in \mathbf{H}$ so that $\varrho(\hat{x}) = x$. Let

$$\mathscr{D} = \{h(\hat{x}) : h \in G\}.$$

The orbit $\mathscr{D}$ does not depend on the choice of $\hat{x}$. Thereby we obtain a punctured plane $\mathbf{H}\backslash\mathscr{D}$ of infinite type. Consider a holomorphic universal covering map $\varrho_0 : \mathbf{H} \to \mathbf{H}\backslash\mathscr{D}$. Let Γ denote the covering group of ϱ_0. From Bers [1], we know that the composition $\varrho \circ \varrho_0 : \mathbf{H} \to S$ is a holomorphic universal covering map, and if we denote by $\dot{G}$ the covering group of this composition, there exists an exact sequence:

$$1 \to \Gamma \to \dot{G} \to G \to 1.$$

Following Bers' construction [1], the map τ, being a lift of the Dehn twist $t_{\tilde{u}}$, satisfies the property that $\tau(\mathscr{D}) = \mathscr{D}$. Thus, τ also defines a map (call it τ also) of $\mathbf{H}\backslash\mathscr{D}$ onto itself, which can be further lifted to a map $\hat{\tau} : \mathbf{H} \to \mathbf{H}$, and through the universal covering map $\varrho \circ \varrho_0 : \mathbf{H} \to S$, $\hat{\tau}$ is projected to a map τ^* on S.

Notice that the conformal structure on $\mathbf{H}\backslash\mathscr{D}$ defined by τ is compatible with the conformal structure on the cylinder $\varrho(\mathscr{N})$ defined by $t_{\tilde{u}}$. As $\varrho \circ \varrho_0$ is holomorphic, the conformal structure on $\mathbf{H}\backslash\mathscr{D}$ is also compatible with the conformal structure of S that is given by τ^*. We see that the map τ^* is represented by the Dehn twist t_u about a vertex $u \in \mathcal{C}_0(S)$. For an alternate approach, see Lemma 2.1 of [11]. Since $h(\mathscr{D}) = \mathscr{D}$ for every $h \in G$, h is also mapped to $h^* \in Mod(S)$. A complete characterization of elements h^* for $h \in G$ can be found in [7].

Let $F_{\widetilde{u}}$ be the set of vertices of $\mathcal{C}(S)$ that are all indistinguishable with $\widetilde{u}$ as the puncture x is filled in. Define a map

$$\chi_{\widetilde{u}} : \mathscr{R}_{\widetilde{u}} \to F_{\widetilde{u}} \tag{2.2}$$

by sending each component Ω to u. By Lemma 2.1 and Lemma 2.2 of [15], for every vertex $\widetilde{u} \in \mathcal{C}_0(S)$, $\chi_{\widetilde{u}}$ is a bijective map that satisfies the equivariance condition

$$\chi_{\widetilde{u}}(h(\Omega)) = h^*(\chi_{\widetilde{u}}(\Omega)) \text{ for any } h \in G \text{ and } \Omega \in \mathscr{R}_{\widetilde{u}}.$$

Furthermore, if $\overline{\Omega}_1$ and $\overline{\Omega}_2 \in \mathscr{R}_{\widetilde{u}}$ are disjoint, then $u_1 = \chi_{\widetilde{u}}(\Omega_1)$ and $u_2 = \chi_{\widetilde{u}}(\Omega_2)$ intersect, whereas if Ω_1 and Ω_2 are adjacent, in the sense that $\overline{\Omega}_1 \cap \overline{\Omega}_2$ is a geodesic in $\{\varrho^{-1}(\widetilde{u})\}$, then $\{u_1, u_2\}$ forms an x-punctured cylinder embedded in S.

The bijection $\chi_{\widetilde{u}} : \mathscr{R}_{\widetilde{u}} \to F_{\widetilde{u}}$ naturally extends (fiberwise) to a bijection

$$\chi : \bigcup\{\mathscr{R}_{\widetilde{u}} : \text{all vertices } \widetilde{u} \in \mathcal{C}_0(\widetilde{S})\} \to \mathcal{C}_0(S) \tag{2.3}$$

satisfying the equivariance condition

$$\chi(h(\Omega)) = h^*(\chi(\Omega)) \tag{2.4}$$

for any $\widetilde{u} \in \mathcal{C}_0(\widetilde{S})$, $\Omega \in \mathscr{R}_{\widetilde{u}}$, and any $h \in G$.

Let $u, v \in \mathcal{C}_0(S)$ be such that $d_{\mathcal{C}}(u, v) = 1$, i.e., u and v are disjoint. Let $\Omega_u, \Omega_v \in \bigcup\{\mathscr{R}_{\widetilde{u}}\}$ be such that $\chi(\Omega_u) = u$ and $\chi(\Omega_v) = v$. Then either $\widetilde{u} = \widetilde{v}$ or $\widetilde{u}, \widetilde{v}$ are disjoint. In former case, $\Omega_u, \Omega_v \in \mathscr{R}_{\widetilde{u}}$, so they are adjacent, which says that $\{u, v\}$ forms an x-punctured cylinder. In later case, $d_{\mathcal{C}}(\widetilde{u}, \widetilde{v}) = 1$. By Lemma 2.4 of [15], $\Omega_u \cap \Omega_v \neq \emptyset$ and $\partial\Omega_u \cap \partial\Omega_v = \emptyset$.

Let $f \in \mathscr{F}$ be any pseudo-Anosov element. There exists an essential hyperbolic element $g \in G$ such that $g^* = f$, which tells us that $axis(g)$ is an oriented geodesic pointing from the repelling fixed point B to the attracting fixed point A of g and, $\varrho(axis(g))$ is a filling closed geodesic on $\widetilde{S}$. So each vertex $\widetilde{u}_0 \in \mathcal{C}_0(\widetilde{S})$ intersects $\varrho(axis(g))$. This is equivalent to that $axis(g)$ intersects $\{\varrho^{-1}(\widetilde{u}_0)\}$ infinitely many times.

Let $\{\mathcal{L}, \mathcal{R}\} = \mathbf{S}^1 \backslash \{A, B\}$, where $\mathcal{L}$ stays on the left side of $axis(g)$, while $\mathcal{R}$ stays on the right side of $axis(g)$. Points on $\mathcal{L}$ and on $\mathcal{R}$ can be ordered in the following way. Let X, $X' \in \mathcal{L}$ be any two points. We declare $X < X'$ (resp. $X \leq X'$) if and only if the arc on $\mathcal{L}$ connecting B and X is contained in (resp. equal to) the arc on $\mathcal{L}$ connecting B and X'. We can further define open, closed, or semi-open intervals on $\mathcal{L}$. For example, we use $(XX']$ to denote the set of points $\{X'' \in \mathcal{L} : X < X'' \leq X'\}$. Analogously, we can introduce similar notations when points lie on $\mathcal{R}$.

Choose $u_0 \in F_{\widetilde{u}_0}$ so that $\Omega'_0 = \chi^{-1}(u_0)$ crosses $axis(g)$. Observe that one component Δ_0 of $\mathbf{H} \backslash \overline{\Omega'_0}$ covers A (the attracting fixed point of g). Let Δ'_0 be the component of $\mathbf{H} \backslash \overline{\Omega'_0}$ that covers B, the repelling fixed point of g. Refer to Figure 3. Note that Δ_0 and Δ'_0 are level one half-planes in $\mathscr{U}_0$.

For every $i \geq 0$, we write $\Delta'_i = g^i(\Delta'_0)$ and obtain a sequence of nested half-planes

$$\Delta'_0 \subset \Delta'_1 \subset \Delta'_2 \subset \cdots \subset \Delta'_m \subset \cdots. \tag{2.5}$$

By (2.3) and (2.4), $u_m = f^m(u_0) \in F_{\widetilde{u}}$ and satisfies $\chi^{-1}(u_m) = g^m(\Omega'_0)$, which lies outside of Δ'_m. Write $\Omega'_m = g^m(\Omega'_0)$.

Let P_i, Q_i denote the endpoints of $\partial\Delta'_i$, where $Q_i \in \mathcal{L}$ and $P_i \in \mathcal{R}$. These points are referred to as labeled points in the sequel which satisfy

$$P_0 < P_1 < P_2 < \cdots < P_m < \cdots \text{ and } Q_0 < Q_1 < Q_2 < \cdots < Q_m < \cdots.$$

The geodesic $\partial\Delta_0'$ connecting P_0 and Q_0 projects to $\widetilde{u}_0$. Thus, $\partial\Delta_0' = axis(h_0)$ for an $h_0 \in G$. It is clear that $g(P_i P_{i+1}) = (P_{i+1} P_{i+2})$ and $g(Q_i Q_{i+1}) = (Q_{i+1} Q_{i+2})$. In particular, we have:

$$g^i(P_0) = P_i \text{ and } g^i(Q_0) = Q_i.$$

It follows that for any $i \geq 0$, P_i and Q_i are fixed points of $h_i = g^i h_0 g^{-i} \in G$.

For $X, X' \in \mathcal{L}$, let $|XX'|$ denote the number of labeled points in $\{Q_j\}$ that are contained in $(XX']$. Likewise, for any $Y, Y' \in \mathcal{R}$, the symbol $|YY'|$ denotes the number of labeled points in $\{P_j\}$ that are contained in $(YY']$. It is readily seen that $|XX| = 0$ and $|YY| = 0$ for all $X \in \mathcal{L}$ and $Y \in \mathcal{R}$, and that $|P_k g^i(P_k)| = i$ and $|Q_k g^i(Q_k)| = i$ for all $i, k \geq 0$.

For convenience, we specify the arc in $\mathcal{L}$ between X and $g(X)$ has length one; which is written as $\delta(X, g(X)) = 1$. Similarly, we declare $\delta(Y, g(Y)) = 1$ for points $Y \in \mathcal{R}$.

Some basic properties are summarized in the following lemma (the same is also true for points on $\mathcal{R}$).

Lemma 2.1. *Let* $X, X', X'' \in \mathcal{L}$. *We have*:

(i) $|Xg(X)| = 1$;

(ii) $|XX'| \leq |XX''|$ *whenever* $X' \leq X''$;

(iii) $|XX''| = |XX'| + |X'X''|$ *whenever* $X \leq X' \leq X''$;

(iv) $|Xg^i(X)| = i$ *for all* $i \geq 0$;

(v) *if* $X < X'$ *and* $\delta(X, X') < 1$, *then* $X' < g(X)$;

(vi) *if* $X < X'$ *and* $\delta(X, X') \leq 1$, *then* $|XX'| \leq 1$; *and*

(vii) *if* $X < X'$ *and* $\delta(X, X') \geq 2$, *then* $|XX'| \geq 2$.

In what follows, we write $\Omega_u = \Omega$, $\tau_u = \tau$ and $\mathscr{U}_u = \mathscr{U}$ to emphasize the dependence of Ω, τ and $\mathscr{U}$ on u. For any $u \in \mathcal{C}_0(S)$, $\Omega_u = \chi^{-1}(u)$ may contain $axis(g)$. If this occurs, from the construction of τ_u, we have $g(\Omega_u) = \Omega_u$, which implies $\tau_u g = g\tau_u$. So $t_u \circ f = f \circ t_u$ or $t_u = f \circ t_u \circ f^{-1} = t_{f(u)}$. It follows that $u = f(u)$ and thus f is reducible, which contradicts that $f \in \mathscr{F}$ is pseudo-Anosov.

