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BEST PROXIMITY POINT THEOREMS FOR $(\mathcal{F}h\alpha\psi)$-PROXIMAL CONTRACTIONS

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Abstract

In this paper, we establish best proximity point theorems for $(\mathcal{F}h\alpha\psi)$-proximal contractions in complete metric spaces. Our results extend and improve some results obtained by Jleli and Samet in [16] and some other known results in the literature. We provide an example to analyze and support our main results.

1. Introduction and Mathematical Preliminaries

The best approximation results provide an approximate solution to the fixed point equation $Tx = x$, when the non-self-mapping T has no fixed point. In particular, a well-known best approximation theorem, due to Fan [12], asserts the fact that if K is a nonempty compact convex subset of a Hausdorff locally convex topological vector space X and $T : K \to X$ is a continuous mapping, then there exists an element satisfying the condition $d(x, Tx) = \inf\{d(y, Tx) : y \in K\}$, where d is a metric on X. The evolution of best proximity point theory has been extended as a generalization of the

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concept of the best approximation. The best approximation theorem guarantees the existence of an approximate solution; the best proximity point theorem is considered for solving the problem to find an approximate solution which is optimal.

Given nonempty closed subsets A and B of X, when a non-self-mapping $T : A \to B$ has not a fixed point, it is quite natural to find an element x^* such that $d(x^*, Tx^*)$ is minimum.

An element x^* is called a *best proximity point* of T if

$$d(x^*, Tx^*) = d(A, B),$$

where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

Because of the fact that $d(x, Tx) > d(A, B)$ for all $x \in A$, the global minimum of the mapping $x \to d(x, Tx)$ is attained at a best proximity point. Clearly, if the underlying mapping is self-mapping, then it can be observed that a best proximity point is essentially a fixed point. The goal of best proximity point theory is to furnish sufficient conditions that assure the existence of such points. For some results in this direction, we refer to [3, 10, 13, 14, 19, 20] and references therein.

On the other side, the most basic fixed point theorem in analysis is due to Banach and appeared in his Ph.D. thesis (1920, published in 1922) [8].

Theorem 1.1 (see [8])**.** *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a map such that*

$$d(Tx, Ty) \leq cd(x, y)$$

for some $0 \leq c < 1$ *and all x and y in X. Then T has a unique fixed point in X.*

Theorem 1.1 is called the *contraction mapping theorem* or *Banach contraction principle*. It is one of the most well-known and useful tools in modern analysis. This principle has been generalized by many authors, in many different ways (see [7, 9, 11, 17, 22, 24]). Recently, Samet et al. [21] introduced the notion of α-ψ-contractive type mappings and proved some fixed point theorems for such mappings within the framework of complete metric spaces. Karapinar and Samet [15] generalized α-ψ-contractive type mappings and obtained some fixed point theorems for generalized α-ψ-contractive type mappings. More recently, Jleli and Samet [16] introduced the notion of α-ψ-proximal contractive type mappings and proved certain best proximity point theorems. Many authors have obtained best proximity point theorems and have done so in a variety of settings (see [1, 4, 5, 18, 23] for examples).

Inspired and motivated by the recent results of Jleli and Samet in [16] and the concept of functions of subclass of type I and the pair $(\mathcal{F}, h)$, an upper class of type I, introduced in [2, 6], we establish new best proximity point results for $(\mathcal{F}h\alpha\psi)$-proximal contractions. We also give an example to support our main results.

Let (X, d) be a metric space. For $A, B \subset X$, we use the following notations subsequently:

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\},$$

$$A_0 = \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\},$$

$$B_0 = \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}.$$

Let Ψ denote the set of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following properties:

(1) ψ is monotone nondecreasing;

(2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$.

(**H**) If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k.

Definition 1.2 [23]**.** Let A and B be two nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the *P-property* if and only if

$$\left.\begin{array}{l} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{array}\right\} \Rightarrow d(x_1, x_2) = d(y_1, y_2), \tag{1.1}$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Definition 1.3 [16]**.** Let A and B be two nonempty subsets of a metric space (X, d). A mapping $T : A \to A$ is called α*-proximal admissible* if there exists a mapping $\alpha : A \times A \to [0, \infty)$ such that

$$\left.\begin{array}{l} \alpha(x_1, x_2) \geq 1 \\ d(w_1, Tx_1) = d(A, B) \\ d(w_2, Tx_2) = d(A, B) \end{array}\right\} \Rightarrow \alpha(w_1, w_2) \geq 1, \tag{1.2}$$

where $x_1, x_2, w_1, w_2 \in A$.

