Far East Journal of Mathematical Sciences (FJMS)
© 2018 Pushpa Publishing House, Allahabad, India
http://www.pphmj.com
http://dx.doi.org/10.17654/MS103071229
Volume 103, Number 7, 2018, Pages 1229-1241 **ISSN: 0972-0871**

THE AUSLANDER-REITEN QUIVER OF $k\tilde{Q}$ WITH THE DUAL QUIVER OF *ADE* TYPE

Mianmian Zhang

Hangzhou Normal University
P. R. China

Abstract

Let Q be a finite quiver of type A_n, $n \geq 1$, D_n, $n \geq 4$, E_6, $\sigma \in Aut(Q)$, k be an algebraic closed field with characteristic does not divide the order of σ. Let $\tilde{Q}$ be the dual quiver of Q, which means that $\text{mod}\, k\tilde{Q}$ is Morita equivalent to $\text{mod}\, kQ\# k\langle\sigma\rangle$. Then $\tilde{Q}$ is also a finite quiver of type A_n, $n \geq 1$, D_n, $n \geq 4$, E_6. In this paper, we give a way to draw the Auslander-Reiten quiver $\Gamma_{\tilde{Q}}$ from the Auslander-Reiten quiver Γ_Q, which is independent of the orientations of Q in each cases.

1. Introduction

The representations of the Dynkin quivers are treated in many books. It is well-known that the Dynkin quivers are representation finite, that the dimension vectors of the indecomposable representations are just the corresponding positive roots and that any representation is the direct sum of indecomposable representations. Ringel in [6] presented three building

Received: October 16, 2017; Revised: December 14, 2017; Accepted: January 29, 2018
2010 Mathematics Subject Classification: 16G10.
Keywords and phrases: Auslander-Reiten quiver, dual quiver, Dynkin quiver.

blocks for dealing with representations of Dynkin quivers. Hubery constructed in [3, 4] the dual quiver with automorphism $(\tilde{Q}, \tilde{\sigma})$, where $\tilde{Q}$ is the Ext-quiver of $kQ\# k\langle\sigma\rangle$ and $\tilde{\sigma}$ is the automorphism of $k\tilde{Q}$ induced from an admissible automorphism σ, i.e., Q has no arrow connecting two vertices in the same σ-orbit, and k is an algebraically closed field of characteristic not dividing n. Hubery uses the dual quiver $(\tilde{Q}, \tilde{\sigma})$ to prove generalizing of Kac's theorem. Moreover, $k\tilde{Q}$ is isomorphic to the basic algebra that is Morita equivalent to $kQ\# k\langle\sigma\rangle$, and $Q, \tilde{Q}$ have the same representation type.

We in [9] investigate the relationship between indecomposable modules over the path algebra kQ and the skew group algebra $kQ\# k\langle\sigma\rangle$, respectively, where k is an algebraically closed field with the characteristic does not divide the order of σ, Q is a finite connected quiver without oriented cycles, and $\sigma \in Aut(Q)$. We prove that a $kQ\# k\langle\sigma\rangle$-module is indecomposable if and only if it is an indecomposable $\langle\sigma\rangle$-equivalent kQ-module. Namely, a method is given in order to induce all indecomposable $kQ\# k\langle\sigma\rangle$-modules from each indecomposable $\langle\sigma\rangle$-equivalent kQ-module. According to this construction, we in [8] prove that the dual quiver $\tilde{\Gamma}_Q$ of the Auslander-Reiten quiver Γ_Q of kQ, the Auslander-Reiten quiver of $kQ\# k\langle\sigma\rangle$, and the Auslander-Reiten quiver $\Gamma_{\tilde{Q}}$ of $k\tilde{Q}$, where $\tilde{Q}$ is the dual quiver of Q, are isomorphism, where Q is a finite quiver of type $A_n, n \geq 1, D_n, n \geq 4, E_6, E_7$ and E_8, $\sigma \in Aut(Q)$, k is an algebraic closed field with characteristic does not divide the order of σ.

We in [8] concerned the shape of Γ_Q and $\Gamma_{\tilde{Q}}$ as quivers, we get $\Gamma_{\tilde{Q}} \cong \tilde{\Gamma}_Q$. In this paper, we concern Γ_Q and $\Gamma_{\tilde{Q}}$ as Auslander-Reiten quivers, which means we need to characterize indecomposable modules as vertices and irreducible morphisms as arrows.

