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FIXED POINT THEOREMS FOR WEAK CONTRACTION MAPPINGS IN A QUASI αb-METRIC SPACE

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Abstract

In this paper, we introduce a quasi αb-metric space as an extension of a quasi b-metric space and prove the existence and uniqueness of fixed point theorems for different weak contraction mappings in the quasi αb-metric space.

1. Introduction

The concept of b-metric was introduced by Bakhtin [6] and applied to the generalization of Banach's fixed point theorem in b-metric spaces by Czerwik [8]. There are several generalizations of the Banach's contraction principle in b-metric spaces [2, 9, 12, 17, 19] and even some authors used generalizations of the Banach's contraction principle in the quasi b-metric spaces [13, 16]. The concept of weak contraction mappings is introduced by Alber and Guerr-Delabriere in Hilbert space [3], many other authors have considered the weak contraction mapping in b-metric space [1, 5, 7, 10, 11, 16], while the quasi αb-metric space is introduced by Nurwahyu [14, 15].

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The Banach contraction theorem and its several extensions have been generalized using developed notion of weak contraction mapping.

The following basic result of weak contraction mapping is given by Rhoades et al. [18]:

Let (X, d) be a metric space and let $T : X \to X$ be a mapping such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \text{ for every } x, y \in X,$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0.$ Then T is called a *weak contraction* (or φ-*weak contraction*).

In this paper, we consider the weak contraction mapping in an extension of a quasi b-metric space and this space is called a *quasi* αb-*metric space*. The aim of this paper is to establish and prove some fixed point theorems in complete quasi αb-metric spaces with using several weak contraction mappings.

2. Preliminaries

Definition 2.1 [6, 8]**.** Let X be a nonempty set and let $b \geq 1$ be a given real number.

If $d : X \times X \to [0, \infty)$ is a mapping which satisfies the following conditions for all $x, y, z \in X$:

(1) $d(x, y) = d(y, x) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \leq b(d(x, z) + d(z, y)),$

d and (X, d) are called a b-*metric* on X and b-*metric space*, respectively. If only (1) and (3) hold, then (X, d) is called a *quasi* b-*metric space*.

Now we introduce a generalization of a quasi b-metric space by modifying the triangle inequality condition in a quasi b-metric space.

Definition 2.2 [14, 15]**.** Let X be a nonempty set and let $0 \leq \alpha < 1$ and $b \leq 1$ be a given real number.

Let $d : X \times X \to [0, \infty)$ be a self mapping on X satisfying the conditions:

(1) $d(x, y) = d(y, x) = 0$ if and only if $x = y$;

(2)

$$d(x, y) \leq \alpha d(y, x) + \frac{1}{2} b(d(x, z) + d(z, y)) \text{ for all } x, y, z \in X. \quad (2.1)$$

Then d is called a *quasi* α*b-metric* on X and (X, d) is called a *quasi* α*b-metric space*.

From the definition of a quasi αb-metric, it can be shown that every quasi b-metric is a quasi αb-metric, but the converse is not true.

Example 2.3 [14, 15]**.** Let $X = \{0, 1, 2\}$. Define $d : X \times X \to R^+$ as follows: $d(0, 0) = d(1, 1) = d(2, 2) = d(0, 2) = d(2, 1) = 0$, $d(1, 0) = 4$, $d(2, 0) = 1$, $d(0, 1) = 2$ and $d(1, 2) = 3$. It is clear that d is a quasi αb-metric with because $\alpha = \frac{1}{2}$ and $b = 4$, because

$$2 = d(0, 1) \leq \frac{1}{2} d(1, 0) + 2(d(0, 2) + d(2, 1)),$$

but for every $c \geq 1$, $2 = d(0, 1) > c(d(0, 2) + d(2, 1))$. So d is not a quasi b-metric.

Example 2.4 [14, 15]**.** Let $X = R$ and define $d : X \times X \to R^+$ as $d(x, y) = \begin{cases} 2x^2 + y^2, & x \neq y, \\ 0, & x = y. \end{cases}$ The first condition of a quasi αb-metric is clear from the definition of function d, while the second condition will be shown as follows.

