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ON THE CLASSIFICATION OF α-KRULL MODULES

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Abstract

It is proved that for any ring R and right R-module M, if M is α-Krull (i.e., for each submodule N of M, either $k\text{-dim}\, N \leq \alpha$ or $k\text{-dim}\,\frac{M}{N} \leq \alpha$ and α is the least ordinal number with this property), then $k\text{-dim}\, M = \alpha$ or $k\text{-dim}\, M = \alpha + 1.$ The main aim of this paper is to characterize modules, which are α-Krull if and only if their Krull dimension is equal to α.

1. Introduction

Throughout this paper, all rings are associative with $1 \neq 0,$ and all modules are unitary right modules. Letting M be an R-module, by $k\text{-dim}\, M$ and $n\text{-dim}\, M,$ we mean the Krull dimension and the Noetherian dimension (dual of Krull dimension of M, see Karamzadeh [9] and Lemonnier [17]) of M over R, respectively. The notation $N \subseteq M$ (resp., $N \subset M$) will mean N

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is a submodule (resp. proper submodule) of M. It is convenient when we are dealing with the latter dimensions, to begin our list of ordinals with –1. In [5], the authors by using the concept of Noetherian dimension introduced and investigated the concept of α-short modules. They called an R-module M to be α-*short* if for every submodule N of M, either $n\text{-dim}\, N \le \alpha$ or $n\text{-dim}\, \frac{M}{N} \le \alpha$, where α is the least ordinal number with this property. Using this concept, they extended almost all basic results of short-modules to α-short modules, see [5]. They showed that in case α is countable, every submodule of M is countably generated. They also observed that any α-short module has Noetherian dimension equal to either α or $\alpha + 1$. In particular, a semiprime ring R is α-short if and only if $n\text{-dim}\, R = \alpha$. This fact raised the natural question, namely, for which R-module M, M is α-short if and only if $n\text{-dim}\, M = \alpha$. In [8], we answered this question. We proved that any semiprime module M (in the sense of [21] and [20]) is α-short if and only if $n\text{-dim}\, M = \alpha$. The concept of α-Krull modules, that is dual of α-short modules, introduced and extensively investigated in [4] and the dual of almost all of single results in [5], were obtained. It is proved that an R-module M is α-Krull if and only if M has Krull dimension equal to either α or $\alpha + 1$. In this paper, we are going to characterize $\mathcal{A}$ (resp. $\mathcal{B}$), the category of R-modules M, that for any ordinal number α, M is α-Krull if and only if $k\text{-dim}\, M = \alpha$ (resp. $k\text{-dim}\, M = \alpha + 1$). To reach this goal, after reviewing some necessary preliminaries, we investigate some basic properties of α-Krull modules. For instance, we show that an R-module M is α-Krull module if and only if there exists a submodule $A(\alpha)$ of M such that $k\text{-dim}\, A(\alpha) \le \alpha$ and $k\text{-dim}\, \frac{M}{B} \le \alpha$ for any submodule $B \nsubseteq A(\alpha)$ and α is the least ordinal number with this property. Finally, we show that $M \in \mathcal{A}$ if and only if either $k\text{-dim}\, M$ is a limit ordinal or $k\text{-dim}\, M = k\text{-dim}\, N$, for any co-critical submodule N of M. Also, we observe that if M is a Noetherian uniserial R-module, then M is either –1-Krull or 0-Krull.

For all concepts and basic properties of rings and modules which are not defined in this paper, we refer the reader to [6] and [18].

2. Preliminaries

We need the following definition, see [4, Definition 3.1].

Definition 2.1. An R-module M is called α-*Krull*, if for each submodule N of M, either $k\text{-dim}\, N \leq \alpha$ or $k\text{-dim}\, \frac{M}{N} \leq \alpha$ and α is the least ordinal number with this property.

Remark 2.2 [4, Remark 3.2]**.** If M is an R-module with $k\text{-dim}\, M = \alpha$, then M is β-Krull for some $\beta \leq \alpha$.

Remark 2.3 [4, Remark 3.3]**.** If M is an α-Krull module, then each submodule and each factor module of M is β-Krull for some $\beta \leq \alpha$.

Lemma 2.4 [4, Corollary 3.5]**.** *Let M be an* α*-Krull module. Then M has Krull dimension and* $k\text{-dim}\, M \geq \alpha$.