Assume that all the modules are unital and finitely generated, Q is a finite quiver without oriented cycles, $\sigma \in Aut(Q)$ and that k is always an algebraically closed field with characteristic does not divide the order of σ. All the concepts and notations on skew group algebras and Auslander-Reiten quivers can be found in [1, 5].

Let X be a kQ-module. Define a kQ-module ${}^{\sigma}X$ by taking the same underlying vector space as X but with the new action:

$$p \cdot x := \sigma^{-1}(p)x, \ \forall p \in kQ.$$

If $\phi : X \to Y$ is a kQ-module homomorphism, then we obtain a homomorphism ${}^{\sigma}\phi : {}^{\sigma}X \to {}^{\sigma}Y$ as follows. We set ${}^{g}\phi = \phi.$ Then

$$\phi(p \cdot x) = \phi(g^{-1}(p)x) = g^{-1}(p)\phi(x) = p \cdot \phi(x).$$

We say two indecomposable kQ-modules X, Y are related (we denote it as $X \sim Y$) if $X \cong {}^{g}Y$ for some $g \in \langle\sigma\rangle$. For a fixed indecomposable kQ-module X, write m_X to be the minimal positive integer satisfying ${}^{\sigma^{m_X}}X \cong X$. In [9], we know that m_X exists, and moreover $r_X = n/m_X$ is an integer. Recall the relation between indecomposable $kQ\# k\langle\sigma\rangle$-modules and kQ-modules as follows, see in [9] for details:

(1) Let X be an indecomposable kQ-module, and $r_X = n/m_X$. Then we can induce r_X non-isomorphic indecomposable $kQ\# k\langle\sigma\rangle$-modules $\{\mathcal{X}^{(0)}, \mathcal{X}^{(1)}, \ldots, \mathcal{X}^{(r_X-1)}\}$ from the kQ-module $X \oplus {}^{\sigma}X \oplus \cdots \oplus {}^{\sigma^{m-1}}X$.

(2) Let X and Y be two indecomposable kQ-modules such that $X \sim Y$. We have that $m_X = m_Y$, $r_X = r_Y$, and $\{\mathcal{X}^{(0)}, \mathcal{X}^{(1)}, \ldots, \mathcal{X}^{(r_X-1)}\}$ is the same as $\{\mathcal{Y}^{(0)}, \mathcal{Y}^{(1)}, \ldots, \mathcal{Y}^{(r_Y-1)}\}$ under isomorphism.

(3) For any indecomposable $kQ\# k\langle\sigma\rangle$-module $\mathcal{Y}$, there exists an indecomposable kQ-module X such that $\mathcal{Y} \cong \mathcal{X}^{(i)}$ for some $i \in \{0, 1, \ldots, r_X - 1\}$.

Let Q be a finite connected quiver without multiple edges, $\sigma \in Aut(Q)$ and n is the order of σ. Recall the construction of the dual quiver $\tilde{Q} = (\tilde{Q}_0, \tilde{Q}_1)$, which is the Ext-quiver of the basic algebra that is Morita equivalent to $kQ\# k\langle\sigma\rangle$ as follows, see in [3] for details:

(1) Let $\mathbf{I}$ be the set of σ-orbits in Q_0 and $d_{\mathbf{i}}$ be the number of vertices in $\mathbf{i} \in \mathbf{I}$.

(2) Label the vertices in $\mathbf{i} \in \mathbf{I}$ as the set of pairs $(\mathbf{i}, j)$, $j \in \mathbb{Z}/d_{\mathbf{i}}\mathbb{Z} = \{0, 1, ..., d_{\mathbf{i}} - 1\}$ such that $\sigma(\mathbf{i}, j) = (\mathbf{i}, j + 1)$.

(3) $\tilde{Q}_0 = \{(\mathbf{i}, \mu) | \mathbf{i} \in \mathbf{I}, 0 \leq \mu < n/d_{\mathbf{i}}\}$.

(4) Let $\rho : (\mathbf{i}, 0) \to (\mathbf{j}, l) \in Q_1$. Set $t = lcm(d_{\mathbf{i}}, d_{\mathbf{j}})$. Let u be the integer such that $\sigma^t(\rho) = \zeta^{tu}\rho$, where ζ is the nth primitive root of 1. We get arrows $(\mathbf{i}, \mu) \to (\mathbf{j}, \nu) \in \tilde{Q}_1$ for $\mu \equiv u + \nu \bmod n/t$. And all the arrows in $\tilde{Q}_1$ can be gotten by this way.

