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A NOTE ON REAL ZEROS OF SELF-RECIPROCAL POLYNOMIALS

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Abstract

In this note, necessary and sufficient conditions are given for real self-reciprocal polynomials to have only real zeros. This result was inspired by a paper by Lakatos where necessary and sufficient conditions for real self-reciprocal polynomials to have all zeros on the unit circle are provided.

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1. Introduction

The properties of self-reciprocal polynomials have many applications in some areas of mathematics, for instance in cryptography and coding theory [2, 4, 5, 12], in Knot theory [8, 13, 15], and in Salem number theory [3].

A polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$ of degree n with complex coefficients is said to be *self-reciprocal polynomial* if $P(z) = z^n P(1/z)$. If $a_i \in \mathbb{R}$, then $P(z)$ is called a *real self-reciprocal polynomial* (following the definition in [6]). It is clear that if $P(z)$ is a real self-reciprocal polynomial, then $a_i = a_{n-i}$, for $i = 0, 1, \ldots, n$.

There are many results in the literature that determine conditions to guarantee that a real self-reciprocal polynomial $P(z)$ has all its zeros on the unit circle [1, 6, 7, 9-11]. Nevertheless, it is difficult to find references in the literature that furnish conditions to guarantee when $P(z)$ has only real zeros. On the other hand, the theory of real zeros of polynomials is very well established [14, 17]. Hence, the purpose of this paper is to furnish necessary and sufficient conditions for a real self-reciprocal polynomial $P(z)$ has all zeros on the real line.

The note is organized as follows: In Section 2, the definition of Chebyshev transform is recalled and some preliminary results are stated. In Section 3, necessary and sufficient conditions for real zeros of self-reciprocal polynomials are given together with some remarks. Finally, some conclusions are drawn on the coefficients of real self-reciprocal polynomials.

2. Preliminary Results

The aim of this section is to provide some definitions and results concerning Chebyshev polynomials and the Chebyshev transform, the crucial ingredients to prove our main result (please see [7, 16] for further details).

The Chebyshev polynomial $T_j(x)$ of the first kind is a polynomial of degree *j* in *x* defined by

$$T_j(\cos x) = \cos jx \text{ when } x = \cos\theta.$$

Taking $z + 1/z = x$, we deduce that $z^j + 1/z^j = C_j(x)$ (see [16, p. 224]), where

$$C_j(x) = 2T_j\left(\frac{x}{2}\right),$$

with $x \in \mathbb{C}$ and $j = 1, 2, \dots$. For our convenience, we define $C_0(x) = T_0(x)$, $x \in \mathbb{C}$.

Now let us denote by $\mathcal{R}_{2n}$ the set of all real self-reciprocal polynomials of degree at most $2n$. If $P \in \mathcal{R}_{2n}$ is a non-zero polynomial, then there is an integer *k*, $k = 0, \dots, n$, such that

$$P(z) = \sum_{j=0}^{2n} a_j z^j = z^n\left[a_{n+k}\left(z^k + \frac{1}{z^k}\right) + \cdots + a_{n+1}\left(z + \frac{1}{z}\right) + a_n\right].$$

The next result is in [7].

Proposition 1. *Every non-zero polynomial* $P \in \mathcal{R}_{2n}$ *has the decomposition*

$$P(z) = a_{n+k}z^{n-k}\prod_{j=1}^{k}(z^2 - \alpha_j z + 1),$$

where $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{C}$ *and* $a_{n+k} \neq 0$ *for some k with* $0 \leq k \leq n$.

In this way, we make sense the following definition:

Definition 1. The *Chebyshev transform* of a non-zero polynomial $P \in \mathcal{R}_{2n}$ is

$$\mathcal{T}P(x) = a_{n+k}\prod_{j=1}^{k}(x - \alpha_j).$$

The Chebyshev transform of the zero polynomial $P \equiv 0$ is defined by $\mathcal{T}0(x) = 0.$

More details and results about the Chebyshev transform may be found in [7].

