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DMA CHART MONITORING OF THE FIRST INTEGER-VALUED AUTOREGRESSIVE PROCESSES OF POISSON COUNTS

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Abstract

In this paper, explicit formulas are proposed to evaluate the performance characteristics of the double moving average control chart (DMA chart) for the first order integer-valued autoregressive (INAR(1)) model and to check the accuracy of explicit formulas by means of Monte Carlo (MC) simulation. The characteristics of control charts are frequently measured by average run length (ARL), which is the expectation of the samples taken before a system signals that it is out of control. The results obtained from the proposed explicit formulas of the average run length (ARL) are in excellent agreement with the results obtained from the MC. The performance of the DMA chart gets better as the value of the span (w) decreases for upward shifts. The numerical results showed that when the process is out of control with increasing shifts, the DMA chart performs better as the value of the span (w) decreases.

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1. Introduction

Statistical quality control (SQC) is often implemented with processes of counts. Count data is used in various fields of practice due to the ease of data collection. The marginal distribution of count processes can often be modeled by a Poisson distribution with parameter λ [19]. One particular area where these counts can be useful is in process monitoring to detect shifts of a process from an in control state to various out of control states. Hence, quality loss can be reduced or prevented through corrective actions to bring the process back to a normal state [1]. Most prominent are two charts of the Shewhart type, namely, the *c* and *u* chart, which both monitor the marginal distribution of the process [5]. The *c* chart has been often used to monitor count data. Although the original control charts were developed for independent count data, such as the *c* and *u* chart, they used only the information in the last sample and ignored information given by the entire data sequence (memory-less property). Thus, these charts are known to have a poor performance in terms of detecting small shifts in the process mean. In the past few decades, an exponentially weighted moving average control chart (EWMA chart) was introduced by Roberts [14], which was effective at detecting small and moderate shifts. A cumulative sum control chart (CUSUM chart) was introduced by Page [6], which is sensitive to small shifts in the process mean. Recently, a moving average control chart (MA chart) was proposed by Khoo [11] and studied in order to monitor the fraction of non-conforming observations. The numerical results showed that the MA chart was more efficient than the *p* chart. Later, Khoo and Wong [12] proposed a double moving average control chart (DMA chart) when observations are from normal distributions. The numerical results showed that the DMA chart improved the average run length (ARL) of the MA chart using a Monte Carlo (MC) simulation. Typically, the count process assumes that they are independent and identically distributed (i.i.d.). However, observations could be serially autocorrelated, which may adversely affect the performance of the control chart under the assumption of independence [13]. A popular class of models for stationary real value processes is time series models, such as autoregressive (AR), moving average (MA) and autoregressive moving average (ARMA) models. These models have a simple autocorrelation structure and other attractive properties. As an example, count type data on bed utilization in a hospital [3] and web page requests arriving at a server [2] turned out to be highly autocorrelated. These counts often refer to rare events, making the Poisson distribution a reasonable choice for the marginal distribution of N_t. In case of real-valued time series, the ARMA model is able to model a great variety of serial dependent structures of stationary processes. Although the arithmetic operations are well defined over $N_0 = \{0, 1, ...\}$, the recursion ARMA model cannot be applied to an integer-valued case, since the multiplication of an integer by a real number usually results in a non-integer value. This motivated us to replace the scalar multiplication in the recursion ARMA model by binomial thinning [7]. Al-Osh and Alzaid [10] introduced the first order integer-valued autoregressive model (INAR(1) model), which is well suited for modeling the autocorrelation structure of a process with a Poisson distribution. The statistical properties of the INAR(1) model were discussed by McKenzie [8].

The average run length (ARL) statistic is one of the indicators used for comparing the efficiency of quality control charts. The performance of a control chart when the process is in control can usually be characterized by the in control average run length (ARL_0) - the average of observations before the control chart gives a false alarm as the in control process has gone to an out of control process. The performance of a control chart when the process is out of control is the average of delay time (ADT) - the average of observations between the process going out of control and the control chart giving an alarm that the process has gone out of control. Ideally, the value of ARL_0 of an acceptable chart should be sufficiently large and the value of ADT should be minimal. Most work that focused on evaluating the ARL for control charts has been studied in the previous literature. Traditionally, Roberts [14] studied the ARL for the EWMA chart using Monte Carlo simulations (MC) for processes following normal distributions, which could be used to find the ARL for a variety of parameter values. This approach is often used to test accuracy with other methods. Crowder [15] studied numerical quadrature methods to solve the exact integral equations (IE) for the ARL for normal distributions. Brook and Evans [4] used approximate formulas for the ARL of the EWMA chart by applying a finite-state Markov chain approach (MCA). Areepong and Novikov [20] derived explicit formulas for the ARL of the EWMA chart. Areepong and Sukparungsee [21] studied analytical ARL of the binomial of the DMA chart. Areepong [22] studied explicit formulas of the ARL for the MA chart to monitor the number of defective products. Areepong and Sukparungsee [23] studied closed-form formulas of the ARL of the MA chart for a nonconforming zero inflated process. Sukparungsee [16] studied average run length of double moving average control chart for zero-inflated count processes. Sukparungsee and Areepong [17] studied explicit expressions for the ARL of the DMA chart for a zero inflated binomial process. According to the explicit formulas, the results are in good agreement with the results obtained from the MC. Recently, Phantu et al. [18] studied explicit expressions of the ARL of the MA chart for a Poisson integer-valued autoregressive model.

