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A REVISIT TO NON-LINEAR DIFFERENTIAL EQUATIONS ASSOCIATED WITH GENOCCHI NUMBERS

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Abstract

In this paper, we obtain non-linear differential equations arising from the generating function of the Genocchi numbers. Also, we derive explicit formulae for the Genocchi numbers which are derived from those non-linear differential equations.

1. Introduction

The *Genocchi polynomials* are defined by the generating function to be

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$$\frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} \quad \text{(see [1, 2, 3, 5, 6, 12, 15, 16, 17]).} \tag{1.1}$$

In the special case $x = 0$, $G_n = G_n(0)$ $(n \geq 0)$ are called *Genocchi numbers*. The first few Genocchi numbers are 0, 1, –1, 0, 1, 0, –3, 0, 17, … .

By the definition of Genocchi numbers, it is well known that

$$\begin{aligned} G_{2n} &= 2(1 - 2^{2n})B_{2n} \\ &= 2nE_{2n-1}, \end{aligned}$$

where B_n and E_n are the Bernoulli numbers and Euler numbers, respectively which are defined by the generating function to be

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

and

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

Genocchi numbers have been studied extensively in many different contexts of mathematics and applied mathematics, for example, complex analytic number theory, p-adic analytic number theory, homotopy theory, quantum physics.

There have been many works related with non-linear differential equations. For example, in [8], Kim and Kim studied non-linear differential equations arising from Frobenius-Euler polynomials, differential equations arising from Bernoulli numbers of second kind, Changee numbers and polynomials, Mittag-Leffer polynomials, degenerate Changhee polynomials are investigated by Kim et al. in [4, 5, 7, 9, 10, 11, 13]. Also, non-linear differential equations related special numbers and polynomials are studied in [12, 14, 15]. In particular, in [6], Kim investigated differential equations associated with Genocchi polynomials.

In this paper, we obtain non-linear differential equations arising from the generating function of the Genocchi numbers. In addition, we derive explicit formulae for the Genocchi numbers which are derived from those non-linear differential equations.

2. Some Properties for Genocchi Numbers

In this section, we assume that

$$F = F(t) = \frac{2}{e^t + 1}, \text{ and } F^N(t) = \underbrace{F \times \cdots \times F}_{N\text{-times}} \text{ for } N \in \mathbb{N}. \tag{2.1}$$

By (2.1), we have

$$F^{(1)} = \frac{dF(t)}{dt} = \frac{-2e^t}{(e^t + 1)^2} = F + \frac{1}{2}F^2, \tag{2.2}$$

$$\begin{aligned} F^{(2)} &= \frac{dF^{(1)}}{dt} = F^{(1)} + \frac{1}{2}2FF^{(1)} \\ &= \left(F + \frac{1}{2}F^2\right) + F\left(F + \frac{1}{2}F^2\right) = F - \frac{3}{2}F^2 + \frac{1}{2}F^3, \end{aligned} \tag{2.3}$$

where $F^{(k)} = \left(\frac{d}{dt}\right)^k F(t)$ and $k \in \mathbb{N}$. From (2.2) and (2.3), we get

$$\begin{aligned} F^{(3)} &= \frac{dF^{(2)}(t)}{dt} = F^{(1)} - 6FF^{(1)} + \frac{3}{2}F^2F^{(1)} \\ &= -F + \frac{7}{2}F^2 - 3F^2 + \frac{3}{2}F^4. \end{aligned}$$

Continuing this process, we set

$$F^{(N)} = \sum_{k=0}^{N} a_k(N)F^{K+1}. \tag{2.4}$$

From now on, we will determine the coefficients $a_i(N)$ in (2.4). Let us take the derivative of (2.4) with respect to *t*. From (2.2) and (2.4), we have

$$F^{(N+1)} = \sum_{k=0}^{N} a_k(N)(k+1)F^k F^{(1)}$$

$$= -\sum_{k=0}^{N} a_k(N)(k+1)F^{k+1} + \sum_{k=0}^{N} a_k(N)\frac{k+1}{2}F^{k+2}$$

$$= -\sum_{k=0}^{N} a_k(N)(k+1)F^{k+1} + \sum_{k=1}^{N+1} a_{k-1}(N)\frac{k}{2}F^{k+1}$$