3. Main Results

Necessary and sufficient conditions for a real self-reciprocal polynomial to possess all zeros on the unit circle were given by Lakatos [7]. Our result is stated as follows:

Theorem 1. *Let $P(z)$ be a real self-reciprocal polynomial of degree $2n$. Then all zeros of $P(z)$ are real if and only if all zeros of its Chebyshev transform $\mathcal{T}P(x)$ are located in $(-\infty, -2] \cup [2, \infty)$.*

Proof. We first assume that all zeros of $P(z)$ are real. Since $P(z)$ is a self-reciprocal polynomial, it follows that if z_j is a zero of $P(z)$, then so does $1/z_j$. Hence,

$$P(z) = a_{2n}\prod_{j=1}^{n}(z - z_j)\left(z - \frac{1}{z_j}\right) = a_{2n}\prod_{j=1}^{n}\left(z^2 - \left(z_j + \frac{1}{z_j}\right)z + 1\right)$$

and

$$\mathcal{T}P(x) = a_{2n}\prod_{j=1}^{n}\left(x - \left(z_j + \frac{1}{z_j}\right)\right).$$

Since z_j and $1/z_j$ are real for each j, $|z_j + 1/z_j| \geq 2$, $j = 1, \ldots, n.$ Consequently, all zeros of $\mathcal{T}P(x)$ are located in $(-\infty, -2] \cup [2, \infty).$

Conversely, suppose that the Chebyshev transform has the form

$$\mathcal{T}P(x) = a_{2n}\prod_{j=1}^{n}(x - \alpha_j),$$

where $a_{2n} \neq 0$ and $\alpha_j \in (-\infty, -2] \cup [2, \infty),\ j = 1, ..., n.$ It follows immediately that

$$P(z) = a_{2n}\prod_{j=1}^{n}(z^2 - \alpha_j z + 1).$$

Therefore, the zeros of $P(z)$ are given by

$$z = \frac{\alpha_j \pm \sqrt{\alpha_j^2 - 4}}{2}.$$

Since $\alpha_j \in (-\infty, -2] \cup [2, \infty)$ for every j, $\alpha_j^2 - 4 \geq 0,\quad j = 1, ..., n.$ More precisely, this means that all zeros of $P(z)$ are real.

Next we state the analogue of Theorem 1 for polynomials of odd degree.

Corollary 1. *Let $P(z)$ be a real self-reciprocal polynomial of degree $2n + 1$, which can be represented by*

$$P(z) = (z + 1)Q(z),$$

where $Q(z)$ is a real self-reciprocal polynomial of even degree. Then all zeros of $P(z)$ are real if and only if all zeros of the Chebyshev transform of $Q(z)$ are located in $(-\infty, -2] \cup [2, \infty)$.

Remark 1. Considering x_k a zero of $\mathcal{T}P(x)$, from $x_k = z_k + 1/z_k$, we see that

$$z_{k,1} = \frac{x_k - \sqrt{x_k^2 - 4}}{2} \quad \text{and} \quad z_{k,2} = \frac{x_k + \sqrt{x_k^2 - 4}}{2},$$

where $z_{k,1}$ and $z_{k,2}$ are the corresponding zeros of $P(z)$. If $|x_k| > 2$, then the multiplicities of $z_{k,1}$ and $z_{k,2}$ are the same as the multiplicities of x_k. In the case that $x_k = \pm 2$, the multiplicities of $z_{k,1}$ and $z_{k,2}$ are doubled. Observe that $\operatorname{sgn}(x_k) = \operatorname{sgn}(z_{k,1}) = \operatorname{sgn}(z_{k,2})$.

Remark 2. Note that $z_{k,1}$ is an increasing function of x_k for $x_k \in (-\infty, -2]$ and a decreasing function of x_k for $x_k \in [2, \infty)$. On the other hand, $z_{k,2}$ is a decreasing function of x_k for $x_k \in (-\infty, -2]$ and an increasing function of x_k for $x_k \in [2, \infty)$.

3.1. Necessary and sufficient conditions on the coefficients

The aim of this subsection is to provide necessary and sufficient conditions on the coefficients of the real self-reciprocal polynomials in order to obtain only real zeros. Our analysis is for polynomials of small degree.

Let

$$P_n(z) = \sum_{i=0}^{n} a_i z^i, \quad a_n \neq 0$$

be a real self-reciprocal polynomial of degree n. There is no loss of generality in assuming that $a_n = 1$.

We summarize the study in the following three cases:

(1) If $n = 2$, then from Theorem 1, it follows that $P_2(z) = z^2 + a_1 z + 1$ has only real zeros iff $|a_1| \geq 2$, i.e., $a_1 \in (-\infty, -2] \cup [2, \infty)$.