We are left with two possibilities: $\Omega_u = \chi^{-1}(u)$ is either a type (I) or a type (II) region with respect to g, as shown in Figure 2(a) or 2(b). Here Ω_u is of type (I) if Ω_u is disjoint from $axis(g)$; Ω_u is of type (II) if $axis(g)$ crosses Ω_u.

In the former case, $axis(g)$ is contained entirely in a component Δ_u of $\mathbf{H}\backslash\overline{\Omega}_u$, where $\Delta_u \in \mathscr{U}_u$ is a level one half-plane. Since $\varrho(\partial\Delta_u)$ is a simple closed geodesic, Ω_u and $g(\Omega_u)$ must be disjoint, and if we write $\sigma_u = \mathbf{H}\backslash\overline{\Delta}_u$, σ_u and $g(\sigma_u)$ are disjoint and stay on one side of $axis(g)$. σ_u is called to be supported on $\mathcal{L}$ (resp. on $\mathcal{R}$) if $\sigma_u \cap \mathbf{S}^1 \subset \mathcal{L}$ (resp. $\sigma_u \cap \mathbf{S}^1 \subset \mathcal{R}$). Write $\{Y_u, X_u\} = \partial\sigma_u \cap \mathbf{S}^1$, where $Y_u < X_u$.

In the latter case, $axis(g)$ crosses Δ_u, and so $g^{-1}(\mathbf{H}\backslash\overline{\Delta}_u)$ is contained in another component Δ_u^* of $\mathbf{H}\backslash\overline{\Omega}_u$, where we note that $\Delta_u, \Delta_u^* \in \mathscr{U}_u$ are level one half-planes. Denote $D_u = \mathbf{H}\backslash\{\overline{\Delta}_u, \overline{\Delta}_u^*\}$. We have $\Omega_u \subset D_u$ and $D_u \cap \mathbf{S}^1$ consists of two open intervals I_1 and I_2, where $I_1 \subset \mathcal{L}$ and $I_2 \subset \mathcal{R}$. By Lemma 2.1 of [16], I_1 can cover at most one labeled point in $\{Q_j\}$, and I_2 can cover at most one labeled point in $\{P_j\}$, and more is true: $g(\Omega_u)$ is either adjacent to Ω_u or disjoint from Ω_u, depending on whether $\tilde{u}$ intersects $\varrho(axis(g))$ only once or more than once. Write $\{Y_u, X_u\} = \partial\Delta_u \cap \mathbf{S}^1$ and $\{Y_u^*, X_u^*\} = \partial\Delta_u^* \cap \mathbf{S}^1$, where $X_u, X_u^* \in \mathcal{L}$ and Y_u, Y_u^*

$\in \mathcal{R}$. It is clear that $g^{-1}(X_u) \leq X_u^*$ and $g^{-1}(Y_u) \leq Y_u^*$, and the equalities hold if and only if $\tilde{u}$ intersects $\varrho(axis(g))$ only once. $\{X_u, X_u^*, Y_u, Y_u^*\}$ are called corner points of D_u.

Regardless of type (I) and type (II) regions described above, in the context, Δ_u is referred to as the distinguished half-plane for u and, if $\chi^{-1}(u)$ is of type (II), Δ_u^* is called the accompanied half-plane of Δ_u.

Example. For the choice $u_0 \in \mathcal{C}_0(S)$ as made in Figure 3, $\Omega_0' = \chi^{-1}(u_0)$ is a type (II) region, $\Delta_0 \in \mathscr{U}_{u_0}$ is the distinguished half-plane for u_0 and $\Delta_0' \in \mathscr{U}_{u_0}$ is the accompanied half-plane of Δ_0.

Consider now a sequence $\{\gamma_j\}$ of distinct geodesics in **H** satisfying:

(i) all γ_j's intersect $axis(g)$.

Let L_j, R_j denote the endpoints of γ_j on $\mathcal{L}$ and on $\mathcal{R}$, respectively. The sequence $\{\gamma_j\}$ is called *partially* ordered if it satisfies (i) and the condition:

(ii) $L_0 \leq L_1 \leq L_2 \leq \cdots$.

It is readily seen that if $\{\gamma_j\}$ is partially ordered and also satisfies the condition:

(iii) for any $j \geq 0$, γ_j and γ_{j+1} are disjoint,

then $\{\gamma_j\}$ is mutually disjoint and thus is ordered in a way that is based on the ordering of $\{Z_j\}$ for $Z_j = \gamma_j \cap axis(g)$. That is, $\gamma_1 \prec \gamma_2$ if and only if Z_2 is closer to A than Z_1.

Lemma 2.2. *Let* $\{\tilde{u}_j\} \in \mathcal{C}_0(\tilde{S})$ *be a sequence of vertices such that* $\tilde{u}_j$ *and* $\tilde{u}_{j+1}$ *are disjoint for all* $j \geq 0$. *Let* (Q_n, Q_{n+1}) *be a pair of any* *successive labeled points on* $\mathcal{L}$. *Then for each* j, *there is* $\gamma_j \in \{\varrho^{-1}(\widetilde{u}_j)\}$ *such that* $\{\gamma_j\}$ *satisfies conditions* (i) *and* (iii) *above and in addition*, $\{L_j\} \subset [Q_n Q_{n+1}]$.

Proof. Since $\varrho(axis(g)) \subset \widetilde{S}$ is a filling geodesic, $\varrho(axis(g))$ intersects each $\widetilde{u}_j$ at least once. As such, we can find a geodesic γ'_j in $\{\varrho^{-1}(\varrho(\widetilde{u}_j))\}$ that intersects $axis(g)$. We may thus find a suitable power i such that $g^i(\gamma'_j)$ meets $[Q_n Q_{n+1}]$. As $axis(g)$ is invariant under the action of g, $\gamma_j = g^i(\gamma'_j)$ is the required geodesic. □

We remark that the choice of γ_j in Lemma 2.2 may not be unique. This occurs when the filling geodesic $\varrho(axis(g))$ intersects $\varrho(\gamma_j) = \widetilde{u}_j$ more than once. Let $\{\gamma_j^{(1)}, ..., \gamma_j^{(q)}\}$ be the collection of such γ_j's. Since $\widetilde{u}_j$ is a simple closed geodesic, $\{\gamma_j^{(1)}, ..., \gamma_j^{(q)}\}$ are mutually disjoint. It turns out that $\{\gamma_j^{(1)}, ..., \gamma_j^{(q)}\}$ is ordered. Suppose that $\gamma_j^{(1)} \prec \cdots \prec \gamma_j^{(q)}$. We then choose $\gamma_j = \gamma_j^{(q)}$ unless otherwise stated.

Lemma 2.3. *Let* $\{\gamma_j\}$ *be obtained from Lemma* 2.2. *For any pair* (γ_i, γ_{i+1}) *of geodesics in* $\{\gamma_j\}$, *if* $R_i < R_{i+1}$, *then* $\delta(R_i, R_{i+1}) \leq 1$.

Proof. By Lemma 2.2, $\{\gamma_j\}$ satisfies (i) and (iii). Suppose that $\delta(R_i, R_{i+1}) > 1$. Then $R_i < g^{-1}(R_{i+1})$, whereas $g^{-1}(L_{i+1}) \leq L_i$. If $g^{-1}(L_{i+1}) < L_i$, then $g^{-1}(\gamma_{i+1})$ intersects γ_i. But this contradicts the condition $d_{\mathcal{C}}(\widetilde{u}_i, \widetilde{u}_{i+1}) = 1$.

Suppose that $g^{-1}(L_{i+1}) = L_i$. Then $g^{-1}(\gamma_{i+1})$ and γ_i share a common fixed point $L_i = Q_n$. Notice that all these points R_i and L_i are fixed points of G. This contradicts that G is discrete. □

The following lemma is a direct consequence of Lemma 2.3.

Lemma 2.4. *Under the same condition as in Lemma* 2.2, *suppose*, *in addition*, *that* $\{\gamma_j\}$ *is partially ordered*. *Then for any* $j, k \geq 0$, $|R_j R_{j+1}| \leq 1$ *and so* $|R_j R_{j+k}| \leq k$.

Proof. The assumption implies that $\{\gamma_j\}$ satisfies (i), (ii) and (iii) above. Hence $\{\gamma_j\}$ is ordered. Thus, $R_0 \leq R_1 \leq R_2 \leq \cdots$. If $R_{j_0} = R_{j_0+1}$ for some j_0, then γ_{j_0} and γ_{j_0+1}, which are the axes of some hyperbolic elements h_{j_0} and h_{j_0+1} of G, must be the same, which contradicts the hypothesis of Lemma 2.2. We conclude that $R_0 < R_1 < R_2 < \cdots$.

Suppose that $|R_j R_{j+1}| > 1$. By Lemma 2.1(vi), $\delta(R_j, R_{j+1}) > 1$. But this contradicts Lemma 2.3.

From Lemma 2.1(iii) and the inequality $|R_j R_{j+1}| \leq 1$, we deduce that

$$|R_j R_{j+k}| = \sum_{i=0}^{k-1} |R_{j+i} R_{j+i+1}| \leq k. \qquad \square$$

Remark. The above inequality remains valid when $\{\gamma_i\}$ contains duplicate elements, that is, it could happen that $\gamma_j = \gamma_{j+1}$ for some j. This occurs when Ω_j and Ω_{j+1} are adjacent, which is equivalent to that u_j and u_{j+1} are the boundary components of an x-punctured cylinder.

3. Geodesics Mapped to Type (I) Regions

In this section, we investigate those consecutive vertices in a geodesic segment in $\mathcal{C}(S)$ that are all mapped to type (I) regions $\{\Omega_j\}$ in $\mathbf{H}$. These regions further determine a sequence of geodesics $\{\gamma_j\}$ that intersects $axis(g)$ as well as some fixed (but arbitrarily chosen) intervals in $\mathcal{L}$. Our aim is to estimate how far the other endpoints of γ_j can reach.

To be more precise, consider a small geodesic segment $[w_0, w_1, ..., w_r, w_{r+1}]$, $r \geq 1$, which joins w_0 to w_{r+1} and satisfies the condition that Ω_{w_j}, $0 \leq j \leq r+1$, are all type (I) regions in **H**, where Ω_{w_j} are obtained from the bijective map (2.3). For convenience, we write $\Omega_j = \Omega_{w_j}$ and $\sigma_j = \mathbf{H} \backslash \overline{\Delta}_j$, where Δ_j are the distinguished half-planes for w_j. Obviously, $\partial \Delta_j = \partial \sigma_j$ is a geodesic in **H** projecting to $\widetilde{w}_j$ under the universal covering map $\varrho : \mathbf{H} \to \widetilde{S}$. Assume that σ_j is supported on $\mathcal{L}$. Denote

$$\{Y_j, X_j\} = \partial \sigma_j \cap \mathcal{L} \text{ with } Y_j < X_j.$$

Lemma 3.1. (i) *All* σ_j*'s are disjoint from* $axis(g)$;

(ii) *all* σ_j*'s are supported on* $\mathcal{L}$;

(iii) *for* $0 \leq j \leq r$, (σ_j, σ_{j+1}) *are pairs of nested half-planes*; *and*

(iv) $\left(\bigcup_{j=0}^{r+1} \overline{\sigma}_j \right) \cap \mathcal{L}$ *is a connected closed interval*.