$$= \sum_{k=0}^{N}\left\{-a_k(N)(k+1) + a_{k-1}(N)\frac{k}{2}\right\}F^{k+1}$$

$$- a_0(N)F + a_N(N)\frac{N+1}{2}F^{N+2}. \tag{2.5}$$

By replacing N by $N+1$ in (2.4), we get

$$F^{(N+1)} = \sum_{k=0}^{N+1} a_k(N+1)F^{k+1}. \tag{2.6}$$

From (2.5) and (2.6), we have

$$-a_0(N) = a_0(N+1),\ a_N(N)\frac{N+1}{2} = a_{N+1}(N+1), \tag{2.7}$$

$$a_k(N+1) = -(k+1)a_k(N) + \frac{k}{2}a_{k-1}(N), \quad 1 \le k \le N. \tag{2.8}$$

By (2.2) and (2.3),

$$a_0(1) = -1, \quad a_1(1) = \frac{1}{2}, \tag{2.9}$$

and by (2.7) and (2.9),

$$a_0(N+1) = -a_0(N) = a_0(N-1) = \cdots = (-1)^N a_0(1) = (-1)^{N+1}.$$

From (2.7) and (2.9), we get

$$a_{N+1}(N+1) = \frac{N+1}{2} a_N(N) = \frac{N+1}{2}\frac{N}{2} a_{N-1}(N-1) = \cdots$$

$$= \frac{(N+1)!}{2^N} a_1(1) = \frac{(N+1)!}{2^{N+1}}.$$

By (2.8), we have

$$a_1(N+1)$$

$$= -2a_1(N) + \frac{1}{2} a_0(N)$$

$$= -2\left(-2a_1(N-1) + \frac{1}{2} a_0(N-1)\right) + \frac{1}{2} a_0(N)$$

$$= (-2)^2 a_1(N-1) + \frac{1}{2}(a_0(N-1) + (-2)a_0(N-1))$$

$$= (-2)^2\left(-2a_1(N-2) + \frac{1}{2} a_0(N-2)\right) + \frac{1}{2}(a_0(N) + (-2)a_0(N-1))$$

$$= (-2)^3 a_1(N-2) + \frac{1}{2}(a_0(N) + (-2)a_0(N-1) + (-2)^2 a_0(N-2))$$

$$= \cdots$$

$$= (-2)^N a_1(1) + \frac{1}{2}(a_0(N) + (-2)a_0(N-1) + \cdots + (-2)^{N-1} a_0(1))$$

$$= \frac{1}{2}(a_0(N) + (-2)a_0(N-1) + \cdots + (-2)^{N-1} a_0(1) + (-2)^N)$$

$$= \frac{1}{2} h_{N,1}$$

$$= \frac{(-1)^N}{2}(1 + 2 + 2^2 + \cdots + 2^N) = \frac{(-1)^N}{2}(2^{N+1} - 1) \qquad (2.10)$$

where

$$h_N = h_{N,1} = a_0(N) + (-2)a_0(N-1) + \cdots + (-2)^{N-1} a_0(1) + (-2)^N$$

$$= (-1)^N (2^{N+1} - 1).$$

From (2.8) and (2.10), we get

$$a_2(N+1)$$

$$= -3a_2(N) + \frac{2}{2}a_1(N)$$

$$= -3\left(-3a_2(N-1) + \frac{2}{2}a_1(N-1)\right) + \frac{2}{2}a_0(N)$$

$$= (-3)^2 a_2(N-1) + \frac{2}{2}(a_1(N-1) + (-3)a_1(N-1))$$

$$= (-3)^2\left(-3a_2(N-2) + \frac{2}{2}a_1(N-2)\right) + \frac{2}{2}(a_1(N) + (-3)a_1(N-1))$$

$$= (-3)^3 a_2(N-2) + \frac{2}{2}(a_1(N) + (-3)a_1(N-1) + (-3)^2 a_1(N-2))$$

$$= \cdots$$

$$= (-3)^{N-1}a_2(2) + \frac{2}{2}(a_1(N) + (-3)a_1(N-1) + \cdots + (-3)^{N-2}a_1(2))$$

$$= \frac{2}{2}(a_1(N) + (-3)a_1(N-1) + \cdots + (-3)^{N-2}a_1(2) + (-3)^{N-1}a_1(1))$$

$$= \frac{2}{2}h_{N,2} \tag{2.11}$$

where

$$h_{N,2} = a_1(N) + (-3)a_1(N-1) + \cdots + (-3)^{N-1}a_1(1)$$

$$= \frac{1}{2}h_{N-1,1} + \frac{(-3)}{2}h_{N-2,1} + \cdots + \frac{(-3)^{N-2}}{2}h_{1,1} + (-3)^{N-1}\frac{1}{2}.$$