From now on, we consider Q such that kQ is of finite representation type. Then it is well-known that Q is one of the types A_n, $n \geq 1$, D_n, $n \geq 4$ and E_6, E_7, E_8. The valuation on each arrow of Γ_Q is trivial, and Γ_Q is a finite connected quiver without multiple edges.

2. Draw the Auslander-Reiten Quiver of $\Gamma_{\tilde{Q}}$

2.1. The quiver Q of type A_{2n-1}, $n \geq 3$ with nontrivial quiver automorphism σ: In this case, the underlying graph of Q is

$$1 \overset{\alpha_1}{\text{——}} 2 \overset{\alpha_2}{\text{——}} \cdots n-1 \overset{\alpha_{n-1}}{\text{——}} n \overset{\alpha_n}{\text{——}} n+1 \overset{\alpha_{n+1}}{\text{——}} \cdots 2n-2 \overset{\alpha_{2n-2}}{\text{——}} 2n-1$$

and the orientations of Q satisfy $\delta(i, i+1) = \delta(2n - i, 2n - (i+1))$, $\forall 1 \leq i \leq n$, where

$$\delta(i, j) = \begin{cases} 1, & i \to j, \\ -1, & i \leftarrow j. \end{cases}$$

Let X_{ij}, $1 \le i \le j \le 2n-1$ be the kQ-module defined as:

$$X_{ij_s} = \begin{cases} k, & i \le s \le j, \\ 0, & \text{otherwise}, \end{cases}$$

$$X_{ij_{\alpha_s}} = \begin{cases} 1_k, & i \le s \le j-1, \\ 0, & \text{otherwise}. \end{cases}$$

It is well-known in [2] that $\{X_{ij}, 1 \le i \le j \le 2n-1\}$ classify all indecomposable kQ-modules up to isomorphism.

Let $\sigma \in Aut(Q)$ be a nontrivial isomorphism. Then the order of σ is 2, and $\{X_{ij}, 1 \le i \le j \le 2n-1\}$ is a disjoint union of the following four sets:

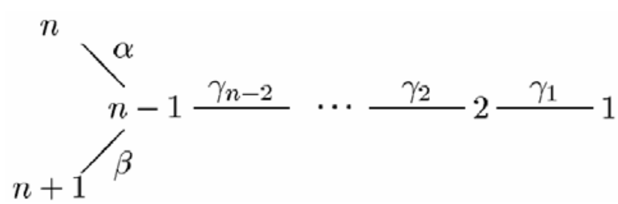
$$\{X_{ij},\ {}^{\sigma}X_{ij} = X_{(2n-j)(2n-i)},\ 1 \le i < j < n\},$$

$$\{X_{in},\ {}^{\sigma}X_{in} = X_{n(2n-i)},\ 1 \le i < n\},$$

$$\{X_{ij},\ {}^{\sigma}X_{ij} = X_{(2n-j)(2n-i)},\ 1 \le i < n < j < 2n-1,\ n-i > j-n\},$$

$$\{X_{i(2n-i)} = {}^{\sigma}X_{i(2n-i)},\ 1 \le i \le n\}.$$

The dual quiver $\widetilde{Q}$ is a quiver of type D_{n+1}, for simplicity, the underlying graph of $\widetilde{Q}$ is



and the orientations of $\widetilde{Q}$ satisfy $\delta(n-1, n+1) = \delta(n-1, n)$ in $\widetilde{Q}$ equals $\delta(n-1, n)$ in Q, and $\delta(i, i+1)$ in $\widetilde{Q}$ equals $\delta(i, i+1)$ in Q, for any $1 \le i \le n$.

Let Y_{ij}, $1 \leq i < j < n$ be the $k\widetilde{Q}$-module defined by

$$Y_{ij_s} = \begin{cases} k, & \text{if } i \leq s \leq j, \\ 0, & \text{if otherwise,} \end{cases}$$

$$Y_{ij_\gamma} = \begin{cases} 1_k, & \text{if } \gamma \in \{\gamma_i, \gamma_{i+1}, \ldots, \gamma_{j-1}\}, \\ 0, & \text{if otherwise.} \end{cases}$$

Let Y_{in}, $1 \leq i < n$ be the $k\widetilde{Q}$-module defined by

$$Y_{in_s} = \begin{cases} k, & \text{if } s \geq i, \\ 0, & \text{if otherwise,} \end{cases}$$

$$Y_{in_\gamma} = \begin{cases} 1_k, & \text{if } \gamma \in \{\gamma_i, \gamma_{i+1}, \ldots, \gamma_{n-1}, \alpha, \beta\}, \\ 0, & \text{if otherwise.} \end{cases}$$