For $x \neq y$, and every $z \in X$, we have

$$d(x, y) = 2x^2 + y^2 \leq \frac{5}{2} x^2 + 2y^2 + 3z^2$$

$$= \frac{1}{2}(2y^2 + x^2) + ((2y^2 + z^2) + (2z^2 + y^2))$$

$$= \frac{1}{2}d(y, x) + \frac{2}{2}(d(x, z) + d(z, y)).$$

So we get

$$d(x, y) \leq \frac{1}{2}d(y, x) + \frac{2}{2}(d(x, z) + d(z, y)).$$

Hence, d is a quasi αb-metric with $\alpha = \frac{1}{2}$ and $b = 2$.

Definition 2.5 [14, 15]**.** Let (X, d) be a quasi αb-metric space, a sequence $\{x_n\}$ in (X, d) converges to $x \in X$ and we write $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$.

Definition 2.6 [14, 15]**.** Let $\{x_n\}$ be a sequence in a quasi αb-metric space (X, d). Then $\{x_n\}$ is called a *Cauchy sequence* if

$$\lim_{n, m\to\infty} d(x_n, x_m) = \lim_{n, m\to\infty} d(x_m, x_n) = 0.$$

Definition 2.7 [14, 15]**.** Let (X, d) be a quasi αb-metric space. Then (X, d) is called *complete* if every Cauchy sequence in X is convergent in X.

Definition 2.8. Let X be a nonempty set and let T be a self mapping on X. An element $x \in X$ is called a *fixed point* of T if $Tx = x$.

Definition 2.9. Let (X, d) and (Y, d) be quasi αb-metric spaces. Then $T : X \to Y$ is called *continuous* on X if every $x, y \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $d(Tx, Ty) < \varepsilon$ for $d(x, y) < \delta$.

3. Main Results

Lemma 3.1. *Let (X, d) be a quasi αb-metric space with $0 \leq \alpha < 1$ and $b \geq 1$, let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.1}$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Proof. By using (2.1), we get

$$\begin{aligned}
& d(x_n,\ x_{n+2}) \\
& \leq \alpha d(x_{n+2},\ x_n) + \frac{b}{2}(d(x_n,\ x_{n+1}) + d(x_{n+1},\ x_{n+2})) \\
& \leq \alpha\left[\alpha d(x_n,\ x_{n+2}) + \frac{b}{2}(d(x_{n+2},\ x_{n+1}) + d(x_{n+1},\ x_n))\right] \\
& \quad + \frac{b}{2}(d(x_n,\ x_{n+1}) + d(x_{n+1},\ x_{n+2})) \\
& d(x_n,\ x_{n+2}) \\
& \leq \frac{\frac{1}{2}\alpha b(d(x_{n+2},\ x_{n+1}) + d(x_{n+1},\ x_n)) + \frac{b}{2}(d(x_n,\ x_{n+1}) + d(x_{n+1}, x_{n+2}))}{1-\alpha^2}.
\end{aligned}$$

From this and (3.1), we have

$$\lim_{n\to\infty} d(x_n,\ x_{n+2}) = 0. \tag{3.2}$$

Similarly, we also have

$$\begin{aligned}
& d(x_{n+2},\ x_n) \\
& \leq \frac{\frac{1}{2}\alpha b(d(x_n,\ x_{n+1}) + d(x_{n+1},\ x_{n+2})) + \frac{b}{2}(d(x_{n+2},\ x_{n+1}) + d(x_{n+1},\ x_n))}{1-\alpha^2}.
\end{aligned}$$

From this and (3.1), we get

$$\lim_{n\to\infty} d(x_{n+2},\ x_n) = 0. \tag{3.3}$$

By repeating this process for $d(x_n,\ x_{n+3})$, we obtain

$$\begin{aligned}
& d(x_n,\ x_{n+3}) \\
& \leq \frac{\frac{1}{2}\alpha b(d(x_{n+3},\ x_{n+2}) + d(x_{n+2},\ x_n)) + \frac{b}{2}(d(x_n,\ x_{n+2}) + d(x_{n+2}, x_{n+3}))}{1-\alpha^2}.
\end{aligned}$$