Proof. (i) follows from the definition of a region to be of type (I). (ii) is derived from Lemma 3.1 of [16]. For (iii), we note that $[w_0, w_1, ..., w_r, w_{r+1}]$ is a geodesic segment, which means that $d_{\mathcal{C}}(w_j, w_{j+1}) = 1$ for $0 \leq j \leq r$. This leads to that

$$\Omega_j \cap \Omega_{j+1} \neq \varnothing, \ \partial\Omega_j \cap \partial\Omega_{j+1} = \varnothing. \tag{3.1}$$

If $\sigma_j \cap \sigma_{j+1} = \varnothing$, then since $\Omega_j \subset \sigma_j$ and $\Omega_{j+1} \subset \sigma_{j+1}$, we see that $\Omega_j \cap \Omega_{j+1} = \varnothing$. This contradicts (3.1). Also, notice that $\{\partial\Omega_j\}$ and $\{\partial\Omega_{j+1}\}$ are collections of geodesic components in **H**. If $\partial\sigma_j \cap \partial\sigma_{j+1} \neq \varnothing$, then from the fact that $\partial\sigma_j \in \{\partial\Omega_j\}$ and $\partial\sigma_{j+1} \in \{\partial\Omega_{j+1}\}$ we deduce that $\partial\Omega_j \cap \partial\Omega_{j+1} \neq \varnothing$. This again contradicts (3.1). We conclude that $\sigma_j \cap$

$\sigma_{j+1} \neq \varnothing$ but $\partial\sigma_j \cap \partial\sigma_{j+1} = \varnothing$, which says (σ_j, σ_{j+1}) forms a pair of nested sets. That is, $\sigma_j \subset \sigma_{j+1}$ or $\sigma_{j+1} \subset \sigma_j$. Hence (iii) holds.

To prove (iv), we assume that $\left(\bigcup_{j=0}^{r+1} \overline{\sigma}_j\right) \cap \mathcal{L} = I_1 \cup I_2$, where I_1 and I_2 are disjoint closed intervals (if both are not empty). Write $I_1 = [a_1, b_1]$. Then clearly, $b_1 = X_q$ for some $0 \leq q \leq r+1$. If $q = r+1$, then $I_2 = \varnothing$. Thus, $\left(\bigcup_{j=0}^{r+1} \overline{\sigma}_j\right) \cap \mathcal{L} = [a_1, b_1]$, and we are done. If $q < r+1$ and for all $i = q+1, \ldots, r+1$, we have $X_i < X_q$, then again $I_2 = \varnothing$. Otherwise, there exists q_0 with $q < q_0 \leq r+1$, such that $X_q < X_{q_0}$. Hence we may find a point y such that $b_1 < y < a_2$ while $y < X_{q_0}$ is arbitrarily close to $X_q = b_1$. So I_2 must be empty, as claimed. □

Remark. Similarly, $\left(\bigcup_{j=0}^{r+1} \sigma_j\right) \cap \mathcal{L}$ is an open connected interval on $\mathcal{L} \subset \mathbf{S}^1$.

A more special case occurs when σ_0 covers Q_n and σ_{r+1} covers Q_{n+1}, where (Q_n, Q_{n+1}) is a pair of successive labeled points in $\{Q_i\}$. This says that $[Q_n Q_{n+1}] \subset \left(\bigcup_{j=0}^{r+1} \sigma_j\right) \cap \mathcal{L}$. By Lemma 3.2 of [16], we have $r \geq 2$. Recall that $g \in G$ is an essential hyperbolic element. From Lemma 2.2, among geodesics in $\{\varrho^{-1}(\varrho(\partial\sigma_j))\}$, where $0 \leq j \leq r+1$, there is a geodesic $\gamma_j \subset \Delta_j$ that intersects $axis(g)$ and meets $[Q_n Q_{n+1}]$.

Observe that for all integers j with $0 \leq j \leq r$, either $\{\varrho^{-1}(\varrho(\partial\sigma_j))\} = \{\varrho^{-1}(\varrho(\partial\sigma_{j+1}))\}$, or $\{\varrho^{-1}(\varrho(\partial\sigma_j))\} \cap \{\varrho^{-1}(\varrho(\partial\sigma_{j+1}))\} = \varnothing$. As members in $\{\varrho^{-1}(\varrho(\partial\sigma_j))\}$ and $\{\varrho^{-1}(\varrho(\partial\sigma_{j+1}))\}$, either $\gamma_j = \gamma_{j+1}$, or γ_j and γ_{j+1} are disjoint.

By assumption, σ_0 covers Q_n and σ_{r+1} covers Q_{n+1}. Since $\gamma_0 \in \{\varrho^{-1}(\varrho(\partial\sigma_0))\}$ and $\gamma_{r+1} \in \{\varrho^{-1}(\varrho(\partial\sigma_{r+1}))\}$, γ_0 is disjoint from $\partial\sigma_0$ and $\partial\gamma_{r+1}$ is disjoint from σ_{r+1}. As a consequence, γ_0 and γ_{r+1} intersect $[Q_n Q_{n+1}]$ but not at Q_n and Q_{n+1}. In other words, $L_0, L_{r+1} \in (Q_n Q_{n+1})$. Note that no two hyperbolic elements of G can share a common fixed point. We see that R_0 and R_{n+1} cannot be any labeled points in $\{P_k\}$.

As mentioned earlier, the choice of j may not be unique. By our convention, γ_j is the one in $\{\varrho^{-1}(\varrho(\partial\sigma_j))\}$ that intersects $axis(g)$, meets $[Q_n Q_{n+1}]$ and is closest to A.

Lemma 3.2. *The finite sequence* $\{\gamma_j\}$, $0 \le j \le r+1$, *is not partially ordered, in the sense that there is an index* j_0, $0 \le j_0 \le r$, *such that* $L_{j_0+1} < L_{j_0}$.

Proof. Suppose that $\{\gamma_j\}$ is partially ordered. That is,

$$Q_n < L_0 \le L_1 \le L_2 \le \cdots \le L_r \le L_{r+1} < Q_{n+1}. \tag{3.2}$$

By Lemma 3.1, for $0 \le j \le r$, (σ_j, σ_{j+1}) are pairs of nested sets, which says that $\sigma_j \subset \sigma_{j+1}$ or $\sigma_{j+1} \subset \sigma_j$. Let $\{\sigma_{j_1}, ..., \sigma_{j_q}\}$ be the sub-sequence of $\{\sigma_1, ..., \sigma_r\}$ that satisfies the property:

$$X_0 < X_{j_1} < X_{j_2} < \cdots < X_{j_q}. \tag{3.3}$$

If no such sub-sequence exists, then for all $1 \le j \le r$, we have $\sigma_j \subset \sigma_0$. Observe that σ_0 cannot cover Q_{n+1} and σ_{r+1} covers Q_{n+1}. We assert that $\sigma_r \subset \sigma_0 \bigcap \sigma_{r+1}$. It turns out that

$$Y_{r+1} < X_0. \tag{3.4}$$

On the other hand, since σ_0 covers Q_n and since γ_0 is disjoint from $\partial\sigma_0$ and γ_0 meets $(Q_n Q_{n+1})$, we have $X_0 < L_0$. Similarly, we notice that γ_{r+1}

is disjoint from $\partial\sigma_{r+1}$ and γ_{r+1} meets $[Q_n Q_{n+1}]$. We see that $L_{r+1} < Y_{r+1}$. Along with (3.4), we get $L_{r+1} < L_0$. So $\{L_j\}$ is not partially ordered.

As such, we may choose a sub-sequence $\{\sigma_{j_1}, ..., \sigma_{j_q}\}$ of $\{\sigma_1, ..., \sigma_r\}$. Since $\varrho(\gamma_{j_1}) = \varrho(\partial\sigma_{j_1})$ and since γ_{j_1} is disjoint from $\partial\sigma_{j_1}$, either $X_{j_1} < L_{j_1}$ or $L_{j_1} < Y_{j_1}$. If the latter occurs, then σ_{j_1} intersects σ_0, which leads to $L_{j_1} < Y_{j_1} < X_0 < L_0$, and this would contradict (3.2). It follows that

$$X_{j_1} < L_{j_1}. \tag{3.5}$$

Likewise, as $\varrho(\gamma_{j_2}) = \varrho(\partial\sigma_{j_2})$, γ_{j_2} is disjoint from $\partial\sigma_{j_2}$, so either $X_{j_2} < L_{j_2}$ or $L_{j_2} < Y_{j_2}$. If the latter occurs, then $L_{j_2} < Y_{j_2} < X_{j_1} < L_{j_1}$, this would also contradict (3.2). So we must have $X_{j_2} < L_{j_2}$. An induction argument shows that

$$X_{j_1} < L_{j_1},\ X_{j_2} < L_{j_2},\ ...,\ X_{j_q} < L_{j_q}. \tag{3.6}$$

There remain two cases to consider:

Case 1. $j_q = r$. In this case, we note that $\sigma_r = \sigma_{j_q}$ and (σ_r, σ_{r+1}) forms a pair of nested sets. If $\sigma_r \subset \sigma_{r+1}$, then from (3.6), $Y_r < X_r < L_r$. Since $\varrho(\gamma_{r+1}) = \varrho(\partial\sigma_{r+1})$, γ_{r+1} is not only disjoint from $\partial\sigma_{r+1}$ but also meets $(Q_n Q_{n+1})$. It follows that $L_{r+1} < Y_{r+1} < Y_r < L_r$. But this contradicts (3.2). If $\sigma_{r+1} \subset \sigma_r$, then since σ_{r+1} covers Q_{n+1}, we have $Q_{n+1} < X_r$. But this situation does not occur.

Case 2. $j_q < r$. In this case, all $\sigma_{j_q+1}, ..., \sigma_r$ are contained in σ_{j_q}. In particular, $\sigma_r \subset \sigma_{j_q}$. But we know that (σ_r, σ_{r+1}) forms a pair of nested sets. If $\sigma_{r+1} \subset \sigma_r$, then $\sigma_{r+1} \subset \sigma_{j_q}$, which contradicts that $j_q < r$.

Whence $\sigma_r \subset \sigma_{r+1}$ and thus $\sigma_{j_q} \bigcap \sigma_{r+1} \neq \emptyset$. It follows that $\sigma_r \subset \sigma_{j_q} \bigcap \sigma_{r+1}$. Now, from (3.6), we have $X_{j_q} < L_{j_q}$. On the other hand, since $\varrho(\gamma_{r+1}) = \varrho(\partial\sigma_{r+1})$, γ_{r+1} is disjoint from $\partial\sigma_{r+1}$, we thus obtain

$$L_{r+1} < Y_{r+1} < Y_r < X_r < X_{j_q} < L_{j_q}.$$

Once again, this would contradict (3.2). □

Another situation is that σ_0 covers Q_n but none of σ_j, $0 \leq j \leq r+1$ covers Q_{n+1}. In this case, we prove:

Lemma 3.3. *Suppose that* $\{\gamma_j\}$ *is partially ordered*: $Q_n < L_0 \leq L_1 \leq \cdots \leq L_{r+1} \leq Q_{n+1}$. *Then for* $0 \leq j \leq r+1$, *we have* $X_j < L_j$.

Proof. Since σ_0 covers Q_n and γ_0 is disjoint from $\partial\sigma_0$, we have $X_0 < L_0$. By Lemma 3.1, we know that (σ_0, σ_1) is a pair of nested sets. If $\sigma_0 \subset \sigma_1$, then clearly $X_1 < L_1$. If $\sigma_1 \subset \sigma_0$, then either $Q_n \leq L_1 < Y_1$ or $X_1 < L_1$. In the former case, $L_1 < X_0 < L_0$. This contradicts that $L_0 \leq L_1$. So we must have $X_1 < L_1$.