Let Y_{ij}, $1 \leq i < n < j < 2n-1$, $n - i > j - n$ be the $k\widetilde{Q}$-module defined by

$$Y_{ij_s} = \begin{cases} k, & \text{if } s \in \{i, i+1, \ldots, 2n-1-j, n, n+1\}, \\ k^2, & \text{if } s \in \{2n-j, 2n-j+1, \ldots, n-1\}, \\ 0, & \text{if otherwise.} \end{cases}$$

In order to describe the maps clearly, we introduce the following graph:

$$\begin{tikzcd}
k \arrow[dr, dash, "A"] & & & & & & & & & \\
& k^2 & \cdots & k^2 \arrow[r, dash, "C"] & k & \cdots & k \arrow[r, dash, "0"] & 0 & \cdots & 0 \\
k \arrow[ur, dash, "B"'] & & & & & & & & &
\end{tikzcd}$$

The broken lines denote several copies of the same vector space with the identity map between them. The linear map $C = Y_{ij_{\gamma_i}}$ is given by either of the matrices $[1, 1], [1, 1]^t$ depending on the orientation of γ_i. Similarly the linear map A is given by either of $[0, 1], [1, 0]^t$ and B by either of $[1, 0], [0, 1]^t$ depending on the orientation of the arrows α and β.

Let $Y_{i(2n-i)}$, $1 \leq i \leq n$ be the $k\tilde{Q}$-module defined by

$$Y_{i(2n-i)_s} = \begin{cases} k, & \text{if } i \leq s \leq n, \\ 0, & \text{if otherwise,} \end{cases}$$

$$Y_{i(2n-i)_\gamma} = \begin{cases} 1_k, & \text{if } \gamma \in \{\gamma_i, \gamma_{i+1}, ..., \gamma_{n-1}, \alpha\}, \\ 0, & \text{if otherwise.} \end{cases}$$

Then ${}^\sigma Y_{i(2n-i)}$, $1 \leq i \leq n$ are the $k\tilde{Q}$-modules defined by

$${}^\sigma Y_{i(2n-i)_s} = \begin{cases} k, & \text{if } s \in \{i, i+1, ..., n-1, n+1\}, \\ 0, & \text{if otherwise,} \end{cases}$$

$${}^\sigma Y_{i(2n-i)_\gamma} = \begin{cases} 1_k, & \text{if } \gamma \in \{\gamma_i, \gamma_{i+1}, ..., \gamma_{n-1}, \beta\}, \\ 0, & \text{if otherwise.} \end{cases}$$

It is a known in [2] that the set of all indecomposable $k\tilde{Q}$-modules up to isomorphism is a disjoint union of the following four sets:

$$\{Y_{ij} = {}^\sigma Y_{ij}, 1 \leq i < j < n\},$$

$$\{Y_{in} = {}^\sigma Y_{in}, 1 \leq i < n\},$$

$$\{Y_{ij} = {}^\sigma Y_{ij}, 1 \leq i < n < j < 2n-1, n-i > j-n\},$$

$$\{Y_{i(2n-i)}, {}^\sigma Y_{i(2n-i)}, 1 \leq i \leq n\}.$$

Lemma 2.1. *No irreducible morphism* $X \to Y$ *exist,* $\forall X, Y \in \{X_{i(2n-i)} = {}^\sigma X_{i(2n-i)}, 1 \leq i \leq n\}$.

Proof. Since X_{nn} is either simple projective or simple injective kQ-module, it follows that $\{X_{i(2n-i)} = {}^\sigma X_{i(2n-i)}, 1 \leq i \leq n\} = \{TrD^j X_{nn}, 1 \leq j \leq n\}$ or $\{X_{i(2n-i)} = {}^\sigma X_{i(2n-i)}, 1 \leq i \leq n\} = \{DTr^j X_{nn}, 1 \leq j \leq n\}$. □

Lemma 2.2 [7].

(1) *If* $0 \to X \to E \to Y \to 0$ *is an almost split sequence in* $\operatorname{mod} kQ(or, \operatorname{mod} kQ\# k\langle\sigma\rangle)$, *then* $0 \to kQ\# k\langle\sigma\rangle \otimes_{kQ} X \to kQ\# k\langle\sigma\rangle \otimes_{kQ} E \to kQ\# k\langle\sigma\rangle Y \to 0 (or, 0 \to X \to E \to Y \to 0)$ *is a direct sum of almost split sequences in* $\operatorname{mod} kQ\# k\langle\sigma\rangle\, (or, \operatorname{mod} kQ)$.