By (2.8) and (2.11), we get

$$a_3(N+1)$$

$$= -4a_3(N) + \frac{3}{2}a_2(N)$$

$$= -4\left(-4a_3(N-1) + \frac{3}{2}a_2(N-1)\right) + \frac{3}{2}a_2(N)$$

$$= (-4)^2 a_3(N-1) + \frac{3}{2}(a_2(N-1) + (-4)a_2(N-1))$$

$$= (-4)^2\left(-4a_3(N-2) + \frac{3}{2}a_2(N-2)\right) + \frac{3}{2}(a_2(N) + (-4)a_2(N-1))$$

$$= (-4)^3 a_3(N-2) + \frac{3}{2}(a_2(N) + (-4)a_2(N-1) + (-4)^2 a_2(N-2))$$

$$= \cdots$$

$$= (-4)^{N-2} a_3(3) + \frac{3}{2}(a_2(N) + (-4)a_2(N-1) + \cdots + (-4)^{N-3}a_2(3))$$

$$= \frac{3}{2}(a_2(N) + (-4)a_2(N-1) + \cdots + (-4)^{N-3}a_2(3) + (-4)^{N-2}a_2(2))$$

$$= \frac{3}{2}h_{N,3}, \tag{2.12}$$

where

$$h_{N,3} = a_2(N) + (-4)a_2(N-1) + \cdots + (-4)^{N-3}a_2(3) + (-4)^{N-2} \cdot \frac{2!}{2^2}$$

$$= \frac{2}{2}h_{N-1,2} + \frac{2\cdot(-4)}{2}h_{N-2,2} + \cdots + \frac{2\cdot(-4)^{N-3}}{2}h_{2,2} + (-4)^{N-2} \cdot \frac{2!}{2^2}.$$

From (2.8) and (2.12), we get

$$a_4(N+1)$$

$$= -5a_4(N) + \frac{4}{2}a_3(N)$$

$$= -5\left(-5a_4(N-1) + \frac{4}{2}a_3(N-1)\right) + \frac{4}{2}a_3(N)$$

$$= (-5)^2 a_4(N-1) + \frac{4}{2}(a_3(N-1) + (-5)a_3(N-1))$$

$$= (-5)^2\left(-5a_4(N-2) + \frac{4}{2}a_3(N-2)\right) + \frac{4}{2}(a_3(N) + (-5)a_3(N-1))$$

$$= (-5)^3 a_4(N-2) + \frac{4}{2}(a_3(N) + (-5)a_3(N-1) + (-5)^2 a_3(N-2))$$

$$= \cdots$$

$$= (-5)^{N-3}a_4(4) + \frac{4}{2}(a_3(N) + (-5)a_3(N-1) + \cdots + (-5)^{N-4}a_3(4))$$

$$= \frac{4}{2}\left(a_3(N) + (-5)a_3(N-1) + \cdots + (-5)^{N-4}a_3(4) + (-5)^{N-3}\frac{3!}{2^3}\right)$$

$$= \frac{4}{2}h_{N,4},$$

where

$$h_{N,4} = a_3(N) + (-5)a_3(N-1) + \cdots + (-5)^{N-4}a_3(4) + (-5)^{N-3}\frac{3!}{2^3}$$

$$= \frac{3}{2}h_{N-1,3} + \frac{3\cdot(-5)}{2}h_{N-2,3} + \cdots + \frac{3\cdot(-5)^{N-4}}{2}h_{3,3} + (-5)^{N-3}\frac{3!}{2^3}.$$