Definition 1.4 [16]**.** Let A and B be two nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be an α-ψ*-proximal contraction*, if there exist two functions $\psi \in \Psi$ and $\alpha : A \times A \to [0, \infty)$ such that

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in A. \tag{1.3}$$

Definition 1.5 [2, 6]**.** We say that the function $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type I, if $x \geq 1 \Rightarrow h(1, y) \leq h(x, y)$ for all $y \in \mathbb{R}^+$.

Example 1.6 [2, 6]**.** Define $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x, y) = (y + l)^x$, $l > 1$;

(b) $h(x, y) = (x + l)^y$, $l > 1$;

(c) $h(x, y) = x^n y$, $n \in \mathbb{N}$;

(d) $h(x, y) = \frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right)y,\ n \in \mathbb{N};$

(e) $h(x, y) = \left[\frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) + l\right]^y,\ l > 1,\ n \in \mathbb{N}$

for all $x, y \in \mathbb{R}^+$. Then h is a function of subclass of type I.

Definition 1.7 [2, 6]**.** Let $h, \mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. Then we say that the pair $(\mathcal{F}, h)$ is an *upper class of type I*, if h is a function of subclass of type I, and (i) $0 \leq s \leq 1 \Rightarrow \mathcal{F}(s, t) \leq \mathcal{F}(1, t)$, (ii) $h(1, y) \leq \mathcal{F}(s, t) \Rightarrow y \leq st$ for all $s, t, y \in \mathbb{R}^+$.

Example 1.8 [2, 6]**.** Define $h, \mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, by:

(a) $h(x, y) = (y + l)^x,\ l > 1$ and $\mathcal{F}(s, t) = st + l;$

(b) $h(x, y) = (x + l)^y,\ l > 1$ and $\mathcal{F}(s, t) = (1 + l)^{st};$

(c) $h(x, y) = x^m y,\ m \in \mathbb{N}$ and $\mathcal{F}(s, t) = st;$

(d) $h(x, y) = y$ and $\mathcal{F}(s, t) = st;$

(e) $h(x, y) = \frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right)y,\ n \in \mathbb{N}$ and $\mathcal{F}(s, t) = st;$

(f) $h(x, y) = \left[\frac{1}{n+1}\left(\sum_{i=0}^{n} x^i\right) + l\right]^y,\ l > 1,\ n \in \mathbb{N}$ and $\mathcal{F}(s, t) = (1 + l)^{st}$

for all $x, y, s, t \in \mathbb{R}^+$. Then the pair $(\mathcal{F}, h)$ is an upper class of type I.

The purpose of this paper is to extend the recent results of Jleli and Samet [16] to new best proximity point results of $(\mathcal{F}h\alpha\psi)$-proximal contractions.

2. Main Results

We begin this section by introducing the following definition.

Definition 2.1. Let A and B be two nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be an $(\mathcal{F}h\alpha\psi)$*-proximal contraction*, if there exist two functions $\psi \in \Psi$ and $\alpha : A \times A \to [0, \infty)$ such that

$$h(\alpha(x, y), d(Tx, Ty)) \leq \mathcal{F}(1, \psi(d(x, y))), \quad \forall x, y \in A, \tag{2.1}$$

where pair $(\mathcal{F}, h)$ is an upper class of type I.

We note that this kind of generalization makes sense, since it extends and covers those corresponding classes of proximal contractive mappings defined in [16]. We state and prove our main results.