(2) *If* $X \to Y$ *is a minimal left or right almost split morphism in* $\operatorname{mod} kQ(or, \operatorname{mod} kQ\# k\langle\sigma\rangle)$, *then* $kQ\# k\langle\sigma\rangle \otimes_{kQ} X \to kQ\# k\langle\sigma\rangle \otimes_{kQ} Y$ $(or, X \to Y)$ *is a direct sum of minimal left or right almost split morphism in* $\operatorname{mod} kQ\# k\langle\sigma\rangle\, (or, \operatorname{mod} kQ)$.

Lemma 2.3 [1]**.**

(a) *Let* C *be an indecomposable module. Then a morphism* $g : B \to C$ *is irreducible if and only if there exists some morphism* $g' : B' \to C$ *such that the induced morphism* $(g, g') : B \oplus B' \to C$ *is a minimal right almost split morphism.*

(b) *Let* A *be an indecomposable module. Then a morphism* $g : A \to B$ *is irreducible if and only if there exists some morphism* $g' : A \to B'$ *such that the induced morphism* $\begin{pmatrix} g \\ g' \end{pmatrix} : A \to B \oplus B'$ *is a minimal left almost split morphism.*

Theorem 2.4. $\Gamma_{\tilde{Q}}$ *can be drawn from* Γ_Q *as follows*:

(1) *Vertices.*

Take two vertices X_{ij}, ${}^{\sigma}X_{ij}$ *to one vertex* Y_{ij}, *for any* $1 \le i < j < n$.

Take two vertices X_{in}, ${}^{\sigma}X_{in}$ *to one vertex* Y_{in}, *for any* $1 \le i < n$.

Take two vertices X_{ij}, ${}^{\sigma}X_{ij}$ *to one vertex* Y_{ij}, *for any* $1 \le i < n < j < 2n - 1$, $n - i > j - n$.

Take one vertex $X_{i(2n-i)}$ *to two vertices* $Y_{i(2n-i)}$, ${}^{\sigma}Y_{i(2n-i)}$, *for any* $1 \le i \le n$.

(2) *Arrows. Let* $X_1 \to X_2$ *be any arrow in* Γ_Q. *Then*

Case 1. *Suppose* ${}^{\sigma}X_1 \neq X_1$, ${}^{\sigma}X_2 = X_2$. *Let* Y_1 *be the vertex corresponding to* X_1, ${}^{\sigma}X_1$, *and* Y_2, ${}^{\sigma}Y_2$ *be the vertices corresponding to* X_2 *as in* (1). *Take arrow* $X_1 \to X_2$ *to* $Y_1 \to Y_2$ *and* ${}^{\sigma}X_1 \to X_2$ *to* $Y_1 \to {}^{\sigma}Y_2$.

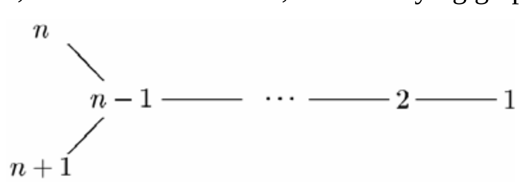
Case 2. *Suppose* ${}^{\sigma}X_1 \neq X_1$, ${}^{\sigma}X_2 \neq X_2$. *Let* Y_1 *be the vertex corresponding to* X_1, ${}^{\sigma}X_1$, *and* Y_2 *be the vertices corresponding to* X_2, ${}^{\sigma}X_2$ *as in* (1). *Take two arrows* $X_1 \to X_2$, ${}^{\sigma}X_1 \to {}^{\sigma}X_2$ *to one arrow* $Y_1 \to Y_2$.

Case 3. *Suppose* ${}^{\sigma}X_1 = X_1$, ${}^{\sigma}X_2 \neq X_2$. *Let* Y_1, ${}^{\sigma}Y_1$ *be the vertex corresponding to* X_1, *and* Y_2 *be the vertices corresponding to* X_2, ${}^{\sigma}X_2$ *as in* (1). *Take arrow* $X_1 \to X_2$ *to* $Y_1 \to Y_2$ *and* $X_1 \to {}^{\sigma}X_2$ *to* ${}^{\sigma}Y_1 \to Y_2$.