Suppose that for some j, $0 \leq j \leq r$, we have $X_j < L_j$. Again, by Lemma 3.1, (σ_j, σ_{j+1}) is a pair of nested sets, either $\sigma_j \subset \sigma_{j+1}$ or $\sigma_{j+1} \subset \sigma_j$. In the former case, since γ_{j+1} is disjoint from $\partial\sigma_{j+1}$, either $L_{j+1} < Y_{j+1}$ or $X_{j+1} < L_{j+1}$. If $L_{j+1} < Y_{j+1}$, then $L_{j+1} < Y_{j+1} < Y_j < X_j < L_j$. This contradicts that $L_j \leq L_{j+1}$. Therefore, $X_{j+1} < L_{j+1}$.

It remains to consider the case where $\sigma_{j+1} \subset \sigma_j$. Notice that γ_{j+1} is disjoint from $\partial\sigma_{j+1}$. We see that either $L_{j+1} < Y_{j+1}$ or $X_{j+1} < L_{j+1}$. In the former case, from the induction hypothesis, we get $L_{j+1} < Y_{j+1} < X_{j+1} < X_j < L_j$. So this case does not occur, and hence we conclude that $X_{j+1} < L_{j+1}$. The lemma is proved. □

It should be noted that $\{\gamma_j\}$ may contain duplicate elements. By removing any duplicates from the sequence, we may assume, throughout the rest of the section, that $\{\gamma_j\}$ contains only distinct geodesics.

Lemma 3.4. *Let the sequence* $\{L_0, L_1, ..., L_{r+1}\}$, $r \geq 2$, *be as in Lemma* 3.2. *We have* $|R_0R_{r+1}| \leq r$ *and hence* $|R_0 g(R_{r+1})| \leq r+1$.

Proof. From Lemma 3.2, there is a smallest integer j_0, $0 \leq j_0 \leq r$, such that $L_{j_0+1} < L_{j_0}$. Since γ_{j_0+1} and γ_{j_0} are disjoint, it must be the case that $R_{j_0+1} < R_{j_0}$.

Let k be the smallest positive integer such that $R_0 \leq P_k$. If $j_0 = 0$, then $L_1 < L_0$ and $R_1 < R_0 \leq P_k$. By Lemma 2.3, $R_2 < P_{k+1}$. Inductively, one shows that $R_{r+1} < P_{k+r}$. Hence $|R_0R_{r+1}| \leq r$.

Assume now that $j_0 > 0$. By applying Lemma 2.3 repeatedly, we conclude that $R_1 \leq P_{k+1}$, $R_2 \leq P_{k+2}$, and so on, $R_{j_0} \leq P_{k+j_0}$. By assumption, $L_{j_0+1} < L_{j_0}$. Since γ_{j_0+1} is disjoint from γ_{j_0}, we must have $R_{j_0+1} < R_{j_0} < P_{k+j_0}$. By Lemma 2.3 again, we obtain $R_{j_0+2} < P_{k+j_0+1}$. Similarly, $R_{j_0+3} < P_{k+j_0+2}$, and so on, inductively, one shows that $R_{j_0+(r-j_0+1)} < P_{k+j_0+(r-j_0)}$. This implies that $|R_0R_{r+1}| \leq r$ and hence $|R_0 g(R_{r+1})| = |R_0R_{r+1}| + |R_{r+1}g(R_{r+1})| \leq r+1$. □

We now discuss the case where $\mathscr{G}' = [w_0, w_1, ..., w_r, w_{r+1}]$ is a geodesic segment so that Ω_j are all type (I) regions that go through two adjacent intervals $[Q_nQ_{n+1}] \cup [Q_{n+1}Q_{n+2}] = [Q_nQ_{n+2}]$, i.e., σ_0 covers Q_n and σ_{r+1} covers Q_{n+2}. In this case, by Lemma 3.3 of [16], $r \geq 5$. Let $\gamma_0 \in \{\varrho^{-1}(\varrho(\partial\sigma_0))\}$ be obtained from Lemma 2.2, which tells that $\gamma_0 \subset \Delta_0$ and γ_0 intersects $axis(g)$ and (Q_nQ_{n+1}). Likewise, let $\gamma_{r+1} \in \{\varrho^{-1}(\varrho(\partial\sigma_o))\}$, $\gamma_{r+1} \subset \Delta_{r+1}$, be obtained from Lemma 2.2; that is, γ_{r+1}

intersects $axis(g)$ and $(Q_{n+1}Q_{n+2})$. Let $\{L_0, R_0\} = \gamma_0 \cap \mathbf{S}^1$ and $\{L_{r+1}, R_{r+1}\} = \gamma_{r+1} \cap \mathbf{S}^1$, where $L_0, L_{r+1} \in \mathcal{L}$ and $R_0, R_{r+1} \in \mathcal{R}$.

Lemma 3.5. *We have* $|R_0R_{r+1}| \leq r$ *and hence* $|R_0 g(R_{r+1})| \leq r + 1$.

Proof. We can write the geodesic segment $\mathscr{G}' = \mathscr{G}_1 \cup \mathscr{G}_2$, where $\mathscr{G}_1 = [w_0, w_1, ..., w_{r_1}, w_{r_1+1}]$ and $\mathscr{G}_2 = [w_{r_1+1}, w_{r_1+2}, ..., w_r, w_{r+1}]$ are geodesic segments with $r_1 \geq 2$ so that σ_0 covers Q_n, σ_{r_1+1} covers Q_{n+1} and σ_{r+1} covers Q_{n+2}.

From the above description, we know that $\mathscr{G}_1 \cap \mathscr{G}_2 = \sigma_{r_1+1}$ and $\mathscr{G}_1$ contains $r_1 + 2 \geq 4$ vertices and $\mathscr{G}_2$ contains $r - r_1 + 1 \geq 4$ vertices.

Let $\{\gamma_0, \gamma_1, ..., \gamma_{r_1+1}\}$ be the (distinct) geodesics obtained from $\mathscr{G}_1$ and from Lemma 2.2, that is, $\gamma_0, \gamma_1, ..., \gamma_{r_1+1}$ all intersect $[Q_nQ_{n+1}]$. Similarly, let $\{\gamma'_{r_1+1}, \gamma'_{r_1+2}, ..., \gamma'_{r+1}\}$ be the (distinct) geodesics obtained from $\mathscr{G}_2$ and from Lemma 2.2. This means that $\gamma'_{r_1+1}, \gamma'_{r_1+2}, ..., \gamma'_{r+1}$ all intersect $[Q_{n+1}Q_{n+2}]$.

We claim that $g(\gamma_{r_1+1}) = \gamma'_{r_1+1}$. Indeed, let $\{\gamma^{(1)}_{r_1+1}, ..., \gamma^{(q)}_{r_1+1}\} \in \{\varrho^{-1}(\varrho(\gamma_{r_1+1}))\}$ be the ordered finite collection of geodesics intersecting $axis(g)$ and (Q_nQ_{n+1}). It is easy to see that $\{g(\gamma^{(1)}_{r_1+1}), ..., g(\gamma^{(q)}_{r_1+1})\} \in \{\varrho^{-1}(\varrho(\gamma_{r_1+1}))\}$ is the collection of ordered geodesics intersecting $axis(g)$ and $(Q_{n+1}Q_{n+2})$. Notice that $g(\gamma_{r_1+1}), \gamma'_{r_1+1} \in \{\varrho^{-1}(\varrho(\gamma_{r_1+1}))\}$. It is clear that $\gamma'_{r_1+1} \in \{g(\gamma^{(1)}_{r_1+1}), ..., g(\gamma^{(q)}_{r_1+1})\}$ and that $\gamma^{(q_0)}_{r_1+1}$ is closest to the attracting fixed point A of g (for some q_0) if and only if so is $g(\gamma^{(q_0)}_{r_1+1})$.

By applying Lemma 3.4 on $\mathscr{G}_1$, we obtain $|R_0 R_{r_1+1}| \le r_1$, and $|R_0 g(R_{r_1+1})| \le r_1 + 1$. Also, by applying Lemma 3.4 on $\mathscr{G}_2$, we get $|g(R_{r_1+1}) R_{r+1}| \le r - r_1 - 1$. Hence, from Lemma 2.1(iii),

$$|R_0 R_{r+1}| = |R_0 g(R_{r_1+1})| + |g(R_{r_1+1}) R_{r+1}| \le (r_1 + 1) + (r - r_1 - 1) = r.$$

It follows that $|R_0 g(R_{r+1})| \le r + 1$, as asserted. □

Next, we consider a general case where $\mathscr{G}' = [w_0, w_1, ..., w_r, w_{r+1}]$ is a geodesic segment whose vertices are mapped to all type (I) regions $\Omega_0, ..., \Omega_{r+1}$, respectively. Assume also that σ_0 covers a labeled point Q_n and σ_{r+1} covers a labeled point Q_{n+d} for a positive integer $d \ge 1$.

As usual, let $\gamma_0 \in \{\varrho^{-1}(\varrho(\partial\sigma_0))\}, \gamma_{r+1} \in \{\varrho^{-1}(\varrho(\partial\sigma_{r+1}))\}$ be obtained from Lemma 2.2, which says γ_0 intersects $axis(g)$ and $(Q_n Q_{n+1})$, and γ_{r+1} intersects $axis(g)$ and $(Q_{n+d-1} Q_{n+d})$. Denote by $\{L_0, R_0\}$ and $\{L_{r+1}, R_{r+1}\}$, respectively, the endpoints of γ_0 and γ_{r+1}, where $L_0, L_{r+1} \in \mathcal{L}$ and $R_0, R_{r+1} \in \mathcal{R}$.

Lemma 3.6. *Under the circumstances, we have*: (i) $3d - 1 \le r$, (ii) $|L_0 L_{r+1}| \le d - 1$, (iii) $|R_0 R_{r+1}| \le r$ *and* (iv) $|R_0 g(R_{r+1})| \le r + 1$.

Proof. Lemma 3.2 of [16] tells us that at least four elements are needed to cover any two successive labeled points. Since $\mathscr{G}'$ covers the labeled points $\{Q_n, Q_{n+1}, ..., Q_{n+d}\}$, we assert that $d + 1 \le (r + 2 - 1)/3 + 1$, which implies that $3d - 1 \le r$. This proves (i).

For (ii), as $\mathscr{G}'$ can be written as a union of $\mathscr{G}_1, \mathscr{G}_2, ..., \mathscr{G}_d$, where the first element σ_0 of $\mathscr{G}_1$ covers Q_n, the last element of $\mathscr{G}_1$, which is also the first element of $\mathscr{G}_2$, covers Q_{n+1}, and so on, the last element of $\mathscr{G}_{d-1}$, which is also the first element of $\mathscr{G}_d$, covers Q_{d-1}, and the last element of $\mathscr{G}_d$ covers Q_{n+d}. Recall that $\gamma_{r+1} \in \{\varrho^{-1}(\varrho(\partial\sigma_{r+1}))\}$, where $\gamma_{r+1} \subset \Delta_{r+1}$, is obtained from Lemma 2.2, which says that γ_{r+1} intersects $axis(g)$ as well as $(Q_{n+d-1}Q_{n+d})$. It follows that $(L_0 L_{r+1}]$ contains at most these labeled points $Q_{n+1}, Q_{n+2}, ..., Q_{n+d-1}$. That is, $|L_0 L_{r+1}| \leq d-1$. This proves (ii).