Theorem 2.2. *Let A and B be two nonempty closed subsets of a complete metric space* (X, d) *such that* A_0 *is nonempty. Let* $\alpha : A \times A \to [0, \infty)$ *and let* $\psi \in \Psi$ *be a strictly increasing map. Suppose that* $T : A \to B$ *is a non-self-mapping satisfying the following conditions*:

(1) $T(A_0) \subseteq B_0$ *and* (A, B) *satisfies the P-property*,

(2) *T is an* α*-proximal admissible map*,

(3) *there exist elements* x_0 *and* x_1 *in* A_0 *such that*

$$d(x_1, Tx_0) = d(A, B), \quad \text{and} \quad \alpha(x_0, x_1) \geq 1, \tag{2.2}$$

(4) *T is a continuous* $(\mathcal{F}h\alpha\psi)$*-proximal contraction*.

Then there exists an element $x^* \in A_0$ *such that*

$$d(x^*, Tx^*) = d(A, B).$$

Proof. From condition (3), there exist elements x_0 and x_1 such that

$$d(x_1, Tx_0) = d(A, B), \quad \text{and} \quad \alpha(x_0, x_1) \geq 1. \tag{2.3}$$

Since $T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B).$$

Now, we have

$$\alpha(x_0, x_1) \geq 1,$$

$$d(x_1, Tx_0) = d(A, B),$$

$$d(x_2, Tx_1) = d(A, B).$$

Since T is α-proximal admissible, this implies that

$$\alpha(x_1, x_2) \geq 1.$$

Thus, we have

$$d(x_2, Tx_1) = d(A, B), \quad \text{and} \quad \alpha(x_1, x_2) \geq 1.$$

Again, since $T(A_0) \subseteq B_0$, there exists $x_3 \in A_0$ such that

$$d(x_3, Tx_2) = d(A, B).$$

Now, we have

$$\alpha(x_1, x_2) \geq 1,$$

$$d(x_2, Tx_1) = d(A, B),$$

$$d(x_3, Tx_2) = d(A, B).$$

Since T is α-proximal admissible, this implies that $\alpha(x_2, x_3) \geq 1$. Thus, we have

$$d(x_3, Tx_2) = d(A, B), \quad \text{and} \quad \alpha(x_2, x_3) \geq 1.$$

Continuing this process, by induction, we can construct a sequence $\{x_n\} \subset A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A, B), \quad \text{and} \quad \alpha(x_n, x_{n+1}) \geq 1, \; \forall n \in \mathbb{N} \cup \{0\}. \tag{2.4}$$

Since (A, B) satisfies the P-property, we conclude from (2.4) that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n), \forall n \in \mathbb{N}. \tag{2.5}$$

From condition (4), that is T is $(\mathcal{F}h\alpha\psi)$-proximal contractions, for all $n \in \mathbb{N}$, we have

$$h(1, d(Tx_{n-1}, Tx_n)) \leq h(\alpha(x_{n-1}, x_n), d(Tx_{n-1}, Tx_n)) \leq \mathcal{F}(1, \psi(d(Tx_{n-1}, Tx_n))).$$

This implies

$$d(Tx_{n-1}, Tx_n) \leq \psi(d(x_{n-1}, x_n)), \forall n \in \mathbb{N}. \tag{2.6}$$

Combining (2.5) with (2.6) yields the following:

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)), \forall n \in \mathbb{N}. \tag{2.7}$$

Suppose that for some positive integer k, we have $x_k = x_{k+1}$. This implies immediately from (2.4) that

$$d(x_k, Tx_k) = d(x_{k+1}, Tx_k) = d(A, B).$$

That is, x_k is a best proximity point of T. So, we can suppose that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Using the monotonicity of ψ, by induction, it follows from (2.7) that

$$d(x_{n+1}, x_n) \leq \psi^n(d(x_1, x_0)), \forall n \in \mathbb{N} \cup \{0\}. \tag{2.8}$$