Continuing this process, we have

$$a_j(N+1) = \frac{j}{2}h_{N,j} \quad (j \in \mathbb{N}), \tag{2.13}$$

where $a_0(N+1) = (-1)^{N+1}$,

$$h_{N,1} = a_0(N) + (-2)a_0(N-1) + \cdots + (-2)^{N-1}a_0(1) + (-2)^N a_0(0)$$

$$= (-1)^N(2^{N+1} - 1), \tag{2.14}$$

and

$$h_{N,j} = a_j(N) + (-j-1)a_j(N-1) + \cdots + (-j-1)^{N-j}a_j(j+1)$$

$$+ (-j-1)^{N-j+1}\frac{(j-1)!}{2^{j-1}}$$

$$= \frac{j}{2} h_{N-1, j-1} + \frac{j(-j-1)}{2} h_{N-2, j-1} + \cdots$$

$$+ \frac{j(-j-1)^{N-j}}{2} h_{j, j} + (-j-1)^{N-j+1} \frac{(j-1)!}{2^{j-1}}. \tag{2.15}$$

Hence, by (2.6), (2.13), (2.14) and (2.15), we obtain the following theorem.

Theorem 2.1. *For* $N \in \mathbb{N}$, *let us consider the following non-linear differential equation with respect to t*:

$$F^{(N+1)} = \sum_{k=0}^{N+1} \frac{k}{2} h_{N, k} F^{k+1}, \tag{2.16}$$

where

$h_{N, 0} = 1$ *for all* $N \in \mathbb{N}$,

$h_{N, 1} = (-1)^N (2^{N+1} - 1)$,

$$h_{N, j} = \sum_{k=0}^{N-j-1} \frac{j(-j-1)^k}{2} h_{N-k-1, j} + (-j-1)^{N-j+1} \frac{(j-1)!}{2^{j-1}} \quad (2 \leq j \leq N).$$

Then $F = F(t) = \frac{2}{e^t + 1}$ *is a solution of* (2.16).

From (1.1), we know that

$$F = \frac{2}{e^t + 1} = \frac{1}{t} \frac{2t}{e^t + 1} = \frac{1}{t}\left(\sum_{n=1}^{\infty} G_n \frac{t^n}{n!}\right) = \sum_{n=2}^{\infty} G_n \frac{t^{n-1}}{n!} + \frac{1}{t}. \tag{2.17}$$

Hence, by (2.17), we get

$$F^{(N)} = (-1)^N N! \frac{1}{t^{N+1}} + \sum_{k=0}^{\infty} \frac{G_{N+k+1}}{N+k+1} \frac{t^{k-N+-1}}{k!}, \tag{2.18}$$

and from (2.18), we have

$$t^{N+1}F^{(N)} = (-1)^N N! + \sum_{k=N+1}^{\infty} \frac{G_k}{k} \frac{t^k}{(k-N-1)!}$$

$$= (-1)^N N! + \sum_{k=N+1}^{\infty} \frac{G_k}{k} \frac{k!}{(k-N-1)!} \frac{t^k}{k!}$$

$$= (-1)^N N! + \sum_{k=N+1}^{\infty} \frac{(k-1)!\, G_k}{(k-N-1)!} \frac{t^k}{k!}$$

$$= (-1)^N N! + \sum_{k=N+1}^{\infty} \binom{k-1}{N} N!\, G_k \frac{t^k}{k!}. \qquad (2.19)$$

The *higher-order Genocchi numbers* are defined by the generating function to be

$$\left(\frac{2t}{e^t+1}\right)^k = \sum_{n=0}^{\infty} G_n^{(k)} \frac{t^n}{n!} \quad \text{(see [1, 3, 5, 16])}. \qquad (2.20)$$

By Theorem 2.1 and (2.20), we have

$$t^{N+1}F^{(N)} = t^{N+1} \sum_{k=0}^{N} \frac{k}{2} h_{N-1,k} F^{k+1}$$

$$= t^{N+1} \sum_{k=0}^{N} \frac{k}{2} h_{N-1,k} \left(\frac{2t}{e^t+1}\right)^{k+1} t^{-k-1}$$

$$= \sum_{k=0}^{N} \frac{N}{2} h_{N-1,k} \left(\frac{2t}{e^t+1}\right)^{k+1} t^{N-k}$$

$$= \sum_{k=0}^{N} \frac{k}{2} h_{N-1,k} \left(\sum_{n=0}^{\infty} G_n^{(k+1)} \frac{t^n}{n!}\right) t^{N-k}$$

$$= \sum_{k=0}^{N} \frac{k}{2} h_{N-1,k} \sum_{n=0}^{\infty} G_n^{(k+1)} \frac{t^{N+n-k}}{n!}$$

$$= \sum_{k=0}^{N} \frac{k}{2} h_{N-1,k} \sum_{n=N-k}^{\infty} G_{n-N+k}^{(k+1)} \frac{n!}{(n-N+k)!} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{N} \frac{k}{2} h_{N-1,k} G_{n-N+k}^{(k+1)} (N-k)! \binom{n}{N-k} \right) \frac{t^n}{n!}. \qquad (2.21)$$

Therefore, by (2.19) and (2.21), we obtain the following theorem.

