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ON BOUNDEDNESS OF INTEGRAL OPERATORS WITH HARDY-TYPE KERNEL ON POWER WEIGHTED LEBESGUE SPACES

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Abstract

We exhibit boundedness of an integral operator with homogeneous kernel, from the power weighted space L^p into the related power weighted L^q space. In turns, the boundedness is used to prove a generalized version of Hardy inequality, in which the classical one can be obtained as a special case.

1. Introduction

The classical Hardy integral inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx, \tag{1}$$

where $p > 1$, has been proved with various ways [2]. A more traditional approach is using integration by parts. Via an alternate route, using homogeneous kernel of degree –1, one can prove a more general inequality (see [1]).

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Okikiolu [3] proved the boundedness of integral operator with homogeneous kernel of degree $\mu - 1$, where $0 \le \mu < 1$. Using special kernel, one can have an inequality similar to (1), with differing exponents, on the left and right hand sides.

In this paper, we will prove a similar result to [3], that is the boundedness of the operator:

$$(Hf)(x) = \int_{-\infty}^{\infty} k(x, y) f(y) dy$$

in a weighted space, where the weight is an appropriate power function. Here $k(x, y)$ is a homogeneous kernel of degree $\mu - 1$, where $0 \le \mu < 1$. Comparing to [3], we do not impose the symmetric condition for the operator. This is a first step to answering an open question raised in [4], on necessary and sufficient condition for boundedness of integral operators with general Hardy-type kernels. Using an appropriate power function as a weight, we prove our main result using similar technique employed in [3], albeit some modification.

In the rest of this paper, we use our main result to obtain the following inequality:

$$\left(\int_0^{\infty} x^{b-q+\mu q}\left(\int_0^x f(y)dy\right)^q dx\right)^{1/q} \le \left(\frac{p}{p-(a+1)}\right)^{1-\mu}\left(\int_0^{\infty} x^a f(x)^p dx\right)^{1/p},$$

where $p > 1$, $0 \le \mu < 1$, $\frac{1}{q} = \frac{1}{p} - \mu$ and for any $a \ge 0$, $b \ge 0$ with some conditions which will be detailed later. Notice that if $\mu = 0$, $a = 0$ and $b = 0$, we have the classical Hardy inequality. We also generalize a variant of Hardy inequality stated in [1, p. 188].

2. Main Results and Discussion

We will work on the power weighted Lebesgue space of the real functions of one variable defined on the whole space $\mathbb{R}$. Suppose $p > 1$ and

$a \geq 0.$ The weighted Lebesgue space in $\mathbb{R}$ is the set (of equivalence classes)

$$L^p(x^a) = \left\{ f \,|\, \int_{-\infty}^{\infty} |f(x)|^p |x|^a dx < \infty \right\}.$$

We write $\| f \|_{L^p(x^a)} = \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^a dx \right)^{1/p}$ as the norm of the space.

We will discuss the boundedness of integral operator

$$(Hf)(x) = \int_{-\infty}^{\infty} k(x, y) f(y) dy$$

on the weighted Lebesgue space, for which the kernel k satisfies the homogeneous condition

$$k(\lambda x, \lambda y) = |\lambda|^{\mu - 1} k(x, y)$$

for all real number $\lambda \neq 0,$ and some $0 \leq \mu < 1.$ Our main result can be stated as follows.

Theorem 1. *Let* $p > 1,\ 0 \leq \mu < 1,$ *and q such that*

$$\frac{1}{q} = \frac{1}{p} - \mu > 0 \tag{2}$$

and $\frac{a}{p} = \frac{b}{q}.$ *Let* $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a homogenous kernel with degree* $\mu - 1.$ *Then the operator H satisfies*

$$\| Hf \|_{L^q(x^b)} \leq K^{1-\mu} \| f \|_{L^p(x^a)},$$

where

$$K = \int_{-\infty}^{\infty} |u|^{\frac{1}{(1-\mu)}\left(\mu - \frac{a+1}{p}\right)} |k(1, u)|^{\frac{1}{1-\mu}} du < \infty.$$