Proposition 2.5 [4, Proposition 3.6]**.** *An R-module M has Krull dimension if and only if M is* α*-Krull for some ordinal* α.

It is well-known that any module with Krull dimension has finite uniform dimension. The following corollary is now evident.

Corollary 2.6. *Every* α*-Krull module has finite uniform dimension.*

Proposition 2.7 [4, Proposition 3.8]**.** *If M is an* α*-Krull R-module, then either* $k\text{-dim}\, M = \alpha$ *or* $k\text{-dim}\, M = \alpha + 1$.

Corollary 2.8 [4, Corollary 3.9]**.** *If M is a* 0*-Krull module, then either M is Artinian or* $k\text{-dim}\, M = 1$.

Proposition 2.9 [4, Proposition 3.12]**.** *Let M be an R-module, with* $k\text{-dim}\, M = \alpha$, *where* α *is a limit ordinal. Then M is* α*-Krull.*

We recall that a nonzero R-module M is said to be α-*critical* if $k\text{-dim}\, M = \alpha$ and $k\text{-dim}\,\frac{M}{N} < \alpha,$ for every nonzero submodule N of M. A module is said to be *critical* if it is α-critical for some ordinal α. Also, a submodule N of a module M is called to be *co-critical* if the factor $\frac{M}{N}$ is critical. It is proved that any nonzero module with Krull dimension has at least one critical submodule possibly not of the same dimension and every critical module is uniform, see [6, Theorem 2.1, Proposition 2.6]. Clearly, an R-module M is 0-critical if and only if M is a simple module.

In view of Proposition 2.7, the following remark is now evident.

Remark 2.10. A nonzero R-module M is –1-Krull if and only if it is simple. Thus, any –1-Krull module is 0-critical.

We also need the following well known results of Krull dimension, see [6, 9, 11] and [18].

Theorem 2.11. *Let M be an R-module and N be a submodule of M. Then* $k\text{-dim}\, M = \sup\left\{k\text{-dim}\, N,\ k\text{-dim}\,\frac{M}{N}\right\}$ *if either side exists.*

Theorem 2.12. *Let M be a module. Then*

(1) $k\text{-dim}\, M = \sup\{k\text{-dim}\, N : 0 \neq N \subseteq M\}.$

(2) $k\text{-dim}\, M \leq \sup\left\{k\text{-dim}\,\frac{M}{E} + 1 : E \subseteq_e M\right\}.$

Lemma 2.13. *If M is an R-module and for each submodule N of M, either N or* $\frac{M}{N}$ *has Krull dimension, then so does M.*

Theorem 2.14. *If M is an R-module with Krull dimension and* $M = \sum_{i\in I} M_i,$ *where* $k\text{-dim}\, M_i \leq \alpha$ *for some ordinal* α *and all* $i \in I,$ *then* $k\text{-dim}\, M \leq \alpha.$

3. The Classification of α-Krull Modules

We cite the following result from [4, Lemma 4.1, Lemma 4.2].

Theorem 3.1. *Let N be a submodule of an R-module M. Then*

(1) *If N is* α*-Krull and* $k\text{-dim}\,\frac{M}{N} \leq \alpha,$ *then M is* α*-Krull.*

(2) *If* $\frac{M}{N}$ *is* α*-Krull and* $k\text{-dim}\,N \leq \alpha,$ *then M is* α*-Krull.*

Next, we give a structure theorem for α-Krull modules.

Theorem 3.2. *The following are equivalent for any R-module M*:

(1) *M is an* α*-Krull module.*

(2) *There exists a submodule* $A(\alpha)$ *of M such that* $k\text{-dim}\,A(\alpha) \leq \alpha$ *and* $k\text{-dim}\,\frac{M}{B} \leq \alpha$ *for any submodule* $B \nsubseteq A(\alpha)$ *and* α *is the least ordinal number with this property.*