Proof. The results follow from Lemma 2.2, Lemma 2.3, and the corresponding one-one correspondences $\mathcal{X}_{ij}^{(0)} \mapsto Y_{ij}$, $1 \leq i < j < n$, $\mathcal{X}_{in}^{(0)} \mapsto Y_{in}$, $1 \leq i < n$, $\mathcal{X}_{ij}^{(0)} \mapsto Y_{ij}$, $1 \leq i < n < j < 2n-1$, $n-i > j-n$, $\mathcal{X}_{i(2n-i)}^{(0)} \mapsto Y_{i(2n-i)}$, $\mathcal{X}_{i(2n-i)}^{(1)} \mapsto {}^{\sigma}Y_{i(2n-i)}$, $1 \leq i \leq n$ between $\operatorname{mod} kQ\# k\langle\sigma\rangle$ and $\operatorname{mod} k\widetilde{Q}$. □

Remark 2.5. It is easy to see that (2) in Theorem 2.4 is naturally induced by (1), which means that the way to draw $\Gamma_{\widetilde{Q}}$ from Γ_Q is independent of the orientations of Q.

2.2. The quiver Q of type D_{n+1}, $n \geq 3$ with nontrivial quiver automorphism σ, $\sigma^2 = id$: In this case, the underlying graph of Q is



and the orientations of Q satisfy $\delta(n-1,\ n+1) = \delta(n-1,\ n)$.

The dual quiver $\widetilde{Q}$ is a quiver of type A_{2n-1}, for simplicity, the underlying graph of $\widetilde{Q}$ is

$$1 \text{——} 2 \text{——} \cdots n-1 \text{——} n \text{——} n+1 \text{——} \cdots 2n-2 \text{——} 2n-1$$

and the orientations of $\widetilde{Q}$ satisfy for any

$$1 \leq i \leq n,\ \delta(i,\ i+1) = \delta(2n-i,\ 2n-(i+1))$$

in $\widetilde{Q}$ equals $\delta(i,\ i+1)$ in Q.

Theorem 2.6. $\Gamma_{\widetilde{Q}}$ *can be drawn from* Γ_Q *as follows*:

(1) *Vertices*.

Take one vertex Y_{ij} *to two vertices* X_{ij}, ${}^{\sigma}X_{ij}$, *for any* $1 \leq i < j < n$.

Take one vertex Y_{in} *to two vertices* X_{in}, ${}^{\sigma}X_{in}$, *for any* $1 \leq i < n$.

Take one vertex Y_{ij} *to two vertices* X_{ij}, ${}^{\sigma}X_{ij}$, *for any* $1 \leq i < n < j < 2n-1,\ n-i > j-n$.

Take two vertices $Y_{i(2n-i)}$, ${}^{\sigma}Y_{i(2n-i)}$ *to one vertex* $X_{i(2n-i)}$, *for any* $1 \leq i \leq n$.

(2) *Arrows*. *Let* $Y_1 \to Y_2$ *be any arrow in* Γ_Q. *Then*

Case 1. *Suppose* ${}^{\sigma}Y_1 \neq Y_1$, ${}^{\sigma}Y_2 = Y_2$. *Let* X_1 *be the vertex corresponding to* X_1, ${}^{\sigma}X_1$, *and* X_2, ${}^{\sigma}X_2$ *be the vertices corresponding to* Y_2 *as in* (1). *Take arrow* $Y_1 \to Y_2$ *to* $X_1 \to X_2$ *and* ${}^{\sigma}Y_1 \to Y_2$ *to* $X_1 \to {}^{\sigma}X_2$.

Case 2. *Suppose* ${}^{\sigma}Y_1 = Y_1$, ${}^{\sigma}Y_2 \neq Y_2$. *Let* X_1, ${}^{\sigma}X_1$ *be the vertex corresponding to* Y_1, *and* X_2 *be the vertices corresponding to* Y_2, ${}^{\sigma}Y_2$ *as in* (1). *Take arrow* $Y_1 \to Y_2$ *to* $X_1 \to X_2$ *and* $Y_1 \to {}^{\sigma}Y_2$ *to* ${}^{\sigma}X_1 \to X_2$.

Case 3. *Suppose* ${}^{\sigma}Y_1 = Y_1$, ${}^{\sigma}Y_2 = Y_2$. *Let* X_1, ${}^{\sigma}X_1$ *be the vertex corresponding to* Y_1, *and* X_2, ${}^{\sigma}X_2$ *be the vertices corresponding to* Y_2 *as in* (1). *Take one arrow* $Y_1 \to Y_2$ *to either two arrows* $X_1 \to X_2$, ${}^{\sigma}X_1 \to {}^{\sigma}X_2$, *or two arrows* $X_1 \to {}^{\sigma}X_2$, ${}^{\sigma}X_1 \to X_2$.