(iii) and (iv) can be proved by using induction arguments. We use Lemma 3.4 to settle the case when $d = 1$.

Write $[Q_n Q_{n+d}]$ as $[Q_n Q_{n+d-1}] \cup [Q_{n+d-1}Q_{n+d}]$. Accordingly, $\mathscr{G}'$ is decomposed into two pieces. Let $[w_0, ..., w_{r_{d-1}+1}], [w_{r_{d-1}+1}, ..., w_{r+1}] \subset \mathscr{G}'$ be geodesic segments whose corresponding type (I) regions cover $[Q_n Q_{n+d-1}]$ and $[Q_{n+d-1}Q_{n+d}]$, respectively. We must have $r_{d-1} \geq 3(d-1)+1 = 3d-2$ and $r - r_{d-1} \geq 4$. By Lemma 3.4, $|R_0 R_{r_{d-1}+1}| \leq r_{d-1}$ and $|g(R_{r_{d-1}+1})R_{r+1}| \leq r - r_{d-1} - 1$. Hence, by Lemma 2.1(iii),

$$\begin{aligned} |R_0 R_{r+1}| &= |R_0 R_{r_{d-1}+1}| + |R_{r_{d-1}+1} g(R_{r_{d-1}+1})| + |g(R_{r_{d-1}+1})R_{r+1}| \\ &\leq r_{d-1} + 1 + (r - r_{d-1} - 1) = r. \end{aligned}$$

Thus, $|R_0 g(R_{r+1})| \leq r+1$, as asserted. □

Remark. From Lemma 3.6(ii), $|L_0 L_{r+1}| \leq d-1$. Thus, $|L_0 g(L_{r+1})| = |L_0 L_{r+1}| + 1 \leq d$. On the other hand, Lemma 3.6(i) yields that $d \leq r+1$. It turns out that $|L_0 g(L_{r+1})| \leq |R_0 g(R_{r+1})|$.

4. Geodesics Mapped to Regions with Mixed Types

Consider a geodesic segment

$$\mathscr{G}_0 = [u, \Gamma, v] \tag{4.1}$$

in $\mathcal{C}(S)$, where $\Gamma = \varnothing$ if $s = 0$; and $\Gamma = \{v_1, ..., v_s\}$ if $s \geq 1$. From the discussion of Section 2, vertices u and v can be mapped to regions Ω_u and Ω_v. If $s \geq 1$, all $v_j, 1 \leq j \leq s$, are mapped to regions Ω_j in **H** with geodesic boundaries. Assume throughout this section that Ω_u, Ω_v are of type (II) and all other regions Ω_j are of type (I) that are supported on $\mathcal{L}$. Let $\Delta_u, \Delta_1, ..., \Delta_s, \Delta_v$ denote the distinguished half-planes for $u, v_1, ..., v_s, v$, respectively. As usual, we write $\{X_u, Y_u\} = \partial\Delta_u \cap \mathbf{S}^1$ and $\{X_v, Y_v\} = \partial\Delta_v \cap \mathbf{S}^1$, where $X_u, X_v \in \mathcal{L}$ and $Y_u, Y_v \in \mathcal{R}$, and for $1 \leq j \leq s$, $\{Y_j, X_j\} = \partial\Delta_j \cap \mathcal{L}$ with $Y_j < X_j$. Denote $\sigma_i = \mathbf{H}\backslash\overline{\Delta}_j$. Our aim in this section is to estimate $|X_u X_v|$ and $|Y_u Y_v|$.

Lemma 4.1. *In the case where* $s = 0$, *if* $X_u \leq X_v$, *then* $|X_u X_v| \leq 1$ *and* $|Y_u Y_v| \leq 1$ $(= s + 1)$.

Proof. The condition $s = 0$ means that Ω_u, Ω_v are consecutive type (II) regions.

Case 1. $\widetilde{u}, \widetilde{v}, \widetilde{u}_0$ are distinct. Then $\Omega_u \cap \Omega_v \neq \varnothing$ and thus $D_u \cap D_v \neq \varnothing$ and no corner points of $\overline{D}_u \cup \overline{D}_v$ are labeled points. Here we recall that $D_u = \mathbf{H}\backslash\{\overline{\Delta}_u, \overline{\Delta}_u^*\}$, $D_v = \mathbf{H}\backslash\{\overline{\Delta}_v, \overline{\Delta}_v^*\}$, $\partial D_u = \partial\Delta_u \cup \partial D_u^*$ and $\partial D_v = \partial\Delta_v \cup \partial D_v^*$. Hence $X_v^* < X_u$ and $Y_v^* < Y_u$. By Lemma 2.5 of [16], $D_v \cap \mathcal{L}$ contains at most one labeled point. It follows that $|X_u X_v| \leq |X_v^* X_v| \leq 1$ and thus that $|Y_u Y_v| \leq |Y_v^* Y_v| \leq 1$.

Case 2. $\widetilde{u} = \widetilde{v} = \widetilde{u}_0$. In this case, $\Omega_u, \Omega_v \in \mathcal{R}_{\widetilde{u}_0}$. Then D_u, D_v are adjacent so that $\overline{D}_u \cap \overline{D}_v$ is a geodesic and $\varrho(\overline{D}_u \cap \overline{D}_v) = \widetilde{u}_0$.

If $\widetilde{u}_0$ intersects $\varrho(axis(g))$ more than once, then $\{Q_i, P_i\} = (\overline{D}_u \cap \overline{D}_v) \cap \mathbf{S}^1$ are labeled points but the four corner points of $\overline{D}_u \cup \overline{D}_v$ are not labeled points. If D_v is on the left side of D_u, then $X_v = X_u^* = Q_i$ and $Y_v = Y_u^* = P_i$. This tells us that $X_v < X_u$. If D_v is on the right side of D_u, then $X_u = X_v^* = Q_i$, $Y_u = Y_v^* = P_i$, $X_v < Q_{i+1}$, $Y_v < P_{i+1}$, $Q_{i-1} < X_u^*$ and $P_{i-1} < Y_u^*$. So $|X_u X_v| = 1$ and $|Y_u Y_v| = 1$.

If $\widetilde{u}_0$ intersects $\varrho(axis(g))$ only once, then again D_u, D_v are adjacent and there exists $i \geq 0$ such that $D_u = \Delta'_{i+1} \backslash \overline{\Delta'_i}$ and $D_v = \Delta'_{i+2} \backslash \overline{\Delta'_{i+1}}$. Again we have $\varrho(\overline{D}_u \cap \overline{D}_v) = \widetilde{u}_0$, $X_v^* = X_u = Q_{i+1}$, $X_u^* = Q_i$ and $X_v = Q_{i+2}$. We see that $|X_u X_v| = |X_v^* X_v| = 1$. Similarly, $|Y_u Y_v| = |Y_v^* Y_v| = 1$.

Case 3. $\widetilde{u} = \widetilde{v} \neq \widetilde{u}_0$. That is, $\{u, v\}$ forms the boundary of an x-punctured cylinder, which means that Ω_u, Ω_v are adjacent and so are D_u and D_v. Assume that D_v is on the right side of D_u. Then $X_u = X_v^*$ and $Y_u = Y_v^*$. Note that these points cannot be labeled points. By Lemma 2.5 of [16], no corner points of $\overline{D}_u \cup \overline{D}_v$ are labeled points. Also, we know that the interiors of $(\overline{D}_u \cup \overline{D}_v) \cap \mathcal{L}$ and $(\overline{D}_u \cup \overline{D}_v) \cap \mathcal{R}$ contain at most two labeled points. It is immediate that $|X_u X_v| = |X_v^* X_v| \leq 1$ and $|Y_u Y_v| = |Y_v^* Y_v| \leq 1$.

Case 4. $\widetilde{u} = \widetilde{u}_0 \neq \widetilde{v}$. If $\widetilde{u}_0$ intersects $\varrho(axis(g))$ only once, then there exists an integer i such that $D_u = \Delta'_i \backslash \overline{\Delta'_{i-1}}$, $X_u = Q_i$ and $Y_u = P_i$. It follows from $d_{\mathcal{C}}(u, v) = 1$ that $D_u \cap D_v \neq \emptyset$ and $\partial D_u \cap \partial D_v = \emptyset$. In particular, $X_v^* < X_u$ and $Y_v^* < Y_u$. Note that the corner points of D_v are not labeled points. We see that $|X_u X_v| \leq |X_v^* X_v| \leq 1$ and $|Y_u Y_v| \leq |Y_v^* Y_v| \leq 1$.

If $\widetilde{u}_0$ intersects $\varrho(axis(g))$ more than once, then $\{X_u, Y_u\}$ are labeled points, but we still have $X_v^* < X_u$ and $Y_v^* < Y_u$. Since $\overline{D}_v \cap \mathcal{L}$ and $\overline{D}_v \cap \mathcal{R}$ contain at most one labeled point, we conclude that $|X_u X_v| \leq |X_v^* X_v| \leq 1$ and $|Y_u Y_v| \leq |Y_v^* Y_v| \leq 1$.

Case 5. $\widetilde{v} = \widetilde{u}_0 \neq \widetilde{u}$. The discussion of this case is the same as Case 4. □

Let j, k be the positive integers such that

$$Q_{j-1} < X_u \leq Q_j \text{ and } P_{k-1} < Y_u \leq P_k.$$

The following two lemmas improve the results in [15, 16].

Lemma 4.2. *If* $s = 1$, *then* $|X_u X_v| \leq 1$ *and* $|Y_u Y_v| \leq 2$ $(= s + 1)$.

Proof. Let $\sigma_1 = \mathbf{H}\backslash\overline{\Delta}_1$. Then $\Delta_u \subset \Delta_1$, which tells us that $Y_1 < X_1 < X_u \leq Q_j$. But we know that $\delta(X_1, X_v) < 1$. Hence $\delta(X_u, X_v) < 1$. This leads to $|X_u X_v| \leq 1$. In particular, $X_v < Q_{j+1}$. Here we assume that σ_1 is supported on $\mathcal{L}$.

Let $\gamma_1 \in \{\varrho^{-1}(\varrho(\partial\sigma_1))\}$ be obtained from Lemma 2.2; which says $\gamma_1 \subset \Delta_1$ and γ_1 intersects $axis(g)$ and $[Q_{j-1}Q_j]$. Let $\{L_1, R_1\}$ be the endpoints of γ_1 lying on $\mathcal{L}$ and $\mathcal{R}$, respectively.

Case 1. $Q_{j-1} \leq L_1 \leq X_u$. Since γ_1 does not intersect $\partial\Delta_u$, $R_1 \leq Y_u \leq P_k$. Now $X_v \leq Q_{j+1}$ implies that $\delta(L_1, X_v) < 2$. We claim that $\delta(R_1, Y_v) < 2$. Indeed, if $\delta(R_1, Y_v) = 2$, then we may find two distinct hyperbolic elements of G sharing a common fixed point R_1, which contradicting that G is discrete. If $\delta(R_1, Y_v) > 2$, then $g^{-2}(\partial\Delta_v)$ intersects γ_1, which would contradict that $d_{\mathcal{C}}(v_1, v) = 1$. We conclude that $\delta(R_1, Y_v) < 2$. So $\delta(Y_u, Y_v) < 2$. This leads to that $|Y_u Y_v| \leq 2$.