Proof. Let M be an α-Krull module and

$$\Delta = \{X \subseteq M : k\text{-dim}\,X \leq \alpha\}.$$

It is clear that $0 \in \Delta$ and so $\Delta \neq \emptyset.$ Let $A(\alpha) = \sum_{X \in \Delta} X,$ indeed $A(\alpha)$ is the α-torsion submodule of M (see 2.18 in [18]). Then $k\text{-dim}\,A(\alpha) \leq \alpha$ by Theorem 2.14. Now let B be a submodule of M such that $B \nsubseteq A(\alpha),$ then $k\text{-dim}\,B \nleq \alpha$ and so $k\text{-dim}\,\frac{M}{B} \leq \alpha,$ for M is α-Krull. In order to show that α is the least ordinal number with this property, it is sufficient to prove the converse. So let M has a submodule $A(\alpha)$ with mentioned conditions and $B \subseteq M,$ if $B \subseteq A(\alpha),$ then $k\text{-dim}\,B \leq \alpha.$ But, if $B \nsubseteq A(\alpha),$ then k-dim $\frac{M}{B} \leq \alpha,$ but α is the least ordinal number with this property and so M is α-Krull. □

Our main aim in this paper is to characterize modules M, that for any ordinal number α, M is α-Krull if and only if $k\text{-dim}\, M = \alpha$. We use the following notations throughout the paper:

(1) $\mathcal{M} =$ the set of all modules with Krull dimension.

(2) $\mathcal{M}_\alpha =$ the set of all α-Krull modules.

(3) $\mathcal{A}_\alpha =$ the set of all α-Krull modules M with $k\text{-dim}\, M = \alpha$ and $\mathcal{A} = \bigcup_\alpha \mathcal{A}_\alpha$.

(4) $\mathcal{B}_\alpha =$ the set of all α-Krull modules M with $k\text{-dim}\, M = \alpha + 1$ and $\mathcal{B} = \bigcup_\alpha \mathcal{B}_\alpha$.

Remark 3.3. It is easy to see that $\mathcal{A}_\alpha \cap \mathcal{B}_\alpha = \varnothing$ and $\mathcal{A}_\alpha \cup \mathcal{B}_\alpha = \mathcal{M}_\alpha$ and so $\{\mathcal{A}_\alpha, \mathcal{B}_\alpha\}$ is a partition of $\mathcal{M}_\alpha$. Similarly, $\mathcal{A} \cap \mathcal{B} = \varnothing$ and $\mathcal{M} = \mathcal{A} \cup \mathcal{B}$, so $\{\mathcal{A}, \mathcal{B}\}$ is a partition of $\mathcal{M}$.

Clearly, $\mathcal{A}_{-1} = \{0\}$ and $\mathcal{B}_{-1}$ is the set of all simple modules.

Theorem 3.4. *Let M be an R-module. If $M \in \mathcal{B}_\alpha$ and $N \subseteq M$, then either $N \in \mathcal{B}_\alpha$ or $\frac{M}{N} \in \mathcal{B}_\alpha$.*

Proof. By the above notations, M is α-Krull and $k\text{-dim}\, M = \alpha + 1$. We have the following cases:

Case 1. If $k\text{-dim}\, N > k\text{-dim}\, \frac{M}{N}$, then $k\text{-dim}\, N = k\text{-dim}\, M = \alpha + 1$, by Theorem 2.11. Also, N is β-Krull for some $\beta \leq \alpha$. If $\beta < \alpha$, by Proposition 2.7, we have $k\text{-dim}\, N \leq \beta + 1 \leq \alpha$. This is a contradiction. Consequently, N is α-Krull with $k\text{-dim}\, N = \alpha + 1$. Therefore, $N \in \mathcal{B}_\alpha$.

Case 2. If $k\text{-dim}\, \frac{M}{N} > k\text{-dim}\, N$, then $k\text{-dim}\, \frac{M}{N} = k\text{-dim}\, M = \alpha + 1$, by Theorem 2.11. Also, $\frac{M}{N}$ is β-Krull for some $\beta \leq \alpha$. If $\beta < \alpha$, by Proposition

2.7, we have $k\text{-dim}\,\frac{M}{N} \leq \beta + 1 \leq \alpha,$ a contradiction. Consequently, $\frac{M}{N}$ is α-Krull with $k\text{-dim}\,\frac{M}{N} = \alpha + 1.$

Therefore, $\frac{M}{N} \in \mathcal{B}_{\alpha}.$

At last,

Case 3. If $k\text{-dim}\,N = k\text{-dim}\,\frac{M}{N},$ then the same argument shows that both N and $\frac{M}{N}$ belong to $\mathcal{B}_{\alpha}.$ □

In view of the proof of Theorem 3.4, the following results are now immediate.