Proof. We borrow the idea from [3] to estimate the operator

$$|(Hf)(x)|$$

$$\leq \int_{-\infty}^{\infty} |k(x, y)||f(y)|dy$$

$$= \int_{-\infty}^{\infty} |f(y)|^{\mu p}|y|^{\mu a}|f(y)|^{1-\mu p}|y|^{\alpha}|k(x, y)|^{\beta}|y|^{-(\mu a+\alpha)}|k(x, y)|^{1-\beta}dy. \tag{3}$$

The numbers α and β will be determined later. The conjugate number p' of p is the number that satisfies $\frac{1}{p} + \frac{1}{p'} = 1.$ Holder inequality is applied to three functions in (3) with parameters that satisfies the relation (2) or $\mu + \frac{1}{q} + \frac{1}{p'} = 1.$ It yields

$$|(Hf)(x)| \leq \left(\int_{-\infty}^{\infty} |f(y)|^{p}|y|^{a}dy\right)^{\mu}(I_1(x))^{1/q}(I_2(x))^{1/p'}$$

with

$$I_1(x) = \int_{-\infty}^{\infty} |f(y)|^{(1-\mu p)q}|y|^{\alpha q}|k(x, y)|^{\beta q}dy,$$

$$I_2(x) = \int_{-\infty}^{\infty} |y|^{-(\mu a+\alpha)p'}|k(x, y)|^{(1-\beta)p'}dy.$$

Using the homogeneous property of k, I_2 is manipulated

$$I_2(x) = \int_{-\infty}^{\infty} |y|^{-(\mu a+\alpha)p'}|x|^{(\mu-1)(1-\beta)p'}|k(1, x^{-1}y)|^{(1-\beta)p'}dy.$$

Write $u = x^{-1}y$ or $y = xu,$ then

$$I_2(x) = \int_{-\infty}^{\infty} |xu|^{-(\mu a+\alpha)p'}|x|^{(\mu-1)(1-\beta)p'}|k(1, u)|^{(1-\beta)p'}|x|du$$

$$= |x|^{-(\mu a+\alpha)p'+(\mu-1)(1-\beta)p'+1}\int_{-\infty}^{\infty}|u|^{-(\mu a+\alpha)p'}|k(1, u)|^{(1-\beta)p'}du$$

$$= |x|^{-(\mu a+\alpha)p'+(\mu-1)(1-\beta)p'+1}K,$$

where the constant K is

$$K = \int_{-\infty}^{\infty}|u|^{-(\mu a+\alpha)p'}|k(1, u)|^{(1-\beta)p'}du. \tag{4}$$

Now we compute the norm of the operator. Using the Fubini Theorem, the norm of the operator can be written

$$\int_0^{\infty}|(Hf)(x)|^q|x|^b dx$$

$$\leq \|f\|_{L^p(x^a)}^{\mu pq}K^{q/p'}\int_{-\infty}^{\infty}|x|^{b-q(\mu a+\alpha)+(\mu-1)(1-\beta)q+q/p'}I_1(x)dx$$

$$\leq \|f\|_{L^p(x^a)}^{\mu pq}K^{q/p'}\int_{-\infty}^{\infty}|f(y)|^{(1-\mu p)q}|y|^{\alpha q}(I_3(y))dy \tag{5}$$

with

$$I_3(y) = \int_{-\infty}^{\infty}|x|^{b-q(\mu a+\alpha)+(\mu-1)(1-\beta)q+q/p'}|k(x, y)|^{\beta q}dx.$$

Using the property of kernel k, we have

$$I_3(y) = \int_{-\infty}^{\infty}|x|^{b-q(\mu a+\alpha)+(\mu-1)(1-\beta)q+q/p'}|x|^{(\mu-1)\beta q}|k(1, x^{-1}y)|^{\beta q}dx.$$

Upon using the variable $u = x^{-1}y$ or $x = u^{-1}y,$ then

$$I_3(y) = \int_{-\infty}^{\infty}|u^{-1}y|^{b-q(\mu a+\alpha)+(\mu-1)q+q/p'}|k(1, u)|^{\beta q}u^{-2}ydu$$

$$= |y|^{b-q(\mu a+\alpha)+(\mu-1)q+q/p'+1}$$

$$\cdot\int_{-\infty}^{\infty}|u|^{-b+q(\mu a+\alpha)-(\mu-1)q-q/p'-2}|k(1, u)|^{\beta q}du.$$