Case 2. $X_u < L_1 \leq Q_j$. We claim that $R_1 < P_{k+1}$. Suppose $R_1 \geq P_{k+1}$. Then $g^{-1}(\gamma_1)$ intersects $\partial\Delta_u$, and this contradicts that $d_{\mathcal{C}}(u, v_1) = 1$. We conclude that $R_1 < P_{k+1}$.

The condition $d_{\mathcal{C}}(v_1, v) = 1$ implies $\delta(Y_1, X_v) < 1$. But $Y_1 < X_1 < X_u < L_1$. We see that $\delta(L_1, X_v) < 1$. Hence $\delta(R_1, Y_v) < 1$ (otherwise, $g^{-1}(\partial\Delta_1)$ would intersect γ_1, which would contradict that $d_{\mathcal{C}}(v_1, v) = 1$). But $R_1 < P_{k+1}$. We see that $Y_v < P_{k+2}$, which implies that $|Y_u Y_v| \leq 2$, as required. □

More generally, in the case of $s \geq 2,$ we have:

Lemma 4.3. *If* $s \geq 2,$ *then* $|X_u X_v| \leq [(s-2)/3] + 2$ *and* $|Y_u Y_v| \leq s + 1,$ *where and below,* $[z]$ *denotes the largest integer less than or equal to z.*

Proof. First we consider the case where the geodesic segment (4.1) can be rewritten as follows:

$$\mathscr{G}_0 = [u, \mathscr{A}, w_0, ..., w_{r+1}, \mathscr{B}, v], \quad r \geq 2,$$

where $\mathscr{A} = \{a_1, ..., a_\alpha\}$ is a sub-sequence of vertices $\{v_1, v_2, ..., v_s\}$ that lying prior to the first vertex w_0 whose corresponding (type (I)) region $\sigma_0 = \mathbf{H} \backslash \overline{\Delta}_0$ covers $Q_j,$ and $\mathscr{B},$ if not empty, is the sub-sequence $\{b_1, ..., b_\beta\}$ of $\{v_1, v_2, ..., v_s\}$ that lies after the first vertex w_{r+1} whose corresponding (type (I)) region $\sigma_{r+1} = \mathbf{H} \backslash \overline{\Delta}_{r+1}$ covers $Q_{j+d-1},$ where $d \geq 2,$ and Q_j and Q_{j+d-1} are the first and last labeled points covered by $\{\Omega_i\}_{1 \leq i \leq s},$ respectively. Note that $\mathscr{A} \neq \varnothing$ and $\mathscr{B}$ may be empty. This gives rise to

$$\alpha \geq 1, \beta \geq 0 \text{ and } \alpha + \beta + r + 2 = s. \tag{4.2}$$

Note that at least four consecutive type (I) regions are needed to cover an interval $[Q_n Q_{n+1}]$ for $j \leq n \leq j + d - 2.$ It follows that

$$d \leq \left[\frac{(r+2)-1}{3}\right] + 1 = \left[\frac{r+1}{3}\right] + 1. \tag{4.3}$$

From (4.2), we obtain $s = \alpha + \beta + r + 2 \geq r + 3.$ Thus, (4.3) yields that

$$d \leq \left[\frac{s-2}{3}\right] + 1. \tag{4.4}$$

If $\mathscr{B} = \varnothing,$ then we claim $X_v < Q_{j+d}.$ Suppose that $X_v \geq Q_{j+d}.$ Then $X_v^* \geq Q_{j+d-1}.$ But $\Omega_v \subset \mathbf{H} \backslash \{\overline{\Delta}_v, \overline{\Delta}_v^*\}.$ This implies that Ω_v is disjoint from

$\Omega_{w_{r+1}}$, or $\partial\Omega_v$ intersects $\partial\Omega_{w_{r+1}}$. Both the cases would contradict $d_{\mathcal{C}}(v, w_{r+1}) = 1$. We conclude that $X_v < Q_{j+d}$ and thus $|X_u X_v| \leq d \leq \left[\frac{s-2}{3}\right] + 1$.

Consider next the case where $\mathscr{B} \neq \emptyset$. Then $\beta \geq 1$. Since $\mathscr{B}$ does not cover Q_{j+d}, if $X_v \geq Q_{j+d+1}$, then Ω_v is disjoint from any Ω_{b_i} for $1 \leq i \leq \beta$, and this would contradict $d_{\mathcal{C}}(v, b_\beta) = 1$. So we conclude that $X_v < Q_{j+d+1}$. It follows from (4.4) that

$$|X_u X_v| \leq d + 1 \leq \left[\frac{s-2}{3}\right] + 2.$$

This proves the first statement.

To establish the second statement, we recall that Δ_{a_i}, $1 \leq i \leq \alpha$, are the distinguished half-planes for a_i. Write $\sigma_{a_i} = \mathbf{H} \backslash \overline{\Delta}_{a_i}$. Let $\gamma_i \in \{\varrho^{-1}(\varrho(\partial\sigma_{a_i}))\}$, where $\gamma_i \subset \Delta_{a_i}$, be obtained from Lemma 2.2, which says γ_i intersects $axis(g)$ and $[Q_{j-1}Q_j]$. Let $\{L_i, R_i\}$ be the endpoints of γ_i, where $L_i \in [Q_{j-1}Q_j] \subset \mathcal{L}$ and $R_i \in \mathcal{R}$.

Case 1. The sequence $\{\partial\Delta_u, \gamma_i\}_{1 \leq i \leq \alpha}$ is partially ordered. Then $\{\partial\Delta_u, \gamma_i\}_{1 \leq i \leq \alpha}$ is ordered $\partial\Delta_u \prec \gamma_1 \prec \cdots \prec \gamma_\alpha$. In particular, $X_u < L_1 \leq L_2 \leq \cdots \leq L_\alpha$. Notice that $Y_u \leq P_k$. By Lemma 2.3, $R_1 \leq P_{k+1}$, and so on, we obtain

$$R_\alpha \leq P_{k+\alpha}. \tag{4.5}$$

Since Ω_u is of type (II) and Ω_1 is of type (I), $\Delta_u \subset \Delta_1$, which says that $\sigma_1 \subset D_u$ and thus that $Y_1 < X_1 < X_u$. As it turns out, $X_1 < L_1$. Now, by the same argument of Lemma 3.3, one shows that

$$X_\alpha < L_\alpha. \tag{4.6}$$

Denote by $\gamma_0 \in \{\varrho^{-1}(\varrho(\partial\sigma_{w_0}))\}$ the geodesic obtained from Lemma 2.2; which says that $\gamma_0 \subset \Delta_{w_0}$ and $L_0 = \gamma_0 \cap \mathcal{L} \in [Q_{j-1}Q_j]$. Notice that $(\sigma_\alpha, \sigma_0)$ is a pair of nested half-planes. By the definition, σ_α does not cover Q_j while σ_0 covers Q_j. We have $\sigma_\alpha \subset \sigma_{w_0}$, which implies that $L_0 < Y_\alpha < X_\alpha$. Together with (4.6), we have $L_0 < L_\alpha$. But γ_0 is disjoint from γ_α. So $R_0 < R_\alpha$. By combining (4.5), we conclude that $R_0 < R_\alpha \le R_{k+\alpha}$. This also yields that $g(R_0) < P_{k+\alpha+1}$; that is,

$$|Y_u g(R_0)| \le \alpha + 1. \tag{4.7}$$

Case 2. $\{\partial\Delta_u, \gamma_i\}_{1\le i\le\alpha}$ is not partially ordered. In this case, by a similar argument of Lemma 3.4, (4.7) remains valid.

Now $g(\gamma_0) \in \{\varrho^{-1}(\varrho(\partial\sigma_{w_0}))\}$ is the geodesic that corresponds to w_0 and is obtained from Lemma 2.2, and moreover, one endpoint $g(L_0)$ of $g(\gamma_0)$ lies in $[Q_jQ_{j+1}]$. From Lemma 3.6, we assert that

$$|g(R_0)R_{r+1}| \le r. \tag{4.8}$$

But from Lemma 2.1(i),

$$|R_{r+1}g(R_{r+1})| = 1. \tag{4.9}$$

Suppose $\mathscr{B} \ne \emptyset$. Recall that Δ_{b_i}, $1 \le i \le \beta$, are the distinguished half-planes for b_i. Write $\sigma_{b_i} = \mathbf{H}\backslash\overline{\Delta}_{b_i}$. Let $\gamma_i' \in \{\varrho^{-1}(\varrho(\partial\sigma_{b_i}))\}$, $1 \le i \le \beta$ and each $\gamma_i' \subset \Delta_{b_i}$, be obtained from Lemma 2.2; that is, each γ_i' intersects $axis(g)$ and $[Q_{j+d-1}Q_{j+d}]$. Let $\{L_i', R_i'\}$ be the endpoints of γ_i', where $L_i' \in [Q_{j+d-1}Q_{j+d}] \subset \mathcal{L}$ and $R_i' \in \mathcal{R}$.

Case 1. $X_\beta < L_\beta' \le Q_{j+d}$ (here we recall that $\{X_\beta, Y_\beta\} = \partial\sigma_{b_\beta} \cap \mathcal{L}$ with $Y_\beta < X_\beta$). We may first assume that $L_\beta' < X_v$. Notice that

$\delta(X_\beta, X_v) < 1$ (otherwise, Ω_v and Ω_{b_β} would be disjoint, contradicting that $d_{\mathcal{C}}(b_\beta, v) = 1$). So $\delta(L'_\beta, X_v) < 1$, and hence $\delta(R'_\beta, Y_v) < 1$ (otherwise, $g^{-1}(\partial\Delta_v)$ crosses γ'_β, contradicting $d_{\mathcal{C}}(b_\beta, v) = 1$). Therefore, $| R'_\beta Y_v | \le 1$. But from Lemma 2.4, we obtain

$$| g(R_{r+1}) R'_\beta | \le \beta. \tag{4.10}$$

It follows that

$$| g(R_{r+1}) R'_\beta | + | R'_\beta Y_v | \le \beta + 1. \tag{4.11}$$

If $L'_\beta \ge X_v$, then we must have $R'_\beta \ge Y_v$. It is clear that $| g(R_{r+1}) Y_v | \le | g(R_{r+1}) R'_\beta | \le \beta < \beta + 1$. Hence (4.11) remains valid.

Case 2. $Q_{j+d-1} \le L'_\beta \le X_\beta < Q_{j+d}$. In this case, $L'_\beta < Y_\beta$ (elements in $\{\varrho^{-1}(\varrho(\gamma'_\beta))\}$ are mutually disjoint). From Lemma 3.3, $\{g(\gamma_{r+1}), \gamma'_1, ..., \gamma'_\beta\}$ is not partially ordered. By the same argument of Lemma 3.4, $| g(R_{r+1}) R'_\beta | \le \beta - 1$. We claim that $| R'_\beta Y_v | \le 2$. Indeed, inequalities $L'_\beta < Y_\beta < X_\beta < Q_{j+d}$ and $\delta(Y_\beta, X_v) < 1$ lead to that $\delta(L'_\beta, X_v) < 2$, which yields that $\delta(R'_\beta, Y_v) < 2$ (otherwise, $g^{-1}(\partial\Delta_v)$ or $g^{-2}(\partial\Delta_v)$ would intersect γ'_β, contradicting $d_{\mathcal{C}}(b_\beta, v) = 1$). So we conclude that $| R'_\beta Y_v | \le 2$, and thus (4.11) remains true.