Thus, by using (3.1), (3.2) and (3.3), we get

$$\lim_{n\to\infty} d(x_n,\ x_{n+3}) = 0.$$

Similarly, we obtain

$$d(x_{n+3}, x_n)$$

$$\leq \frac{\frac{1}{2}\alpha b(d(x_{n+2}, x_{n+3}) + d(x_n, x_{n+2})) + \frac{b}{2}(d(x_{n+2}, x_n) + d(x_{n+3}, x_{n+2}))}{1-\alpha^2}.$$

Thus, by using (3.1), (3.2) and (3.3), we get

$$\lim_{n\to\infty} d(x_{n+3}, x_n) = 0.$$

Thus, by using induction for $k = 1, 2, 3, 4, 5, \ldots,$ we obtain

$$d(x_n, x_{n+k})$$

$$\leq \frac{\frac{1}{2}\alpha b(d(x_{n+k}, x_{n+k-1}) + d(x_{n+k-1}, x_n)) + \frac{b}{2}(d(x_n, x_{n+k-1}) + d(x_{n+k-1}, x_{n+k}))}{1-\alpha^2}$$

and

$$d(x_{n+k}, x_n)$$

$$\leq \frac{\frac{1}{2}\alpha b(d(x_{n+k-1}, x_{n+k}) + d(x_n, x_{n+k-1})) + \frac{b}{2}(d(x_{n+k-1}, x_n) + d(x_{n+k}, x_{n+k-1}))}{1-\alpha^2}.$$

In this way, we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(x_{n+k}, x_n) = 0.$$

So for $m > n \geq 0,$ we have

$$\lim_{n\to\infty} d(x_n, x_k) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(x_m, x_n) = 0.$$

Hence, $\{x_n\}$ is a Cauchy sequence in X.

Definition 3.2. Let (X, d) be a quasi αb-metric space with $0 \leq \alpha < 1$ and $b \geq 1.$ A mapping $T : X \to X$ is called a *weak contraction* on X if there exists a function $\varphi : [0, \infty) \to [0, \infty),$ where φ is continuous on X, $\varphi(t) = 0$ iff $t = 0$ and satisfying the following condition:

$$d(Tx, Ty) \leq d(x, y) - \delta\varphi(d(x, y)) \text{ for all } x, y \in X, \tag{3.4}$$

where $0 < \delta \leq 1.$

Example 3.3. Let $X = R$ and define $d : X \times X \to R^+$ as $d(x, y) = \begin{cases} 2x^2 + y^2, & x \neq y, \\ 0, & x = y. \end{cases}$ From Example 2.4, d is a quasi αb-metric with parameter $\alpha = \frac{1}{2}$ and $b = 2$.

Let $T : X \to X$ be a mapping defined by $T(x) = \frac{x}{4}$, $\varphi(t) = 15t$ for all $t \geq 0$,

$$d(Tx, Ty) = d\left(\frac{x}{4}, \frac{y}{4}\right) = \frac{x^2}{8} + \frac{y^2}{16} = (2x^2 + y^2) - \frac{15}{16}(2x^2 + y^2)$$

$$= d(x, y) - \frac{1}{16}\varphi(d(x, y)).$$

Then T is a weak contraction on a quasi αb-metric space (X, d).

Theorem 3.4. *Let (X, d) be a complete quasi αb-metric space with $0 \leq \alpha < 1$, $b \geq 1$ and T be a weak contraction mapping on X. Then T has a unique fixed point in X.*

Proof. Let $\{x_n\}$ be a sequence in X. We define $x_{n+1} = Tx_n$ for $n = 0, 1, 2, 3, \ldots$. By using (3.4), we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq d(x_{n-1}, x_n) - \delta\varphi(d(x_{n-1}, x_n)). \tag{3.5}$$

Since $\varphi \geq 0$, $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$.

Thus, $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence, so $\{d(x_n, x_{n+1})\}$ is a convergent sequence in R^+. Consequently, there exists $L \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = L$.