Remark 2.7. Theorem 2.6 is an inverse of Theorem 2.4.

2.3. The quiver Q of type E_6 with nontrivial quiver automorphism σ: Q is a quiver of type E_6 :

$$
\begin{tikzcd}
 & & 3 \arrow[d, dash] & & \\
1 \arrow[r, dash] & 2 \arrow[r, dash] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6
\end{tikzcd}
$$

with the orientations satisfy $\delta(4, 2) = \delta(4, 5)$, $\delta(1, 2) = \delta(6, 5)$.

Then $\tilde{Q}$ is also a quiver of type E_6 with the orientations satisfy $\delta(4, 2) = \delta(4, 5)$ in $\tilde{Q}$ is equals $\delta(4, 2) = \delta(4, 5)$ in Q, $\delta(3, 4)$ in $\tilde{Q}$ equals $\delta(1, 2) = \delta(6, 5)$ in Q, $\delta(1, 2) = \delta(6, 5)$ in $\tilde{Q}$ equals $\delta(3, 4)$ in Q.

Theorem 2.8. $\Gamma_{\tilde{Q}}$ *can be drawn from* Γ_Q *by the correspondences of vertices as follows*:

$$\begin{smallmatrix} & & 0 & & \\ 0 & 0 & 1 & 0 & 0 \end{smallmatrix} \longleftrightarrow \left\{ \begin{smallmatrix} & & 0 & & \\ 0 & 1 & 0 & 0 & 0 \end{smallmatrix}, \begin{smallmatrix} & & 0 & & \\ 0 & 0 & 0 & 1 & 0 \end{smallmatrix} \right\},$$

$$\begin{smallmatrix} & & 0 & & \\ 0 & 1 & 1 & 1 & 0 \end{smallmatrix} \longleftrightarrow \left\{ \begin{smallmatrix} & & 0 & & \\ 0 & 1 & 1 & 0 & 0 \end{smallmatrix}, \begin{smallmatrix} & & 0 & & \\ 0 & 0 & 1 & 1 & 0 \end{smallmatrix} \right\},$$

$$\begin{smallmatrix} & & 0 & & \\ 1 & 1 & 1 & 1 & 1 \end{smallmatrix} \longleftrightarrow \left\{ \begin{smallmatrix} & & 1 & & \\ 0 & 1 & 1 & 0 & 0 \end{smallmatrix}, \begin{smallmatrix} & & 1 & & \\ 0 & 0 & 1 & 1 & 0 \end{smallmatrix} \right\},$$

$$\begin{smallmatrix} & & 1 & & \\ 0 & 0 & 0 & 0 & 0 \end{smallmatrix} \longleftrightarrow \left\{ \begin{smallmatrix} & & 0 & & \\ 1 & 0 & 0 & 0 & 0 \end{smallmatrix}, \begin{smallmatrix} & & 0 & & \\ 0 & 0 & 0 & 0 & 1 \end{smallmatrix} \right\},$$

$$\begin{smallmatrix} & & 1 & & \\ 0 & 0 & 1 & 0 & 0 \end{smallmatrix} \longleftrightarrow \left\{ \begin{smallmatrix} & & 0 & & \\ 1 & 1 & 0 & 0 & 0 \end{smallmatrix}, \begin{smallmatrix} & & 0 & & \\ 0 & 0 & 0 & 1 & 1 \end{smallmatrix} \right\},$$

$$\begin{smallmatrix} & & 1 & & \\ 0 & 1 & 1 & 1 & 0 \end{smallmatrix} \longleftrightarrow \left\{ \begin{smallmatrix} & & 0 & & \\ 1 & 1 & 1 & 0 & 0 \end{smallmatrix}, \begin{smallmatrix} & & 0 & & \\ 0 & 0 & 1 & 1 & 1 \end{smallmatrix} \right\},$$

$$\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 1 & 1 & 1 \end{smallmatrix} \longleftrightarrow \left\{ \begin{smallmatrix} & & 1 & & \\ 1 & 1 & 1 & 0 & 0 \end{smallmatrix}, \begin{smallmatrix} & & 1 & & \\ 0 & 0 & 1 & 1 & 1 \end{smallmatrix} \right\},$$

$$\begin{smallmatrix} & & 1 & & \\ 0 & 1 & 2 & 1 & 0 \end{smallmatrix} \longleftrightarrow \left\{ \begin{smallmatrix} & & 0 & & \\ 1 & 1 & 1 & 1 & 0 \end{smallmatrix}, \begin{smallmatrix} & & 0 & & \\ 0 & 1 & 1 & 1 & 1 \end{smallmatrix} \right\},$$