Corollary 3.5. *Let* $M \in \mathcal{B}_{\alpha}$ *and* $N \subseteq M$. *Then we have the following*:

(1) *If* $k\text{-dim}\,N > k\text{-dim}\,\frac{M}{N},$ *then* $N \in \mathcal{B}_{\alpha}.$

(2) *If* $k\text{-dim}\,N < k\text{-dim}\,\frac{M}{N},$ *then* $\frac{M}{N} \in \mathcal{B}_{\alpha}.$

(3) *If* $k\text{-dim}\,N = k\text{-dim}\,\frac{M}{N},$ *then* $N, \frac{M}{N} \in \mathcal{B}_{\alpha}.$

Corollary 3.6. *If* $N \subseteq M$ *and* $N, \frac{M}{N} \in \mathcal{A},$ *then* $M \in \mathcal{A}.$

Corollary 3.7. *If* $M_1, M_2, ..., M_n \in \mathcal{A},$ *then* $M_1 \oplus M_2 \oplus \cdots \oplus M_n \in \mathcal{A}.$

Remark 3.8. The converse of Corollary 3.7 is not true, in general. For example, if $M = M_1 \oplus M_2,$ where M_1 and M_2 are simple modules, then $M_1, M_2 \in \mathcal{B}_{-1}.$ But M is a semisimple module with Krull dimension and so $M \in \mathcal{A}.$ Note that any semisimple module with Krull dimension has finite uniform dimension and so is both Artinian and Noetherian.

Remark 3.9. Let M be an α-critical module. If $\alpha = \beta + 1,$ then $M \in \mathcal{B}_{\beta}$ and if α is a limit ordinal, then $M \in \mathcal{A}_{\alpha}.$

Theorem 3.10. *Let M be an R-module, which is* $M \in \mathcal{M}_{\alpha}$. *Then the following statements are equivalent*:

(1) $M \in \mathcal{B}_{\alpha}$.

(2) $\frac{M}{A(\alpha)}$ *is* $(\alpha+1)$*-critical, where* $A(\alpha)$ *is the torsion submodule of M.*

Proof. If $M \in \mathcal{B}_{\alpha}$, then $k\text{-dim}\, M = \alpha + 1$. But $k\text{-dim}\, A(\alpha) \leq \alpha$, by Theorem 2.14. This implies that $k\text{-dim}\, \frac{M}{A(\alpha)} = k\text{-dim}\, M = \alpha + 1$. Now let $A(\alpha) \subsetneq B \subseteq M$, then $k\text{-dim}\, B \not\leq \alpha$ (note $A(\alpha)$ is the summation of all submodules of M with Krull dimension at most α) and so $k\text{-dim}\, \frac{M}{B} = k\text{-dim}\, \frac{M/A(\alpha)}{B/A(\alpha)} \leq \alpha$, since M is α-Krull. It follows that $\frac{M}{A(\alpha)}$ is $(\alpha+1)$-critical. Conversely, if $\frac{M}{A(\alpha)}$ is $(\alpha+1)$-critical, then it is α-Krull, by Remark 3.9. Moreover, $k\text{-dim}\, A(\alpha) \leq \alpha$ and so M is α-Krull, by Theorem 3.1. □

In the next theorem, we determine $\mathcal{B}$, the category of R-modules M, that is α-Krull if and only if $k\text{-dim}\, M = \alpha + 1$, for every ordinal number α.

Theorem 3.11. *Let M be an R-module. Then the following are equivalent*:

(1) $M \in \mathcal{B}$.

(2) $k\text{-dim}\, M$ *is not a limit ordinal and M has a co-critical submodule N such that* $k\text{-dim}\, N < k\text{-dim}\, \frac{M}{N}$.

Proof. If $M \in \mathcal{B}$, then there exists an ordinal number α such that $M \in \mathcal{B}_{\alpha}$, so $k\text{-dim}\, M = \alpha + 1$ is not a limit ordinal. Also, $A(\alpha)$ is co-critical and $k\text{-dim}\, A(\alpha) \leq \alpha < \alpha + 1 = k\text{-dim}\, \frac{M}{A(\alpha)}$, by Theorem 3.10.