Observe that the zeros of multiplicity only occur in the cases $a_1 = 2$ ($z = -1$ is zero of multiplicity two) and $a_1 = -2$ ($z = 1$ is zero of multiplicity two).

(2) If $n = 3$, then we have that $P_3(z) = z^3 + a_1 z^2 + a_1 z + 1 = (z + 1) Q_2(z)$, where $Q_2(z) = z^2 + (a_1 - 1) z + 1$. From Corollary 1, $P_3(z)$ possesses only real zeros iff $|a_1 - 1| \geq 2$, i.e., $a_1 \in (-\infty, -1] \cup [3, \infty)$.

The zeros of multiplicity only occur in the following cases: $z = 1$ is zero of multiplicity two when $a_1 = -1$ and $z = -1$ is zero of multiplicity three if $a_1 = 3$.

(3) If $n = 4$, then $P_4(z) = z^4 + a_1 z^3 + a_2 z^2 + a_1 z + 1$ has only real zeros iff $(a_1, a_2) \in R_1 \cup R_2$, where

$$R_1 = \left\{(a_1, a_2) \in \mathbb{R}^2 \,|\, |a_1| \leq -1 - \frac{a_2}{2}\right\}$$

and

$$R_2 = \left\{(a_1, a_2) \in \mathbb{R}^2 \,|\, \max\{4, \sqrt{\max\{4a_2 - 8, 0\}}\} \leq |a_1| \leq 1 + \frac{a_2}{2}\right\},$$

as we can see in Figure 1.

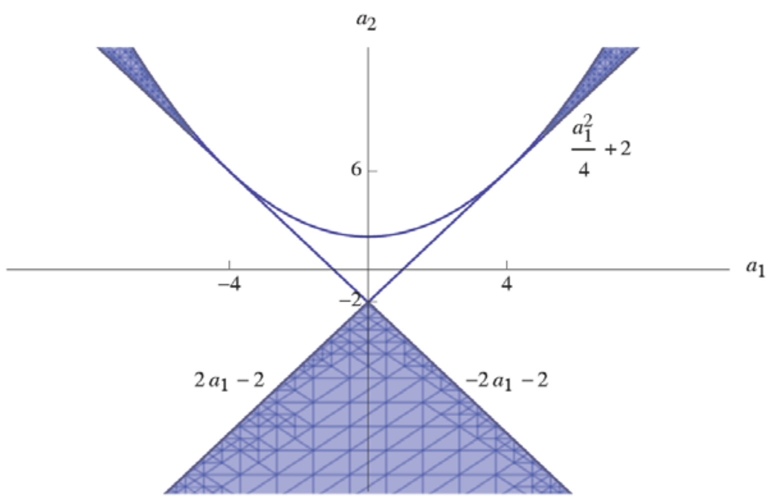

Figure 1. Region $R_1 \cup R_2$.

Remark 3. Considering $|a_1| > 4$, the zeros of multiplicity occur in the case that $a_2 = \frac{a_1^2}{4} + 2$, where zeros $z = \dfrac{-\frac{a_1}{2} - \sqrt{\left(\frac{a_1}{2}\right)^2 - 4}}{2}$ and

$z = \dfrac{-\frac{a_1}{2} + \sqrt{\left(\frac{a_1}{2}\right)^2 - 4}}{2}$ have multiplicity two. Moreover, if $a_1 < -4$ and $a_2 = -2 - 2a_1$, then $z = 1$ is a zero of multiplicity two. If $a_1 > 4$ and $a_2 = 2a_1 - 2$, then $z = -1$ is a zero of multiplicity two. If $a_1 = 4$ and $a_2 = 6$ or $a_1 = -4$ and $a_2 = 6$, $z = -1$ or $z = 1$, respectively, zeros are of multiplicity four.

Remark 4. If $a_1 = 0$, then $P_4(z)$ is an even function. So, $(0, a_2) \in R_1$ and then $P_4(z)$ has four real zeros z_1, z_2, z_3, z_4 with $|z_1| = |z_2|$ and $|z_3| = |z_4|$. Furthermore, if $a_2 = -2$, then $z = 1$ and $z = -1$ are both zeros of multiplicity two.

Remark 5. The intersection point of the lines $-2a_1 - 2$ and $2a_1 - 2$ is $(0, -2)$. The parabola $\frac{a_1^2}{4} + 2$ and the lines $-2a_1 - 2$ and $2a_1 - 2$ intersect, respectively, at the points $(-4, 6)$ and $(4, 6)$.

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