In both the cases, we have established (4.11). Now (4.7), (4.8), (4.9), (4.10) and (4.11) combine to yield

$$\begin{aligned} | Y_u Y_v | &= | Y_u g(R_0) | + | g(R_0) R_{r+1} | \\ &\quad + | R_{r+1} g(R_{r+1}) | + (| g(R_{r+1}) R'_\beta | + | R'_\beta Y_v |) \\ &\le \alpha + 1 + r + 1 + (\beta + 1). \end{aligned} \tag{4.12}$$

It follows from (4.12) and (4.2) that $| Y_u Y_v | \le s + 1$.

Similarly, one shows that $|Y_u Y_v| \leq s+1$ when $\mathscr{B} = \emptyset$. Next, we consider some special cases.

If $\mathscr{G}_0 = [u, \mathscr{A}, v]$ for $\mathscr{A} = \{a_1, \ldots, a_\alpha\} = \{v_1, \ldots, v_s\}$, then $s = \alpha$ and $\bigcup \sigma_{a_i}$ does not cover Q_j. This implies that $(X_u X_v]$ cover at most one labeled point which is Q_j, which says $|X_u X_v| \leq 1$. By a similar argument of (4.7),

$$|Y_u Y_v| \leq \alpha + 1 = s + 1.$$

If $\mathscr{G}_0 = [u, \mathscr{A}, w_0, \mathscr{B}, v]$ for $\mathscr{A}$ and $\mathscr{B}$ sub-sequences of $\{v_1, \ldots, v_s\}$, then

$$s = \alpha + 1 + \beta \text{ and } d = 1. \tag{4.13}$$

In this case, it is easy to see that $|X_u X_v| \leq 2\ (= d+1)$. By the argument of (4.7), we can deduce that $|Y_u g(R_0)| \leq \alpha + 1$. But the same argument of (4.11) yields that $|g(R_0) Y_v| \leq \beta + 1$. It follows from (4.13) that

$$|Y_u Y_v| = |Y_u g(R_0)| + |g(R_0) Y_v| \leq (\alpha + 1) + (\beta + 1) = s + 1. \qquad \square$$

Finally, we can easily handle a special case where all regions involved are type (II) regions.

Lemma 4.4. *Let* $[u_0, u_1, \ldots, u_r, u_{r+1}]$, $r \geq 0$, *be a geodesic connecting* u_0 *and* u_{r+1}. *Suppose that these vertices* u_i, $0 \leq i \leq r+1$, *are mapped to type* (II) *regions* Ω_i *with respect to g. We have* $|X_0 X_{r+1}| \leq r+1$ *and* $|Y_0 Y_{r+1}| \leq r+1$, *where* $\{X_i, Y_i\}$ *are endpoints of* $\partial \Delta_i$ *and* $X_i \in \mathcal{L}$ *and* $Y_i \in \mathcal{R}$.

Proof. From Lemma 2.1(iii), we have

$$|X_0 X_{r+1}| = \sum_{j=0}^{r} |X_j X_{j+1}| \text{ and } |Y_0 Y_{r+1}| = \sum_{j=0}^{r} |Y_j Y_{j+1}|. \tag{4.14}$$

By Lemma 4.1, for $0 \leq j \leq r$, we know that

$$| X_j X_{j+1} | \leq 1 \text{ and } | Y_j Y_{j+1} | \leq 1.$$

It then follows from (4.14) that $| X_0 X_{r+1} | \leq r+1$ and $| Y_0 Y_{r+1} | \leq r+1$, as asserted. □

5. Proof of Theorem 1.1 and Theorem 1.2

Let $f \in \mathscr{F}$ be any pseudo-Anosov element. We know that f can be written as $f = g^*$, where $g \in G$ is an essential hyperbolic element. Let $\widetilde{u}_0 \in \mathcal{C}_0(\widetilde{S})$ and let $u_0 \in F_{\widetilde{u}_0}$ be such that $\Omega'_0 = \Omega_{u_0}$ is a type (II) region with respect to g. Then all regions $\Omega'_0, \Omega'_1 = g(\Omega'_0), \dots, \Omega'_m = g^m(\Omega'_0)$ are of type (II).

We now prove that (1.1) holds for all integers $m \geq 12$ (in [15, 16] (1.1) was established when $0 \leq m \leq 11$). Suppose that

$$[u_0, v_1, v_2, \dots, v_s, u_m], \text{ where } m \geq 12 \text{ and } u_m = f^m(u_0), \tag{5.1}$$

is a geodesic in $\mathcal{C}(S)$ joining u_0 to u_m. Let

$$\Omega'_0, \Omega_1, \Omega_2, \dots, \Omega_s, \Omega'_m \tag{5.2}$$

be the regions corresponding to $u_0, v_1, \dots, v_s, u_m$, respectively. These regions can be classified as type (I) and type (II) regions. First consider two special cases:

Case 1. Besides Ω'_0 and Ω'_m, all $\Omega_1, \Omega_2, \dots, \Omega_s$ are also type (II) regions. By Lemma 4.4, we obtain

$$| X_0 X_m | \leq s+1 \text{ and } | Y_0 Y_m | \leq s+1. \tag{5.3}$$

If $\widetilde{u}_0$ intersects $\varrho(axis(g))$ more than once, then $Q_0 < X_0 < Q_1$ and $P_0 < Y_0 < P_1$. Thus, $Q_m < X_m < Q_{m+1}$ and $P_m < Y_m < P_{m+1}$ (see Figure 3). If $\widetilde{u}_0$ intersects $\varrho(axis(g))$ once, then $X_0 = Q_1$ and $Y_0 = P_1$.

Hence $X_m = Q_{m+1}$ and $Y_m = P_{m+1}$. In both the cases, we have $|X_0X_m| = |Y_0Y_m| = m$. From (5.3), we obtain $s+1 \geq m$. That is,

$$d_{\mathcal{C}}(u_0, u_m) = s + 1 \geq m. \tag{5.4}$$

Case 2. Except Ω_0' and Ω_m', all $\Omega_1, \ldots, \Omega_s$ are type (I) regions. Then they must stay on one side of $axis(g)$. Suppose that all $\sigma_i = \mathbf{H}\backslash\overline{\Delta}_i$, $1 \leq i \leq s$, are supported on $\mathcal{L}$. By Lemma 4.3, $|X_0X_m| \leq \left[\frac{s-2}{3}\right] + 2$ and $|Y_0Y_m| \leq s+1$. Since $\Omega_s \cap \Omega_m' \neq \emptyset$ and Ω_s is of type (I), $\sigma_s \subset D_m$ for $D_m = \mathbf{H}\backslash\{\overline{\Delta}_m', \overline{\Delta}_m\}$. This implies that

$$Q_m < Y_s < X_s < X_m \leq Q_{m+1}.$$

By assumption, we know that $Q_0 < X_0 \leq Q_1$ and $Q_m < X_m \leq Q_{m+1}$. Notice that $X_0 = Q_1$ if and only if $X_m = Q_{m+1}$. Hence $|X_0X_m| = m$. It turns out that

$$m \leq \left[\frac{s-2}{3}\right] + 2 \leq \frac{s-2}{3} + 2.$$

So $s \geq 3m - 4$, which together with $m > 3$ leads to that

$$d_{\mathcal{C}}(u_0, u_m) = s + 1 \geq 3m - 3 > m. \tag{5.5}$$

In general, $\{\Omega_1, \ldots, \Omega_s\}$ contain both type (I) and type (II) regions. Rewrite (5.2) as

$$\Omega_{p(0)} = \Omega_0', \Gamma_{p(0)}, \Omega_{p(1)}, \Gamma_{p(1)}, \ldots, \Omega_{p(M)}, \Gamma_{p(M)}, \Omega_m', M \geq 1, \tag{5.6}$$

where $\Omega_{p(i)}$, $0 \leq i \leq M$, are all type (II) regions and $\Gamma_{p(i)}$ consists of consecutive type (I) regions if not empty. Suppose that $\Gamma_{p(i)} \neq \emptyset$. Write $\Gamma_{p(i)} = \{\omega_{p(i)+1}, \ldots, \omega_{p(i)+r(i)}\}$, where every $\omega_{p(i)+j}$ is a type (I) region and is contained in $\sigma_{p(i)+j} = \mathbf{H}\backslash\overline{\Delta}_{p(i)+j}$. Here we recall that $\Delta_{p(i)+j}$ is the distinguished half-plane for $v_{p(i)+j}$. By Lemma 3.1, any pair

$(\sigma_{p(i)+j}, \sigma_{p(i)+j+1})$ for successive regions $\omega_{p(i)+j}, \omega_{p(i)+j+1}$ in $\Gamma_{p(i)}$ is a pair of nested sets, which means that they are supported on $\mathcal{L}$ or on $\mathcal{R}$. Whence all elements in $\Gamma_{p(i)}$ are supported on $\mathcal{L}$ or on $\mathcal{R}$. Throughout we assume that the first type (I) region in (5.6) is supported on $\mathcal{L}$.

The integer function $p(i)$ in (5.6) satisfies the recursive condition:

$$p(0) = 0, \text{ and for } i \geq 1, \; p(i) - p(i-1) = r(i-1) + 1. \tag{5.7}$$

It is obvious that $s = \sum_{j=0}^{M} r(j) + M = \sum_{j=0}^{M-1} r(j) + r(M) + M$. We thereby obtain

$$\sum_{j=0}^{M-1} r(j) = s - r(M) - M. \tag{5.8}$$

Recall that $\{X_{p(i)}, Y_{p(i)}\}$ are endpoints of $\partial\Delta_{p(i)}$, where $X_{p(i)} \in \mathcal{L}$ and $Y_{p(i)} \in \mathcal{R}$ and $\Delta_{p(i)}$ is the distinguished half-plane for $v_{p(i)}$. By Lemma 2.1(iii),

$$|X_{p(0)}X_{p(M)}| = \sum_{i=0}^{M-1} |X_{p(i)}X_{p(i+1)}|$$

and

$$|Y_{p(0)}Y_{p(M)}| = \sum_{i=0}^{M-1} |Y_{p(i)}Y_{p(i+1)}|. \tag{5.9}$$

Let K denote the number of zeros in $\{r(0), r(1), ..., r(M-1)\}$. From the construction, $\tilde{u}_0$ intersects $\varrho(axis(g))$ at least once. We deduce that

$$Q_0 < X_{p(0)} = X_0 \leq Q_1 \text{ and } P_0 < Y_{p(0)} = Y_0 \leq P_1. \tag{5.10}$$

See Figure 3. For each $0 \leq i \leq M-1$ with $r(i) \geq 2$, we define

$$b_i = \begin{cases} r(i) + 1 & \text{if } \Gamma_{p(i)} \text{ is supported on } \mathcal{L}, \\ \left\lceil \dfrac{r(i) - 2}{3} \right\rceil + 2 & \text{if } \Gamma_{p(i)} \text{ is supported on } \mathcal{R} \end{cases}$$

and if $r(i) = 1$, we define

$$b_i = \begin{cases} r(i) + 1 & \text{if } \Gamma_{p(i)} \text{ is supported on } \mathcal{L}, \\ 1 & \text{if } \Gamma_{p(i)} \text{ is supported on } \mathcal{R}. \end{cases}$$

Since the condition $r(i) \ge 2$ guarantees that $[(r(i) - 2)/3] + 2 \le r(i) + 1$. In the case of $r(i) = 1$, it is automatic that $1 < r(i) + 1$. We see that $b_i \le r(i) + 1$ for all $r(i) > 0$. There are two cases to consider:

Case 1. $P_m \le Y_{p(M)} < P_{m+1}$. From (5.9), (5.10) and Lemmas 4.1-4.3, we know that

$$\begin{aligned} m = | Y_{p(0)} Y_{p(M)} | &= K + \sum_{i=0}^{M-1} \{| Y_{p(i)} Y_{p(i+1)} |; r(i) \ge 1\} \\ &\le K + \sum_{i=0}^{M-1} \{b_i; r(i) \ge 1\}. \end{aligned} \tag{5.11}$$

From the definition of b_i and (5.11), we obtain

$$\begin{aligned} m &\le K + \sum_{i=0}^{M-1} \{r(i) + 1; r(i) \ge 1\} \\ &= K + M - K + \sum_{i=0}^{M-1} \{r(i); r(i) \ge 1\} \\ &= M + \sum_{i=0}^{M-1} \{r(i); r(i) \ge 1\}. \end{aligned} \tag{5.12}$$

But

$$s = \sum_{j=0}^{M} r(j) + M = M + \sum_{i=0}^{M-1} r(i) + r(M).$$

So

$$\sum_{i=0}^{M-1} r(i) = s - M - r(M). \tag{5.13}$$

Since $r(M) \geq 0$, (5.13) and (5.12) combine to yield

$$m \leq M + (s - M - r(M)) = s - r(M) \leq s.$$

Hence

$$d_{\mathcal{C}}(u_0, u_m) = s + 1 \geq m + 1. \tag{5.14}$$

For $Q_m \leq X_{p_M} < Q_{m+1}$, the argument is the same.