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d). Let $\varepsilon > 0$ be fixed. Since $\sum_{n=1}^{\infty} \psi^n(d(x_1, x_0)) < \infty$, there exists a positive integer $\ell = \ell(\varepsilon)$ such that

$$\sum_{k \geq \ell} \psi^k(d(x_1, x_0)) < \varepsilon. \tag{2.9}$$

Let $m > n > l$, using the triangular inequality, by (2.8) and (2.9), we obtain

$$d(x_n,\ x_m) \leq \sum_{k=n}^{m-1} d(x_k,\ x_{k+1})$$

$$\leq \sum_{k=n}^{m-1} \psi^k(d(x_1,\ x_0))$$

$$\leq \sum_{k \geq \ell} \psi^k(d(x_1,\ x_0)) < \varepsilon.$$

This we show that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d). Since (X, d) is complete and A is closed, there exists an element $x^* \in A$ such that $x_n \to x^*$ as $n \to \infty$. On the other hand, T is a continuous mapping. Then we have $Tx_n \to Tx^*$ as $n \to \infty$. Therefore

$$d(x^*, Tx^*) = d(x_{n+1}, Tx_n) = d(A, B), \text{ as } n \to \infty.$$

This completes our proof. □

If we remove the continuity hypothesis in Theorem 2.2, assuming the property (**H**) in A, we obtain the following theorem.

Theorem 2.3. *Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0, \infty)$ and let $\psi \in \Psi$ be a strictly increasing map. Suppose that $T : A \to B$ is a non-self-mapping satisfying the following conditions*:

(i) $T(A_0) \subseteq B_0$ *and* (A, B) *satisfies the P-property,*

(ii) *T is an α-proximal admissible map,*

(iii) *there exist elements* x_0 *and* x_1 *in* A_0 *such that*

$$d(x_1, Tx_0) = d(A, B), \text{ and } \alpha(x_0, x_1) \geq 1, \tag{2.10}$$

(iv) (**H**) *holds and T is an* $(\mathcal{F}h\alpha\psi)$*-proximal contraction.*

Then there exists an element $x^* \in A_0$ *such that*

$$d(x^*, Tx^*) = d(A, B).$$

Proof. Similar to the proof of Theorem 2.2, there exists a Cauchy sequence $\{x_n\} \subset A$ such that (i) holds and $x_n \to x^* \in A$ as $n \to \infty$. From property (**H**), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq 1$ for all k. Since T is an $(\mathcal{F}h\alpha\psi)$-proximal contraction, we get that

$$h(1, d(Tx_{n_k}, x^*)) \leq h(\alpha(x_{n_k}, x^*), d(Tx_{n_k}, x^*)) \leq \mathcal{F}(1, \psi(d(x_{n_k}, x^*))), \forall k.$$

This implies

$$d(Tx_{n_k}, Tx^*) \leq \psi(d(x_{n_k}, x^*)), \forall k.$$

Letting $k \to \infty$ in the above inequality, we have that $Tx_{n_k} \to Tx^*$. The continuity of the metric function d implies that

$$d(A, B) = d(x_{n_k+1}, Tx_{n_k}) = d(x^*, Tx^*) \text{ as } n \to \infty.$$

This completes our proof of the theorem. □

Remark 2.4. If we take $h(x, y) = xy$, $\alpha(x, y) = 1$ and $\mathcal{F}(s, t) = st$, then we will obtain the results in Jleli and Samet [16].

Example 2.5. Consider the Euclidean ordered space $X = \mathbb{R}$ with the usual metric. Suppose $A = [-2, -1]$, $B = [0, 1]$. Define $T : A \to B$ by

$$Tx = -\frac{1}{4}x - \frac{1}{4}, \forall x \in A.$$

Let $\alpha(x, y) = 1$ and $\psi(t) = \frac{t}{2}$, for all $t > 0$. We can see that $T(A) \subseteq B$. It is clear that $d(A, B) = 1$. Define $h, \mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by

$$h(x, y) = x^n y, \quad n \in \mathbb{N}$$

and

$$\mathcal{F}(s, t) = st.$$

Consider

$$\mathcal{F}(1, \psi(d(x, y))) - h(\alpha(x, y), d(Tx, Ty))$$

$$= \psi(d(x, y)) - \alpha(x, y)^n d(Tx, Ty)$$

$$= \frac{d(x, y)}{2} - d(Tx, Ty) \geq 0.$$

Therefore

$$h(\alpha(x, y), d(Tx, Ty)) \leq \mathcal{F}(1, \psi(d(x, y))), \quad \forall x, y \in A.$$

Thus, T is an $(\mathcal{F}h\alpha\psi)$-proximal contraction. All the conditions of Theorem 2.2 hold true and T has a best proximity point. Here $x^* = -1$ is the best proximity point of T.

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