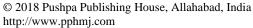
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INJECTIVITY OF CELLULAR AUTOMATA

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Abstract

Our purpose is to analyze and prove as strictly and clearly as possible that cellular automata \mathcal{A} 's of finite type with a quiescent state q are injective if and only if either \mathcal{A} contains two mutually erasable configurations c_1 , c_2 in Moore [2] or two not distinguished configurations d_1 , d_2 in Myhill [4].

1. Preliminaries

A cellular automaton is defined as a quadruple $\mathcal{A} = \{\mathbb{Z}^2, S, N, f\}$, where $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ is the cell set for $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ the rational integers, $S = \{s_1, s_2, ..., s_t\}$ is the set of states, $N : \mathbb{Z}^2 \to 2^{\mathbb{Z}^2}$ is the neighborhood function defined by for each $i = (i_1, i_2)$ in \mathbb{Z}^2 by

$$N(i) = \{j = (j_1, j_2) \mid |i_{\lambda} - j_{\lambda}| \le 1 \text{ for } \lambda = 1, 2\},$$

in particular $|N(i)| = 9$,

and $f: S^9 \to S$ is a map, we call f the *local map*.

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We define the set C of configurations by the set of mapping \mathbb{Z}^2 to S, i.e.,

$$C = S^{\mathbb{Z}^2} = \{c = (\cdots, c(i), \cdots) | i \in \mathbb{Z}^2, c(i) \in S\},$$

where *c* is called a *configuration* on \mathbb{Z}^2 .

Since |N(i)| = 9, if we give N(i) an arbitrary linear order and let c(N(i)), a subconfiguration of c on N(i), endow with the induced ordering of N(i), we may apply f to c(N(i)) to get the global map

$$F:C\to C$$

defined by

$$F(c)(i) = f(c(N(i)))$$
 for $c \in C$ and $i \in \mathbb{Z}^2$.

For Δ_n an $n \times n$ -square subset of \mathbb{Z}^2 we define C_n the set of $n \times n$ -configurations by

$$C_n = \{c \mid_{\Delta_n} | c \in C\}.$$

Since F is homogeneous, that is, it commutes with any parallel transition of \mathbb{Z}^2 , our definition of C_n does not depend essentially on the choice of Δ_n in \mathbb{Z}^2 . Further for $(n+2)\times(n+2)$ -square subset $\Delta_{(n+2)}$ of \mathbb{Z}^2 obtained by extending Δ_n one cell on four sides of Δ_n , we get

$$C_{n+2} = \{c \,|_{\Delta_{n+2}} \,|\, c \in C\}.$$

Then, we can define

$$C_{n+2} \setminus C_n = \{c \mid_{\Delta_{n+2} \setminus \Delta_n} | c \in C\},\$$

the set of square frames of sides n+2. We call $e \in C_{n+2} \setminus C_n$ an edge or a frame of $c_n = c|_{\Delta_n}$ or C_n .

For any c in C and any c_n in C_n ,

$$E = c \setminus c_n$$

is called an *environment* of c_n and we denote their relationship by

$$c = c_n \vee E$$
.

For e with $E = e \vee E'$ and $c_n = (\cdots, c_n(i), \cdots)$ in C_n , we observe that

$$c_n(N(i)) \subseteq c_n \vee e$$
,

which enable us to define a map

$$F_{n,e}:C_n\to C_n$$

$$c_n = (\cdots, c_n(i), \cdots) \mapsto c'_n = (\cdots, c'_n(i), \cdots)$$

with

$$c'_n(i) = f(c_n(N(i))).$$

A state q in S is said to be quiescent if

$$f(\underline{q, q, ..., q}) = q.$$

A configuration c in C is said to be a *finite type* if for some c_n in C_n , $n \in N$, there exists E an environment of c_n of which states are all q and

$$c = c_n \vee E$$
.

Throughout this paper we assume that S contains q and any c in C is a finite type. Thus, we understand C denotes the set of configurations of finite type.