Conversely, if $k\text{-dim}\, M = \alpha + 1$ and N is a co-critical submodule of M such that $k\text{-dim}\, N < k\text{-dim}\, \frac{M}{N}$, then $k\text{-dim}\, \frac{M}{N} = \alpha + 1$ so $\frac{M}{N}$ is $(\alpha + 1)$-critical and hence it is α-Krull, by Remark 3.9. Also, $k\text{-dim}\, N \leq \alpha$ and so M is α-Krull, by Theorem 3.1, therefore, $M \in \mathcal{B}$. □

The next theorem, which is an immediate consequence of Theorem 3.11, is the main result of this paper and as we promised, determines $\mathcal{A}$, consisting of all modules, which are α-Krull if and only if their Krull dimension are equal to α, for every ordinal number α.

Theorem 3.12. *Let M be an R-module. Then the following are equivalent*:

(1) $M \in \mathcal{A}$.

(2) *Either* $k\text{-dim}\, M$ *is a limit ordinal or* $k\text{-dim}\, N = k\text{-dim}\, M$, *for any co-critical submodule N of M*.

We know that every Noetherian module has Krull dimension. The next result is devoted to Noetherian α-Krull modules.

Theorem 3.13. *A Noetherian module M is α-Krull if and only if either* $k\text{-dim}\, N \leq \alpha$ *or* $k\text{-dim}\, \frac{M}{N} \leq \alpha$, *for any co-critical submodule N of M and* α *is the least ordinal number with this property*.

Proof. The "only if" part is true by definition. For the "if" part, let

$$\sum = \left\{N \subsetneq M : k\text{-dim}\, N > \alpha,\, k\text{-dim}\, \frac{M}{N} > \alpha\right\}.$$

If $\sum \neq \varnothing$, then it has a maximal element say N_0, since M is Noetherian. Clearly, $k\text{-dim}\, \frac{M}{N_0} > \alpha$. Now, if $N_0 \subsetneq A \subseteq M$, then $k\text{-dim}\, A \geq k\text{-dim}\, N_0$ $> \alpha$. But $A \notin \sum$, by maximality of N_0 and so $k\text{-dim}\, \frac{M}{A} \leq \alpha$, thus k-dim

$\frac{M/N_0}{A/N_0} \leq \alpha.$ This shows that $\frac{M}{N_0}$ is critical, i.e., N_0 is a co-critical submodule of M, with $k\text{-dim}\, N_0 > \alpha$ and $k\text{-dim}\, \frac{M}{N_0} > \alpha$. This is a contradiction. □

We note that if M is a Noetherian module with $k\text{-dim}\, M = \alpha$, then for any ordinal $\beta \leq \alpha$, there exists a β-co-critical submodule N of M $\left(\text{i.e., } \frac{M}{N} \text{ is } \beta\text{-critical}\right)$. It suffices to take N to be maximal with respect to this property that $k\text{-dim}\, \frac{M}{N} \geq \beta$, which is similar to its dual in [13] and the comment which follows [15, Proposition 1.11].

In view of the above comment and previous theorem, we have the following fact.

Corollary 3.14. *Let M be a Noetherian* α*-Krull module. Then for each ordinal* $\beta < \alpha$, *there exists a submodule N of M such that* $\frac{M}{N} \in \mathcal{B}_\beta$.

Proof. For any $\beta < \alpha$, we have $\beta + 1 \leq \alpha$ and so by the above comment, M has a $(\beta + 1)$-co-critical submodule N. Thus $\frac{M}{N}$ is $(\beta + 1)$-critical and $\frac{M}{N} \in \mathcal{B}_\beta$, by Remark 3.9. □

Recall that a module is uniserial if its submodules are linearly ordered under inclusion. Let us recall the next theorem, see [3, Theorem 4.17].

Theorem 3.15. *Let M be a Noetherian uniserial R-module. Then* $k\text{-dim}\, M \leq 1$. *Moreover, if* $k\text{-dim}\, M = 1$, *then M is* 1*-critical.*

The next result is now immediate.

Corollary 3.16. *If M is a Noetherian uniserial R-module, then M is either* –1*-Krull or* 0*-Krull.*

It is easy to see that if M is an R-module with Krull dimension, which for every $0 \subsetneq N \subsetneq M$, $k\text{-dim}\, N \leq k\text{-dim}\, \frac{M}{N}$, then $M \in \mathcal{A}$. This raises the below natural question that we conclude the paper with.

Question 3.17. For which R-modules M with Krull dimension, $k\text{-dim}\, N \leq k\text{-dim}\, \frac{M}{N}$, for any $0 \subsetneq N \subsetneq M$?

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