Theorem 2.2. *For* $n \geq 0$ *and* $N \in \mathbb{N}$, *we have*

$$\sum_{k=0}^{N} \frac{k}{2} h_{N-1,k} G_{n-N+k}^{(k+1)} (N-k)! \binom{n}{N-k}$$

$$= \begin{cases} (-1)^N N! & \text{if } 0 \leq n \leq N, \\ \sum_{n=N+1}^{\infty} \binom{n-1}{N} N! G_n & \text{if } n \geq N+1. \end{cases}$$

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References

[1] S. Araci, E. Sen and M. Acigoz, Theorems on Genocchi polynomials of higher order arising from Genocchi basis, Taiwanese J. Math. 18(2) (2014), 473-482.

[2] S. Gaboury, R. Tremblay and B. J. Fugère, Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials, Proc. Jangjeon Math. Soc. 17(1) (2014), 115-123.

[3] D. Kang, J. H. Jeong, B. M. Lee, S. H. Rim and S. H. Choi, Some identities of higher order Genocchi polynomials arising from higher order Genocchi basis, J. Comput. Anal. Appl. 17(1) (2014), 141-146.

[4] D. S. Kim and T. Kim, Some identities for Bernoulli numbers of the second kind arising from a nonlinear differential equation, Bull. Korean Math. Soc. 52 (2015), 2001-2010.

[5] S. Kim, B. M. Kim and J. Kwon, Differential equations associated with Genocchi polynomials, Global J. Pure Appl. Math. 12(5) (2016), 4579-4585.

[6] T. Kim, Some identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials, Adv. Stud. Contemp. Math. 20(1) (2010), 23-28.

[7] T. Kim, Identities involving Frobenius-Euler polynomials arising from non-linear differential equations, J. Number Theory 132(12) (2012), 2854-2865.

[8] T. Kim and D. S. Kim, A note on nonlinear Changhee differential equations, Russ. J. Math. Phys. 23 (2016), 88-92.

[9] T. Kim, D. S. Kim, L. C. Jang and H. I. Kwon, Differential equations associated with Mittag-Leffer polynomials, Global J. Pure Appl. 12(4) (2016), 2839-2847.

[10] T. Kim, D. V. Dolgy, D. S. Kim and J. J. Seo, Differential equations for Changhee polynomials and their applications, J. Nonlinear Sci. Appl. 9 (2016), 2857-2864.

[11] T. Kim, D. S. Kim and J. J. Seo, Differential equations associated with degenerate Bell polynomials, Inter. J. Pure Appl. Math. 108(3) (2016), 551-559.

[12] T. Kim, S. H. Rim, D. V. Dolgy and S. H. Lee, Some identities of Genocchi polynomials arising from Genocchi basis, J. Inequal. Appl. 2013(43) (2013), 6 pp.

[13] T. Kim and J. J. Seo, Revisit nonlinear differential equations arising from the generating functions of degenerate Bernoulli numbers, Adv. Stud. Contemp. Math. 26(3) (2016), 401-406.

[14] H. I. Kwon, T. Kim and J. J. Seo, A note on Daehee numbers arising from differential equations, Global J. Pure Appl. Math. 12(3) (2016), 2349-2354.

[15] B. Kurt, The multiplication formulae for the Genocchi polynomials, Proc. Jangjeon Math. Soc. 13(1) (2010), 89-96.

[16] D. Lim, Some identities of degenerate Genocchi polynomials, Bull. Korean Math. Soc. 53(2) (2016), 569-579.

[17] H. Ozden, *p*-adic distribution of the unification of the Bernoulli, Euler and Genocchi polynomials, Appl. Math. Comput. 218(3) (2011), 970-973.