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INJECTIVITY OF CELLULAR AUTOMATA

Hiroyuki Ishibashi

Chuou 2-3-8, Moroyama
Iruma, Saitama, Japan

Abstract

Our purpose is to analyze and prove as strictly and clearly as possible that cellular automata $\mathcal{A}$'s of finite type with a quiescent state q are injective if and only if either $\mathcal{A}$ contains two mutually erasable configurations c_1, c_2 in Moore [2] or two not distinguished configurations d_1, d_2 in Myhill [4].

1. Preliminaries

A cellular automaton is defined as a quadruple $\mathcal{A} = \{\mathbb{Z}^2, S, N, f\}$, where $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ is the cell set for $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ the rational integers, $S = \{s_1, s_2, ..., s_t\}$ is the set of states, $N : \mathbb{Z}^2 \to 2^{\mathbb{Z}^2}$ is the neighborhood function defined by for each $i = (i_1, i_2)$ in $\mathbb{Z}^2$ by

$$N(i) = \{j = (j_1, j_2) \mid |i_\lambda - j_\lambda| \leq 1 \text{ for } \lambda = 1, 2\},$$

$$\text{in particular } |N(i)| = 9,$$

and $f : S^9 \to S$ is a map, we call f the *local map*.

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We define the set C of configurations by the set of mapping $\mathbb{Z}^2$ to S, i.e.,

$$C = S^{\mathbb{Z}^2} = \{c = (\cdots, c(i), \cdots) | i \in \mathbb{Z}^2, c(i) \in S\},$$

where c is called a *configuration* on $\mathbb{Z}^2$.

Since $|N(i)| = 9$, if we give $N(i)$ an arbitrary linear order and let $c(N(i))$, a subconfiguration of c on $N(i)$, endow with the induced ordering of $N(i)$, we may apply f to $c(N(i))$ to get the global map

$$F : C \to C$$

defined by

$$F(c)(i) = f(c(N(i))) \text{ for } c \in C \text{ and } i \in \mathbb{Z}^2.$$

For Δ_n an $n \times n$-square subset of $\mathbb{Z}^2$ we define C_n the set of $n \times n$-configurations by

$$C_n = \{c|_{\Delta_n} | c \in C\}.$$

Since F is homogeneous, that is, it commutes with any parallel transition of $\mathbb{Z}^2$, our definition of C_n does not depend essentially on the choice of Δ_n in $\mathbb{Z}^2$. Further for $(n+2) \times (n+2)$-square subset $\Delta_{(n+2)}$ of $\mathbb{Z}^2$ obtained by extending Δ_n one cell on four sides of Δ_n, we get

$$C_{n+2} = \{c|_{\Delta_{n+2}} | c \in C\}.$$

Then, we can define

$$C_{n+2} \backslash C_n = \{c|_{\Delta_{n+2} \backslash \Delta_n} | c \in C\},$$

the set of square frames of sides $n + 2$. We call $e \in C_{n+2} \backslash C_n$ an *edge* or a *frame* of $c_n = c|_{\Delta_n}$ or C_n.

For any c in C and any c_n in C_n,

$$E = c \backslash c_n$$

is called an *environment* of c_n and we denote their relationship by

$$c = c_n \vee E.$$

For *e* with $E = e \vee E'$ and $c_n = (\cdots, c_n(i), \cdots)$ in C_n, we observe that

$$c_n(N(i)) \subseteq c_n \vee e,$$

which enable us to define a map

$$F_{n,e} : C_n \to C_n$$

$$c_n = (\cdots, c_n(i), \cdots) \mapsto c'_n = (\cdots, c'_n(i), \cdots)$$

with

$$c'_n(i) = f(c_n(N(i))).$$

A state *q* in *S* is said to be *quiescent* if

$$f\underbrace{(q, q, ..., q)}_{9} = q.$$

A configuration *c* in *C* is said to be a *finite type* if for some c_n in C_n, $n \in N$, there exists *E* an environment of c_n of which states are all *q* and

$$c = c_n \vee E.$$

Throughout this paper we assume that *S* contains *q* and any *c* in *C* is a finite type. Thus, we understand *C* denotes the set of configurations of finite type.

2. Statement of the Theorem

Definition. *n*-mutually erasable:

For *n* in $\mathbb{N}$ two configurations d_1, d_2 in C_{n-4} with $d_1 \neq d_2$ are said to be *n-mutually erasable* if there exist

$$d' \text{ in } C_{n-4}, \quad g, g' \text{ in } C_{n-2} \backslash C_{n-4} \text{ and } h \text{ in } C_n \backslash C_{n-2}$$

such that

$$\begin{tikzcd}[column sep=small]
C_{n-2} \ni d_1 \vee g \arrow[dr, "F_{n-2,h}"] & \\
& d' \vee g' \in C_{n-2}. \\
C_{n-2} \ni d_2 \vee g \arrow[ur, "F_{n-2,h}"'] &
\end{tikzcd}$$

Remark. (a) Let d_1, d_2 be n-mutually erasable. Then, for any l in $C_{n+2} \setminus C_n$, there exists h' in $C_n \setminus C_{n-2}$ such that

$$\begin{tikzcd}[column sep=small]
C_n \ni c_1 = d_1 \vee g \vee h \arrow[dr, "F_{n,l}"] & \\
& d' \vee g' \vee h' \in C_n. \\
C_n \ni c_2 = d_2 \vee g \vee h \arrow[ur, "F_{n,l}"'] &
\end{tikzcd}$$

(b) Note that $d_1' = d_1 \vee g$ and $d_2' = d_2 \vee g$ are also $(n+2)$-mutually erasable by taking h, l for g, h and thus this procedure can be continued to get their extensions $\tilde{c}_1, \tilde{c}_2$ in C.