By using continuity of φ and inequality in (3.5), we get

$$L \leq L - \delta \lim_{n\to\infty} \varphi(d(x_{n-1}, x_n))$$

$$= L - \delta\varphi(\lim_{n\to\infty} d(x_{n-1}, x_n)) = L - \delta\varphi(L).$$

Since $0 < \delta \leq 1$, so we get $\varphi(L) = 0$ and this implies that $L = 0$.

Hence, we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Similarly, we have $d(x_{n+1}, x_n) \leq d(x_n, x_{n-1})$. Thus, we have $\{d(x_{n+1}, x_n)\}$ is a non-increasing sequence and $\{d(x_{n+1}, x_n)\}$ is a convergent sequence in R^+.

Hence, there exists $K \geq 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = K$.

By using continuity of φ and inequality in (3.5), we get

$$K \leq K - \delta \lim_{n\to\infty} \varphi(d(x_n, x_{n-1})) = K - \delta\varphi(\lim_{n\to\infty} d(x_n, x_{n-1})) = K - \delta\varphi(K).$$

So $\varphi(K) = 0$. This implies that $K = 0$ and $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

Now, by using Lemma 3.1, we obtain that $\{x_n\}$ is a Cauchy sequence in the complete space X. Therefore, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

Now we have to show that x^* is a fixed point of T:

$$\begin{aligned} d(x^*, Tx^*) &\leq \alpha d(Tx^*, x^*) + \frac{b}{2}(d(x^*, x_n) + d(x_n, Tx^*)) \\ &\leq \alpha\left(\alpha d(x^*, Tx^*) + \frac{b}{2}(d(Tx^*, x_n) + d(x_n, x^*))\right) \\ &\quad + \frac{b}{2}(d(x^*, x_n) + d(x_n, Tx^*)). \end{aligned}$$

So we have

$$\begin{aligned} & d(x^*, Tx^*) \\ &\leq \frac{\frac{\alpha b}{2}(d(Tx^*, x_n) + d(x_n, x^*)) + \frac{b}{2}(d(x^*, x_n) + d(x_n, Tx^*))}{1 - \alpha^2} \\ &= \frac{\frac{\alpha b}{2}(d(Tx^*, Tx_{n-1}) + d(x_n, x^*)) + \frac{b}{2}(d(x^*, x_n) + d(Tx_{n-1}, Tx^*))}{1 - \alpha^2} \end{aligned}$$

$$\leq \frac{\frac{\alpha b}{2}(d(x^*, x_{n-1}) - \delta\varphi(d(x^*, x_{n-1}) + d(x_n, x^*)) + \frac{b}{2}(d(x^*, x_n) + d(x_{n-1}, x^*)) - \delta\varphi(d(x_{n-1}, x^*))}{1-\alpha^2}$$

$$\leq \frac{\frac{\alpha b}{2}(d(x^*, x_{n-1}) + d(x_n, x^*)) + \frac{b}{2}(d(x, x_n) + d(x_{n-1}, x^*))}{1-\alpha^2}.$$

Since $\lim_{n\to\infty} x_n = x^*$, so $\lim\limits_{n\to\infty} d(x_n, x^*) = \lim\limits_{n\to\infty} d(x^*, x_n) = 0.$

Thus, for $n \to \infty$, we get

$$d(x^*, Tx^*) = 0.$$

This is only possible if $Tx^* = x^*$. Hence, x^* is the fixed point of T.

Next, we have to show that the fixed point of T is unique.

Suppose there exists $y^* \in X$ such that $Ty^* = y^*$. So we get

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq d(x, y^*) - \delta\varphi(d(x, y^*)),$$

$$\delta\varphi(d(x^*, y^*)) \leq 0.$$

Since $0 < \delta \leq 1$, it is possible if $\varphi(d(x^*, y^*)) = 0$ and $d(x^*, y^*) = 0$, so we get $x^* = y^*$. So T has a unique fixed point in X.

Theorem 3.5. *Let* (X, d) *be a quasi* αb*-metric space with* $0 \leq \alpha < 1$ *and* $b \geq 1$, *and let* $T : X \to X$ *be a continuous mapping which satisfies the following conditions*:

$$\varphi(d(Tx, Ty)) \leq \varphi(d(Tx, y))\varphi(d(x, Ty))\varphi(d(y, Ty)) + \varphi(d(x, y)) - \delta d(x, y) \tag{3.6}$$

for all $x, y \in X$, *where* $0 < \delta \leq 1$, $\varphi : [0, \infty) \to [0, \infty)$ *is a continuous mapping, nondecreasing and* $\varphi(0) = 0$.