We now choose α and β such that

$$\alpha q + b - q(\mu a + \alpha) + (\mu - 1)q + q/p' + 1 = a, \tag{6}$$

$$-b + q(\mu a + \alpha) - (\mu - 1)q - q/p' - 2 = -(\mu a + \alpha)p'. \tag{7}$$

In this case, we have

$$\alpha = \frac{a - ap'\mu + 1}{p' + q}.$$

Meanwhile in order to have $\beta q = (1 - \beta)p'$, we set $\beta = \dfrac{p'}{p' + q} = \dfrac{1}{1 - \mu}$.

Note that the constant K in (4) can be written as

$$K = \int_{-\infty}^{\infty} |u|^{\frac{1}{1-\mu}\left(\mu - \frac{a+1}{p}\right)} |k(1, u)|^{\frac{1}{1-\mu}} du.$$

Therefore, equation (5) transforms into

$$\int_{-\infty}^{\infty} |(Hf)(x)|^q |x|^b dx \le \| f \|_{L^p(x^a)}^{\mu pq} K^{q/p'} \| f \|_{L^p(x^a)}^{p} K$$

$$= \| f \|_{L^p(x^a)}^{q} K^{1+q/p'}$$

or

$$\| (Hf)(x) \|_{L^q(x^b)} \le K^{\left(\frac{1}{q} + \frac{1}{p'}\right)} \| f \|_{L^p(x^a)}$$

or

$$\| (Hf)(x) \|_{L^q(x^b)} \le K^{1-\mu} \| f \|_{L^p(x^a)}.$$

Then we obtain the result.

Furthermore, from the equation (6), we have

$$a = b - q\mu a$$

or $\dfrac{a}{p} = \dfrac{b}{q}$. □

We now apply the above result for a special case. If we set

$$k(x,\ y) = \frac{1}{x^{1-\mu}}\chi_E(x,\ y),$$

where $E = \{(x,\ y) | \, y < x\},$ then we have the operator

$$Tf(x) = \frac{1}{x^{1-\mu}}\int_0^x f(y)dy$$

and the following inequality.

Corollary 2. *Let* $p > 1,\ 0 \le \mu < 1$ *and* $\frac{1}{q} = \frac{1}{p} - \mu.$ *For all* $a \ge 0$ *and* $b \ge 0$ *such that* $\frac{a}{p} = \frac{b}{q}$, *and f with* $f(x) \ge 0$ *in* $[0, \infty),$ *then*

$$\left(\int_0^\infty x^{b-q+\mu q}\left(\int_0^x f(y)dy\right)^q dx\right)^{1/q} \le \left(\frac{p}{p-(a+1)}\right)^{1-\mu}\left(\int_0^\infty x^a f(x)^p dx\right)^{1/p}.$$

We can consider the inequality as a generalized Hardy integral inequality, in which the classical Hardy inequality is a special case, for $a = b = 0$ and $\mu = 0.$ See page 188 in [1].

For the next example, if we set

$$k(x,\ y) = \begin{cases} x^{\beta-1+\mu}y^{-\beta} & y \le x \\ 0 & y > x \end{cases}$$

and $f(x) = x^\gamma h(x)$ for suitable β and γ, then we have the following inequality.

Corollary 3. *Let* $p > 1,\ 0 \le \mu < 1$ *and* $\frac{1}{q} = \frac{1}{p} - \mu.$ *For all* $a \ge 0$ *and* $b \ge 0$ *such that* $\frac{a}{p} = \frac{b}{q}$, *and* $h(x) \ge 0$ *in* $[0, \infty),$ *then*

$$\left(\int_0^\infty x^{b-r-1}\left(\int_0^\infty h(y)\,dy\right)^q dx\right)^{1/q} \le K\left(\int_0^\infty x^{a+(1-\mu)\,p-\frac{(r-1)a}{b}} h(x)^p\,dx\right)^{1/p},$$

where $K = \left(\dfrac{q}{q\mu - r + 1}\right)^{1-\mu}$.

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