$$\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 2 & 1 & 1 \end{smallmatrix} \longleftrightarrow \left\{ \begin{smallmatrix} & & 1 & & \\ 1 & 1 & 1 & 1 & 0 \end{smallmatrix}, \begin{smallmatrix} & & 1 & & \\ 0 & 1 & 1 & 1 & 1 \end{smallmatrix} \right\},$$

$$\begin{smallmatrix} & & 1 & & \\ 1 & 2 & 2 & 2 & 1 \end{smallmatrix} \longleftrightarrow \left\{ \begin{smallmatrix} & & 1 & & \\ 1 & 1 & 2 & 1 & 0 \end{smallmatrix}, \begin{smallmatrix} & & 1 & & \\ 0 & 1 & 2 & 1 & 1 \end{smallmatrix} \right\},$$

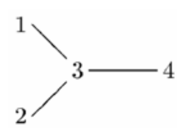
$$\begin{smallmatrix} & & 1 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix} \longleftrightarrow \left\{ \begin{smallmatrix} & & 1 & & \\ 0 & 1 & 2 & 2 & 1 \end{smallmatrix}, \begin{smallmatrix} & & 1 & & \\ 1 & 2 & 2 & 1 & 0 \end{smallmatrix} \right\},$$

$$\begin{smallmatrix} & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix} \longleftrightarrow \left\{ \begin{smallmatrix} & & 1 & & \\ 1 & 2 & 2 & 1 & 1 \end{smallmatrix}, \begin{smallmatrix} & & 1 & & \\ 1 & 1 & 2 & 2 & 1 \end{smallmatrix} \right\}.$$

2.4. Other cases. Only two cases left of Q with nontrivial quiver automorphism σ:

(1) Q is a quiver of type $A_3 : 1 - 2 - 3$ and the orientations satisfy $\delta(2, 1) = \delta(2, 3)$.

(2) Q is a quiver of type D_4 :



with the orientations satisfy $\delta(3, 1) = \delta(3, 2) = \delta(3, 4)$. In this case, $\sigma^3 = id$.

The above two cases have the common that the dual quiver $\tilde{Q} = Q^{op}$. Where Q^{op} is denoted by the quiver having the same set of vertices, and for each arrow $i \to j$ in Q, there is an arrow $j \to i$. By the definition of Auslander-Reiten quiver, we can draw the Auslander-Reiten quiver $\Gamma_{\tilde{Q}}$ more simply as follows.

Theorem 2.9. *Let Q be the above two cases*. *Then* $\Gamma_{\tilde{Q}} = \Gamma_{Q}^{op}$.

Acknowledgement

The author thanks the anonymous referees for their valuable suggestions which led to the improvement of the manuscript.

References

[1] M. Auslander, I. Reiten and S. O. Smal∅, Representation Theory of Artin Algebras, Cambridge University Press, 1997.

[2] P. Gabriel and A. V. Roiter, Representations of finite-dimensional algebras, Algebra, VIII, Encyclopedia Math. Sci. 73, Springer, 1992, pp. 1-177.

[3] A. W. Hubery, Representation of Quivers Respecting a Quiver Automorphism and a Theorem of Kac. Ph.D. Dissertation, the University of Leeds department of Pure Mathematics, 2002.

[4] A. W. Hubery, Quiver representations respecting a quiver automorphism: a generalisation of a theorem of Kac, J. London Math. Soc. 69 (2004), 79-96.

[5] S. Montgomery, Hopf Algebras and their Actions on Rings, CBMS, Lecture in Math, Providence, RI 82, 1993.

[6] C. M. Ringel, Representation theory of Dynkin quivers three contributions, Frontiers of Mathematics in China 11(4) (2016), 765-814.

[7] I. Reiten and C. Riedtmann, Skew Group Algebras in the Representation Theory of Artin Algebras, J. Algebra 92 (1985), 224-282.

[8] M. Zhang, The Dual Quiver of the Auslander Reiten Quiver of Path Algebras, Algebras and Representation Theory 15(2) (2012), 203-210.

[9] M. Zhang and F. Li, Representations of skew group algebras induced from isomorphically invariant modules over path algebras, J. Algebra 312(2) (2009), 567-581.

[10] S. C. Zhang, Homological dimension of crossed product algebras, Chinese Ann. Math. 22A(6) (2001), 767-772.