Then T has a unique fixed point in X.

Proof. Let $\{x_n\}$ be a sequence in X.

Define $x_{n+1} = Tx_n$ for $n = 0, 1, 2, 3, \ldots.$

By using (3.6), we have

$$\begin{aligned}
&\varphi(d(x_n, x_{n+1}))\\
&= \varphi(d(Tx_{n-1}, Tx_n))\\
&\le \varphi(d(x_n, x_n))\varphi(d(x_{n-1}, x_{n+1}))\varphi(d(x_n, x_{n+1}))\\
&\quad + \varphi(d(x_{n-1}, x_n)) - \delta d(x_{n-1}, x_n)\\
&\le \varphi(0)\varphi(d(x_{n-1}, x_{n+1}))\varphi(d(x_n, x_{n+1})) + \varphi(d(x_{n-1}, x_n)) - \delta d(x_{n-1}, x_n).
\end{aligned}$$

Since $\varphi(0) = 0$, thus

$$\varphi(d(x_n, x_{n+1})) \le \varphi(d(x_{n-1}, x_{n+1})) \le \delta d(x_{n-1}, x_n). \tag{3.7}$$

Also, from $0 < \delta \le 1$, we obtain

$$\varphi(d(x_n, x_{n+1})) \le \varphi(d(x_{n-1}, x_n)).$$

Since φ is nondecreasing,

$$d(x_n, x_{n+1}) \le d(x_{n-1}, x_n).$$

Thus, $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence in R^+, so $\{d(x_n, x_{n+1})\}$ is convergent in R^+. Hence, there exists $L \ge 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = L$.

By using continuity of φ and (3.7), we get

$$\lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) \le \lim_{n\to\infty} \varphi(d(x_{n-1}, x_n)) - \delta \lim_{n\to\infty} \varphi d(x_{n-1}, x_n),$$

$$\varphi(\lim_{n\to\infty} d(x_n, x_{n+1})) \le \varphi(\lim_{n\to\infty} d(x_{n-1}, x_n)) - \delta \lim_{n\to\infty} d(x_{n-1}, x_n),$$

$$\varphi(L) \le \varphi(L) - \delta L.$$

Since $0 < \delta \le 1$, so we get $L \le 0$. Since $L \le 0$, so we get $L = 0$.

Hence, we get

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.8}$$

Now we have to show that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$:

$$\begin{aligned}\varphi(d(x_{n+1}, x_n)) &= \varphi(d(Tx_n, Tx_{n-1}))\\ &\le \varphi(d(Tx_n, x_{n-1}))\varphi(d(x_n, Tx_{n-1}))\varphi(d(x_{n-1}, Tx_{n-1}))\\ &\quad + \varphi(d(x_n, x_{n-1})) - \delta d(x_n, x_{n-1})\\ &= \varphi(d(x_{n+1}, x_{n-1}))\varphi(d(x_n, x_n))\varphi(d(x_{n-1}, x_n))\\ &\quad + \varphi(d(x_n, x_{n-1})) - \delta d(x_n, x_{n-1})\\ &= \varphi(d(x_{n+1}, x_{n-1}))\varphi(0)\varphi(d(x_{n-1}, x_n))\\ &\quad + \varphi(d(x_n, x_{n-1})) - \delta d(x_n, x_{n-1}).\end{aligned}$$

Since $\varphi(0) = 0$, thus we get

$$\varphi(d(x_{n+1}, x_n)) \le \varphi(d(x_n, x_{n-1})) - \delta d(x_n, x_{n-1}). \tag{3.9}$$

Thus, from $0 < \delta \le 1$, we have

$$\varphi(d(x_{n+1}, x_n)) \le \varphi(d(x_n, x_{n-1}))$$

and since φ is nondecreasing, we get

$$d(x_{n+1}, x_n) \le d(x_n, x_{n-1}).$$

Thus, $\{d(x_{n+1}, x_n)\}$ is a non-increasing sequence in R^+ and so $\{d(x_{n+1}, x_n)\}$ is a convergent sequence in R^+. Hence, there exists $K \ge 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = K$.