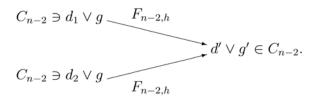
2. Statement of the Theorem

Definition. *n***-mutually erasable:**

For n in \mathbb{N} two configurations d_1 , d_2 in C_{n-4} with $d_1 \neq d_2$ are said to be n-mutually erasable if there exist

$$d'$$
 in C_{n-4} , g , g' in $C_{n-2} \setminus C_{n-4}$ and h in $C_n \setminus C_{n-2}$

such that



Remark. (a) Let d_1 , d_2 be n-mutually erasable. Then, for any l in $C_{n+2} \setminus C_n$, there exists h' in $C_n \setminus C_{n-2}$ such that

$$C_n \ni c_1 = d_1 \lor g \lor h$$

$$F_{n,l}$$

$$C_n \ni c_2 = d_2 \lor g \lor h$$

$$F_{n,l}$$

(b) Note that $d_1' = d_1 \vee g$ and $d_2' = d_2 \vee g$ are also (n+2)-mutually erasable by taking h, l for g, h and thus this procedure can be continued to get their extensions \tilde{c}_1 , \tilde{c}_2 in C.

Definition. *n***-not distinguished:**

Two configurations d_1 , d_2 in C_{n-4} with $d_1 \neq d_2$ are said to be *n-not distinguished* if there exist

- (1) E an environment of C_{n-4} , and
- (2) c' in C

such that

$$C \ni c_1 = d_1 \lor E$$
 F $c' \in C$. $C \ni c_2 = d_2 \lor E$

Now we state our theorem.

Theorem. The following (I), (I_n) and (I'_n) are equivalent:

- (I) *F* is not injective.
- (I_n) There are n-mutually erasable configurations d_1 , d_2 in C_{n-4} for some n in \mathbb{N} .
- (I'_n) There are n-not distinguished configurations d_1 , d_2 in C_{n-4} for some n in \mathbb{N} .

3. Proof for the Theorem

Using the tools prepared in the previous section we now prove our theorem, which will be done in each step of (a) (I) to (I'_n) , (b) (I'_n) to (I_n) , and (c) (I_n) to (I).

For (a). By (I) we have c_1 , c_2 in C such that $c_1 \neq c_2$ and $F(c_1) = F(c_2)$. Further, since c_1 , c_2 are finite type, there are d_1 , d_2 in C_{n-4} for some n in $\mathbb N$ such that for some E an environment of C_{n-4} of which states are all q we have

$$c_i = d_i \vee E$$
, $i = 1, 2$,

where $d_1 \neq d_2$, since $c_1 \neq c_2$. Thus (I'_n) holds.

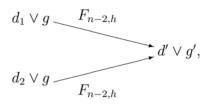
For (b). By (I'_n) we have d_1, d_2 in C_{n-4} with $d_1 \neq d_2$ and E an environment of C_{n-4} such that

$$F(d_1 \vee E) = F(d_2 \vee E).$$

Then, expressing E as

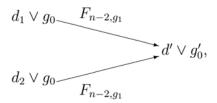
$$E = g \lor h \lor E'$$

for g in $C_{n-2} \setminus C_{n-4}$, h in $C_n \setminus C_{n-2}$, and E' an environment of C_n , we have d' in C_{n-4} and g' in $C_{n-2} \setminus C_{n-4}$ such that

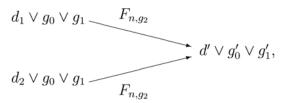


which is (I_n) .

For (c). By (I_n) we have d_1,d_2,d' in C_{n-4} with $d_1\neq d_2,g_0,g'_0$ in $C_{n-2}\setminus C_{n-4}$ and g_1 in $C_n\setminus C_{n-2}$ such that



Further by (a) of Remark in Section 2, for any g_2 in $C_{n+2} \setminus C_n$ there exists g_1' in $C_n \setminus C_{n-2}$ such that



Here, since g_2 is arbitrarily chosen, we may repeat this method to get g_3, g_4, \dots Thus, if we choose E an arbitrary environment of C_n and set

$$c_i = d_i \vee g_0 \vee g_1 \vee E$$
, for $i = 1, 2$,

we have

$$F(c_1) = F(c_2).$$

Since $d_1 \neq d_2$, we see $c_1 \neq c_2$. Thus (I) holds and we have completed our proof for the theorem.

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