Definition. *n*-not distinguished:

Two configurations d_1, d_2 in C_{n-4} with $d_1 \neq d_2$ are said to be *n-not distinguished* if there exist

(1) E an environment of C_{n-4}, and

(2) c' in C

such that

$$\begin{tikzcd}[column sep=small]
C \ni c_1 = d_1 \vee E \arrow[dr, "F"] & \\
& c' \in C. \\
C \ni c_2 = d_2 \vee E \arrow[ur, "F"'] &
\end{tikzcd}$$

Now we state our theorem.

Theorem. *The following* (I), $(\mathrm{I_n})$ *and* $(\mathrm{I'_n})$ *are equivalent*:

(I) *F is not injective*.

$(\mathrm{I_n})$ *There are n-mutually erasable configurations* d_1, d_2 *in* C_{n-4} *for some n in* $\mathbb{N}$.

$(\mathrm{I'_n})$ *There are n-not distinguished configurations* d_1, d_2 *in* C_{n-4} *for some n in* $\mathbb{N}$.

3. Proof for the Theorem

Using the tools prepared in the previous section we now prove our theorem, which will be done in each step of (a) (I) to $(\mathrm{I'_n})$, (b) $(\mathrm{I'_n})$ to $(\mathrm{I_n})$, and (c) $(\mathrm{I_n})$ to (I).

For (a). By (I) we have c_1, c_2 in C such that $c_1 \neq c_2$ and $F(c_1) = F(c_2)$. Further, since c_1, c_2 are finite type, there are d_1, d_2 in C_{n-4} for some n in $\mathbb{N}$ such that for some E an environment of C_{n-4} of which states are all q we have

$$c_i = d_i \vee E, \quad i = 1, 2,$$

where $d_1 \neq d_2$, since $c_1 \neq c_2$. Thus $(\mathrm{I'_n})$ holds.

For (b). By $(\mathrm{I'_n})$ we have d_1, d_2 in C_{n-4} with $d_1 \neq d_2$ and E an environment of C_{n-4} such that

$$F(d_1 \vee E) = F(d_2 \vee E).$$

Then, expressing E as

$$E = g \vee h \vee E'$$

for g in $C_{n-2} \backslash C_{n-4}$, h in $C_n \backslash C_{n-2}$, and E' an environment of C_n, we have d' in C_{n-4} and g' in $C_{n-2} \backslash C_{n-4}$ such that

$$\begin{tikzcd}
d_1 \vee g \arrow[dr, "F_{n-2,h}"] & \\
& d' \vee g', \\
d_2 \vee g \arrow[ur, "F_{n-2,h}"'] &
\end{tikzcd}$$

which is $(\mathrm{I_n})$.

For (c). By $(\mathrm{I_n})$ we have d_1, d_2, d' in C_{n-4} with $d_1 \neq d_2$, g_0, g_0' in $C_{n-2} \backslash C_{n-4}$ and g_1 in $C_n \backslash C_{n-2}$ such that

$$\begin{tikzcd}
d_1 \vee g_0 \arrow[dr, "F_{n-2,g_1}"] & \\
& d' \vee g_0', \\
d_2 \vee g_0 \arrow[ur, "F_{n-2,g_1}"'] &
\end{tikzcd}$$

Further by (a) of Remark in Section 2, for any g_2 in $C_{n+2} \backslash C_n$ there exists g_1' in $C_n \backslash C_{n-2}$ such that

$$\begin{tikzcd}
d_1 \vee g_0 \vee g_1 \arrow[dr, "F_{n,g_2}"] & \\
& d' \vee g_0' \vee g_1', \\
d_2 \vee g_0 \vee g_1 \arrow[ur, "F_{n,g_2}"'] &
\end{tikzcd}$$

Here, since g_2 is arbitrarily chosen, we may repeat this method to get $g_3, g_4, \ldots.$ Thus, if we choose E an arbitrary environment of C_n and set

$$c_i = d_i \vee g_0 \vee g_1 \vee E, \text{ for } i = 1, 2,$$

we have

$$F(c_1) = F(c_2).$$

Since $d_1 \neq d_2$, we see $c_1 \neq c_2$. Thus (I) holds and we have completed our proof for the theorem.

References

[1] B. Durand, Global properties of cellular automata, Cellular Automata Complex Systems, E. Goles and S. Martinez, eds., Kluwer Academic Publishers, 1999, pp. 1-22.

[2] Jarkko Kari, Preface [Part 1: Special issue: Current research trends in cellular automata theory], Nat. Comput. 16(3) (2017), 365-366.

[3] E. F. Moore, Machine models of self-reproduction, Proc. Symp. Appl. Math. 14 (1963), 17-34.

[4] J. Myhill, The converse to Moore's Garden-of-Eden theorem, Proc. Amer. Math. Soc. 14 (1963), 685-686.

[5] H. Nishio and T. Saito, Information dynamics of cellular automata I – An algebraic study, Fundamenta Informaticae 58(3-4) (2003), 399-420.