From (3.9), we get

$$\lim_{n\to\infty} \varphi(d(x_{n+1}, x_n)) \le \lim_{n\to\infty} \varphi(d(x_n, x_{n-1})) - \delta \lim_{n\to\infty} \varphi(x_n, x_{n-1}).$$

By using continuity of φ, we get

$$\varphi(\lim_{n\to\infty} d(x_{n+1}, x_n)) \le \varphi(\lim_{n\to\infty} d(x_n, x_{n-1})) - \delta \lim_{n\to\infty} d(x_n, x_{n-1}),$$

$$\varphi(K) \le \varphi(K) - \delta K.$$

Since $0 < \delta \le 1$, we get $K \le 0$. Since $K \ge 0$, thus it is only possible when $K = 0$.

Hence, we get

$$\lim_{n\to\infty} d(x_{n+1},\ x_n) = 0. \tag{3.10}$$

From (3.8) and (3.10), we have $\lim_{n\to\infty} d(x_n,\ x_{n+1}) = 0$ and $\lim_{n\to\infty} d(x_{n+1},\ x_n) = 0.$ So using Lemma 3.1, we obtain $\{x_n\}$ is a Cauchy sequence in complete X.

So there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

Now we have to show that x^* is a fixed point of T.

Since T continuous on X, we obtain

$$Tx^* = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

Thus, $Tx^* = x^*$. Hence, x^* is the fixed point of T.

Uniqueness. Suppose there exists $y^* \in X$ such that $Ty^* = y^*$.

We have to show that the fixed point of T is unique.

By using (3.6), we get

$$\begin{aligned}
&\varphi(d(x^*,\ y^*)) = \varphi(d(Tx^*,\ Ty^*)) \\
&\leq \varphi(d(Tx^*,\ y^*))\varphi(d(x^*,\ Ty^*))\varphi(d(x^*,\ Ty^*)) + \varphi(d(x^*,\ y^*)) - \delta d(x^*,\ y^*) \\
&= \varphi(d(Tx^*,\ y^*))\varphi(d(x^*,\ Ty^*))\varphi(d(y^*,\ y^*)) + \varphi(d(x^*,\ y^*)) - \delta d(x^*,\ y^*) \\
&= \varphi(d(Tx^*,\ y^*))\varphi(d(x^*,\ Ty^*))\varphi(0) + \varphi(d(x^*,\ y^*)) - \delta d(x^*,\ y^*) \\
&= \varphi(d(x^*,\ y^*)) - \delta d(x^*,\ y^*).
\end{aligned}$$

So we get

$$\varphi(d(x^*,\ y^*)) \leq \varphi(d(x^*,\ y^*)) - \delta d(x^*,\ y^*).$$

Thus,

$$\delta d(x^*,\ y^*) \leq 0.$$

In the same way, from

$$\varphi(d(y^*,\ x^*)) = \varphi(d(Ty^*,\ Tx^*)),$$

we have

$$\delta d(y^*, x^*) \le 0.$$

So it is possible if $d(x^*, y^*) = d(y^*, x^*) = 0$.

Hence, $x^* = y^*$, so T has a unique fixed point in X.

Theorem 3.6. *Let (X, d) be a complete quasi αb-metric space with $0 \le \alpha < 1$ and $b \ge 1$ and let $T : X \to X$ be a mapping satisfying the following condition*:

$$d(Tx, Ty) \le \varphi(x)\varphi(y) \text{ for all } x, y \in X, \tag{3.11}$$

where $\varphi : X \to [0, \infty)$ be a mapping on X with $\varphi(Tx) \le \delta\varphi(x)$, $0 < \delta < 1$.

Then T has a unique fixed point in X.

Proof. Let $\{x_n\}$ be a sequence in X.

We define $x_{n+1} = Tx_n$ for $n = 0, 1, 2, 3, \ldots$.