Case 2. $X_{p(M)} < Q_m$ and $Y_{p(M)} < P_m$. Since $d_{\mathcal{C}}(v_s, u_m) = 1$, $\Gamma_{p(M)} \neq \varnothing$. That is, if we denote $\Gamma_{p(M)} = \{\omega_{p(M)+1}, \ldots, \omega_{p(M)+r(M)}\}$, then $r(M) \geq 1$. It is obvious that $s = p(M) + r(M)$ and suppose that ω_s is supported on $\mathcal{L}$, then $\{X_s, Y_s\} := \partial\sigma_s \cap \mathbf{S}^1 \subset \mathcal{L}$ with $Y_s < X_s$.

From construction (here we refer to Figure 3), $\Omega'_m = g^m(\Omega'_0)$ and $\Omega'_0 \subset \mathbf{H}\backslash\{\overline{\Delta}'_0, \overline{\Delta}_0\}$. This tells us that $\partial\Delta_0$ lies between $\partial\Delta'_0$ and $\partial\Delta'_1$. Thus, $\partial\Delta_m$ lies between $\partial\Delta'_m$ and $\partial\Delta'_{m+1}$ (here we recall that Δ_m is the distinguished half-plane for u_m). That is to say,

$$Q_m < X_m \leq Q_{m+1} \text{ and } P_m < Y_m \leq P_{m+1}. \tag{5.15}$$

By hypothesis, $d_{\mathcal{C}}(v_s, u_m) = 1$. This yields that $\omega_s \cap \Omega'_m \neq \varnothing$. From (5.15), we conclude that

$$Q_m < Y_s < X_s < X_m \leq Q_{m+1}.$$

Let L be the smallest integer such that $X_{p(M)} < Q_L \leq Q_m$. Then $L \leq m$. Since $Q_0 < X_{p(0)} \leq Q_1$, we have

$$L - 2 \leq |\, X_{p(0)} X_{p(M)} \,| \leq L - 1 \leq m - 1. \tag{5.16}$$

On the other hand, Lemmas 4.1-4.3 and (5.9) yield that

$$\begin{aligned}|X_{p(0)}X_{p(M)}| &= \sum_{i=0}^{M-1} |X_{p(i)}X_{p(i+1)}| \le K + \sum_{i=0}^{M-1}\{b_i;\ r(i) \ge 1\} \\ &\le K + \sum_{i=0}^{M-1}\{r(i)+1;\ r(i)\ge 1\} \\ &= K + \sum_{i=0}^{M-1}\{r(i);\ r(i)\ge 1\} + (M-K) \\ &= M + \sum_{i=0}^{M}\{r(i);\ r(i)\ge 1\} - r(M). \qquad (5.17)\end{aligned}$$

From (5.2) and (5.6), we know that $M + \sum_{i=0}^{M}\{r(i);\ r(i)\ge 1\} \le s$, which simplifies to

$$\sum_{i=0}^{M}\{r(i);\ r(i)\ge 1\} \le s - M. \qquad (5.18)$$

Putting (5.18) and (5.17) together, we conclude that

$$|X_{p(0)}X_{p(M)}| \le M + (s-M) - r(M). \qquad (5.19)$$

From (5.16), $|X_{p(0)}X_{p(M)}|$ is either $L-1$ or $L-2$. By (5.19), we obtain

$$s \ge L - 2 + r(M). \qquad (5.20)$$

Since $\Gamma_{p(M)}$ covers at least $m - L + 1$ labeled points $\{Q_L, ..., Q_m\}$ and by Lemma 3.2 of [16], at least four successive regions in $\Gamma_{p(M)}$ are needed to cover a pair of any successive labeled points in $\{Q_L, ..., Q_m\}$. Note also that the first region in $\Gamma_{p(M)}$ does not cover Q_L. We conclude that

$$m - L + 1 \le \left\lceil \frac{r(M)-2}{3} \right\rceil + 1 \le \frac{r(M)-2}{3} + 1. \qquad (5.21)$$

(5.21) simplifies to $3(m - L) \leq r(M) - 2$ or

$$r(M) - 1 \geq 3m - 3L + 1. \tag{5.22}$$

From (5.20) and (5.22), we obtain $s \geq L + 3m - 3L = 3m - 2L$. But $L \leq m$. Hence $s \geq 3m - 2m = m$, that is, $s + 1 \geq m + 1$, which says that

$$d_{\mathcal{C}}(u_0, u_m) \geq m + 1. \tag{5.23}$$

By combining (5.4), (5.5), (5.14) and (5.23), we conclude that $d_{\mathcal{C}}(u_0, u_m) \geq m$, which proves Theorem 1.2. Theorem 1.1 follows immediately from Theorem 1.2. □

6. Unboundedness of Sequence of Stable Translation Lengths

According to Theorem 1.2, for any pseudo-Anosov element $f \in \mathscr{F}$, we can find a vertex $u \in \mathcal{C}_0(S)$ such that for all positive integers *m* and *n*, we have $d_{\mathcal{C}}(u, f^{mn}(u)) \geq mn$. This particularly implies that

$$\frac{d_{\mathcal{C}}(u, (f^m)^n(u))}{n} \geq m \text{ for any integers } n.$$

Thus, $\tau_{\mathcal{C}}(f^m) \geq m$. Notice that *m* is also arbitrary. We conclude that $\tau_{\mathcal{C}}(f^m) \to +\infty$ as $m \to +\infty$. This proves the following result:

Theorem 6.1. *There exists a sequence* $\{f_1, f_2, \ldots\} \subset \mathscr{F}$ *of pseudo-Anosov elements such that* $\tau_{\mathcal{C}}(f_m) \to +\infty$ *as* $m \to +\infty$.

Remark. By a slight modification, we can show that elements f_i in the sequence can be chosen as primitive elements.

7. Bi-infinite Geodesics Invariant under Pseudo-Anosov's $f \in \mathscr{F}$

Let $\mathscr{L}$ denote the set of primitive oriented filling closed geodesics on $\widetilde{S}$ and $\mathscr{L}^*$ the subset of $\mathscr{L}$ consisting of those filling geodesics intersecting every simple closed geodesic more than once. It is not difficult to see that both $\mathscr{L}^*$ and $\mathscr{L}\backslash\mathscr{L}^*$, are not empty. For every $\gamma \in \mathscr{L}\backslash\mathscr{L}^*$, let $\mathscr{L}_\gamma$ be the collection of simple closed geodesics on $\widetilde{S}$ intersecting γ only once.

An infinite path $[\ldots, u_{-m}, \ldots, u_0, \ldots, u_m, \ldots]$, where all $u_i \in \mathcal{C}_0(S)$, is called a *bi-infinite geodesic* if u_{-m} and u_m both tend to points in $\partial\mathcal{C}(S)$ and for any *m*, the subpath $[u_{-m}, \ldots, u_0, \ldots, u_m]$ is a geodesic segment connecting u_{-m} and u_m.

Theorem 7.1. *Let S be of type* $(p, 1)$ *with* $p > 1$. *Let* $f \in \mathscr{F}$ *be a pseudo-Anosov element, and let* $\gamma \subset \mathscr{L}$ *be determined by f. Assume that* $\gamma \in \mathscr{L}\backslash\mathscr{L}^*$. *Then f preserves at least one bi-infinite geodesic in* $\mathcal{C}(S)$. *Furthermore, there is an injective map*:

$$I : \mathscr{L}_\gamma \to \{f\text{-invariant bi-infinite geodesics in } \mathcal{C}(S)\}$$

so that $I(\mathscr{L}_\gamma)$ *consists of disjoint bi-infinite geodesics.*

Proof. Fix $\gamma \in \mathscr{L}\backslash\mathscr{L}^*$ and for every $\widetilde{u}_0 \in \mathscr{L}_\gamma$, let $u_0 \in F_{\widetilde{u}_0}$ be such that Ω_{u_0} is a type (II) region with respect to *g*, where $g^* = f$. We then define

$$I(\widetilde{u}_0) = [\ldots, f^{-m}(u_0), \ldots, f^{-1}(u_0), u_0, f(u_0), \ldots, f^m(u_0), \ldots]. \tag{7.1}$$

For any other $u_0' \in F_{\widetilde{u}_0}$ with $\Omega_{u_0'} \cap axis(g) \neq \emptyset$, we have $\widetilde{u}_0' = \widetilde{u}_0$. Hence $\Omega_{u_0'} \in \mathscr{R}_{\widetilde{u}_0}$. It follows that there is an integer *j* such that $\Omega_{u_0'} = g^j(\Omega_{u_0})$, that is $u_0' = f^j(u_0)$ which tells us that the map *I* is well-defined. From (5.4), (5.5), (5.14) and (5.23), one shows that $I(\widetilde{u}_0)$ for every $\widetilde{u}_0 \in \mathscr{L}_\gamma$ is an *f*-invariant bi-infinite geodesic in $\mathcal{C}(S)$.

To show that *I* is injective, we suppose $I(\widetilde{u}_0) = I(\widetilde{v}_0)$ for some $\widetilde{u}_0$, $\widetilde{v}_0 \in \mathscr{L}_\gamma$. Let $v_0 \in F_{\widetilde{v}_0}$ be such that Ω_{v_0} is a type (II) region with respect to g. From the definition (7.1), we have $v_0 = f^i(u_0)$ for some integer i. Since $f \in \mathscr{F}$, we see that u_0 and $v_0 \in F_{\tilde{u}_0}$ which says $\tilde{v}_0 = \tilde{u}_0$. Similar arguments also yield that $I(\mathscr{L}_\gamma)$ consists of disjoint bi-infinite geodesics in $\mathcal{C}(S)$. □

Question. Is the map I also surjective?

Remark. Bowditch [2] proved that for a surface $S_{p,n}$ with $3p + n - 4 > 0$, there exists a positive integer m such that for any pseudo-Anosov mapping class $f \in Mod(S_{p,n})$, f^m preserves some bi-infinite geodesic in $\mathcal{C}(S_{p,n})$.

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