By using (3.11), we have

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\le \varphi(x_{n-1})\varphi(x_n) = \varphi(Tx_{n-2})\varphi(Tx_{n-1}) \\ &\le \delta^2\varphi(x_{n-2})\varphi(x_{n-1}) = \delta^2\varphi(Tx_{n-3})\varphi(x_{n-2}) \\ &\le \delta^4\varphi(x_{n-3})\varphi(x_{n-2}). \end{aligned}$$

By continuing this process, so we get

$$d(x_n, x_{n+1}) \le \delta^{2n-2}\varphi(x_0)\varphi(x_1).$$

Since $0 < \delta < 1$, we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Similarly, in this way, we get

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \le \varphi(x_n)\varphi(x_{n-1}) = \varphi(Tx_{n-1})\varphi(Tx_{n-2})$$

$$\leq \delta^2 \varphi(x_{n-1})\varphi(x_{n-2}) = \delta^2 \varphi(Tx_{n-2})\varphi(Tx_{n-3})$$

$$\leq \delta^4 \varphi(x_{n-2})\varphi(x_{n-3}).$$

By continuing this process, we get

$$d(x_{n+1}, x_n) \leq \delta^{2n}\varphi(x_0)\varphi(x_1).$$

Since $0 < \delta < 1$,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

By using Lemma 3.1, we conclude that $\{x_n\}$ is a Cauchy sequence in complete quasi αb-metric space (X, d). So there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

Now we have to show that x^* is a fixed point of T:

$$d(x^*, Tx^*) \leq \alpha d(Tx^*, x^*) + \frac{b}{2}(d(x^*, x_n) + d(x_n, Tx^*))$$

$$\leq \alpha\left(\alpha d(x^*, Tx^*) + \frac{b}{2}(d(Tx^*, x_n) + d(x_n, x^*))\right)$$

$$+ \frac{b}{2}(d(x^*, x_n) + d(x_n, Tx^*)).$$

So we have

$$d(x^*, Tx^*)$$

$$\leq \frac{\frac{\alpha b}{2}(d(Tx^*, x_n) + d(x_n, x^*)) + \frac{b}{2}(d(x^*, x_n) + d(x_n, Tx^*))}{1-\alpha^2}$$

$$= \frac{\frac{\alpha b}{2}(d(Tx^*, Tx_{n-1}) + d(x_n, x^*)) + \frac{b}{2}(d(x^*, x_n) + d(Tx_{n-1}, Tx^*))}{1-\alpha^2}$$

$$\leq \frac{\frac{\alpha b}{2}(\delta\varphi(x^*)\varphi(x_{n-1}) + d(x_n, x^*)) + \frac{b}{2}(d(x^*, x_n) + \delta\varphi(x_{n-1})\varphi(x^*))}{1-\alpha^2}$$

$$\leq \frac{\frac{\alpha b}{2}(\delta^n\varphi(x^*)\varphi(x_0) + d(x_n, x^*)) + \frac{b}{2}(d(x^*, x_n) + \delta^n\varphi(x_0)\varphi(x^*))}{1-\alpha^2}.$$

Since $0 < \delta < 1$ and $\lim_{n\to\infty} x_n = x^*$,

$$\lim_{n\to\infty} d(x_n, x^*) = \lim_{n\to\infty} d(x^*, x_n) = 0.$$

Thus, for $n \to \infty$, we get

$$d(x^*, Tx^*) = 0.$$

Similarly, we get

$$d(Tx^*, x^*) = 0.$$

So from $d(x^*, Tx^*) = 0$ and $d(Tx^*, x^*) = 0$, we get $Tx^* = x^*$.

Hence, x^* is the fixed point of T

Uniqueness. Suppose there exists $y^* \in X$ such that $Ty^* = y^*$. Therefore,

$$\varphi(x^*) = \varphi(Tx^*) \leq \delta\varphi(x^*),$$

$$\varphi(x^*)(1-\delta) \leq 0.$$

Since $0 < \delta < 1$ and $\varphi(x^*) \geq 0$, it is only possible when $\varphi(x^*) = 0$. Therefore,

$$\varphi(y^*) = \varphi(Ty^*) \leq \delta\varphi(y^*),$$

$$\varphi(y^*)(1-\delta) \leq 0.$$

Since $0 < \delta < 1$ and $\varphi(y^*) \geq 0$, it is only possible when $\varphi(y^*) = 0$.

By using (3.11), we have

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq \varphi(x^*) \cdot \varphi(y^*).$$

Since $\varphi(x^*) = 0$ and $\varphi(y^*) = 0$, so we get

$$d(x^*, y^*) \leq 0.$$

Similarly, we have

$$d(y^*, x^*) \leq 0.$$

From this, it is only possible when $d(x^*, y^*) = d(y^*, x^*) = 0$.

Hence, $x^* = y^*$, so T has a unique fixed point in X.

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References

[1] A. Aghajani, M. Abbas and J. R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered *b*-metric space, Math. Slov. 4 (2014), 941-960.

[2] M. Akkouchi, A common fixed point theorem for expansive mappings under strict implicit conditions on b-metric spaces, Acta Univ. Palacki. Olomuc. Mathematica 50 (2011), 5-15.

[3] Y. A. Alber and S. Gueer-Delabriere, Principle of weakly contractive maps in Hilbert space, Oper. Theory. Adv. Appl. 98 Birkhauser, Basel, 1997, pp. 7-22.

[4] A. H. Ansari, S. Chandok and C. Ionescu, Fixed point theorems on *b*-metric spaces for weak contractions with auxiliary functions, J. Inequal. Appl. 2014 (2014), 429.

[5] H. Aydi, M. F. Bota, E. Karapinar and S. Moradi, A common fixed point for weak φ-contractions on *b*-metric spaces, Fixed Point Theory 13 (2012), 337-346.

[6] I. A. Bakhtin, The contraction mapping principle in quasi *b*-metric spaces, Funct. Anal. Unianowsk Gos. Ped. Inst. 30 (1989), 26-37.

[7] B. S. Chaudhurya, P. N. Duttab, S. Duttac and P. Maity, Weak contraction principle in *b*-metric spaces, J. Math. Inform. 6 (2016), 15-19.

[8] S. Czerwik, Contraction mappings in *b*-metric space, Acta. Math. Inform. Univ. Ostraviensis 1 (1993), 5-11.

[9] A. K. Dubey, R. Shukla and R. P. Dubey, Some fixed point results in *b*-metric Space, Asian J. Math. Appl. 2014 (2014), 1-6.

[10] H. Huang and S. Xu, Fixed point theorems of contractive mappings in cone *b*-metric spaces and applications. Fixed Point Theory Appl. 2013 (2013), 112.

[11] A. Isufati, Fixed points for weakly contractive mappings in *b*-metric Spaces, AKTET J. Inst. Alb-Shkenca VI 1 (2013), 32-36.

[12] J. M. Joseph, D. D. Roselin and M. Marudai, Fixed point theorems on multi valued mapping in b-metric space, SpringerPlus 5 (2016), 217.

[13] C. Klin-eam and C. Suanoom, Dislocated quasi-b-metric spaces and fixed point theorems for cyclic contractions, Fixed Point Theory Appl. 74 (2015), 1-12.

[14] B. Nurwahyu, Fixed point theorems for generalized contraction mappings in quasi α*b*-metric space, Far East J. Math. Sci. (FJMS) 101(8) (2017), 1813-1832.

[15] B. Nurwahyu, Fixed point theorems for the multivalued contraction mapping in quasi α*b*-metric Space, Far East J. Math. Sci. (FJMS) 102(9) (2017), 2105-2119.

[16] M. Rahman and M. Sarwar, Dislocated quasi *b*-metric Space and fixed point theorems, Elec. J. Math. Anal. Appl. 4 (2016), 16-24.

[17] B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal. 47 (2001), 2683-2693.

[18] B. E. Rhoades, H. K. Pathak and S. N. Mishra, Some weakly contractive mapping theorems in partially ordered spaces and applications, Demonstratio Math. 45 (2012), 621-636.

[19] W. Shatanawi, A. Pitea and R. Lasovic, Contraction conditions using comparison function on *b*-metric spaces, Fixed Point Theory Appl. 2014 (2014), 135.