In the literature, one can find at least four numerical procedures to evaluate average run length. The MC is simple to program and based on a large number of sample trajectories, so it is very time consuming. Moreover, it is difficult to use for optimization, though it is convenient in terms of controlling the accuracy of analytical approximations. An IE is the most advanced method currently available but it requires intensive programming to implement, even for the case of a Gaussian distribution. The MCA is considered a popular technique based on the approximation of matrix inversions. In addition, there are no theoretical results on the accuracy of this procedure in terms of the rate of convergence. The martingale approach is simple and convenient for approximation but it could also be implemented for the case of light-tailed distributions in which the moment generating functions exist. However, the results for ARL_0 and ADT usually cannot be obtained analytically and require intensive programming with specialized software even for the case of a normal distribution.

In this paper, we propose explicit formulas to evaluate the ARL_0 and ADT of the DMA chart when observations are Poisson count process with the first order integer-valued autoregressive (INAR(1)) model. Additionally, the explicit formulas of the ARL_0 and ADT can be generated as a set of optimal parameters that depend on the span (w) parameter for designing the DMA chart with a minimum of ADT.

2. Binomial Thinning

The binomial thinning operator introduced by Steutel and Harn [9] preserves the status of an integer random variable when N operates on by a parameter $\alpha \in (0, 1)$, which has proven to be an adequate alternative to scalar multiplication. If N is a discrete random variable with range $\{0, ..., n\}$, the thinning operation is defined as

$$\alpha \circ N = \sum_{i=1}^{N} X_i, \tag{1}$$

where X_i are i.i.d. Bernoulli counting sequence random variables $P(X_i = 1) = \alpha$ and $P(X_i = 0) = 1 - \alpha$. The operator $(\circ)$ is a random operator. The random variable $\alpha \circ N$ has a binomial distribution with parameter N and α counts the number of survivors from the count N remaining after thinning. Notice that the thinning operator confers greater dispersion on the number of survivors than the ordinary multiplication operator. For instance, in integer time series models, N may often be an equi-dispersed Poisson random variable with an equal mean and variance λ.

Suppose N_{t-1} is an integer random variable arising at time $t - 1$ and subject to binomial thinning to produce the number of survivors in the next period. Then, conditional on N_{t-1} for $\alpha \circ N$ is an integer random variable with variance $\alpha(1 - \alpha) N_{t-1}$, whereas αN_{t-1} has zero conditional variance and the mean and variance of the unconditional counterparts are $\alpha\lambda$ and $\alpha\lambda^2$. Expectation and variance of $\alpha \circ N$ can be obtained easily by applying well-known rules for conditional moments as follows:

$$E[\alpha \circ N] = \alpha E[N] \text{ and } V[\alpha \circ N] = \alpha^2 V[N] + \alpha(1-\alpha)E[N].$$

3. The First Order Integer-valued Autoregressive Model

The first order integer-valued autoregressive (INAR(1)) model is perfectly suited for modeling count data. The INAR(1) model makes use of thinning operators for coherency in the nature of count data. Some of the best results have been achieved with the INAR(1) model based on a binomial thinning operator, which were introduced by McKenzie [7] and Al-Osh and Alzaid [10]. This thinning operator is generated by counting series of Bernoulli distributed random variables. This model has many modifications and generalizations with respect to their order and marginal distribution, and it is quite suitable for use in counting certain random events. The INAR(1) model is defined by

$$N_t = \alpha \circ N_{t-1} + \varepsilon_t, \tag{2}$$

where N_t is the observable count at time t, α is the first order integer-valued autoregressive parameter, $(\circ)$ is the thinning operation at time t performed independently of each other and ε_t is an innovation. The INAR(1) model is the best fitting model for Poisson marginal distributions and ε_t follows the Poisson distribution with mean $\mu(1-\alpha)$ then $\varepsilon_t \sim Poi(\mu(1-\alpha))$ distribution. According to the above situation, it can be modeled as the INAR(1) model, in which the expectation and variance of the INAR(1) model are

$$E[N_t] = V[N_t] = \frac{\mu}{1-\alpha}.$$

Generally, the INAR(1) model could be changed in any unexpected occurrence, and then the change-point model of this process can be described by the following. Assume μ_0, α_0 and $\mu_0/(1-\alpha_0)$ are in control parameters. μ_1, α_1 and $\mu_1/(1-\alpha_1)$ are out of control parameter where $\mu_1 = (1+\delta)\mu_0$, $\alpha_1 = (1+\delta)\alpha_0$, $\mu_1/(1-\alpha_1) = (1+\delta)\mu_0/(1-\alpha_0)$ and δ is the magnitude of the shift for out of control processes.

4. Double Moving Average Control Chart

A double moving average control chart (DMA chart) was proposed by Khoo and Wong [12]. The observations of DMA statistic are the collected double moving average of the MA statistic. The DMA statistic is defined by

$$DMA_i = \begin{cases} \dfrac{MA_i + MA_{i-1} + MA_{i-2} + \cdots}{i}; & i \le w, \\ \dfrac{MA_i + MA_{i-1} + \cdots + MA_{i-w+1}}{w}; & w < i < 2w-1, \\ \dfrac{MA_i + MA_{i-1} + \cdots + MA_{i-w+1}}{w}; & i \ge 2w-1, \end{cases} \tag{3}$$

where MA_i is the statistic of the MA chart and w is the span at time t. The MA statistic can be written as follows:

$$MA_i = \begin{cases} \dfrac{N_i + N_{i-1} + N_{i-2} + \cdots}{i}; & i < w, \\ \dfrac{N_i + N_{i-1} + \cdots + N_{i-w+1}}{w}; & i \ge w. \end{cases} \tag{4}$$

Let observations $N_1, N_2, \ldots,$ be i.i.d. random variable with INAR(1) model, which are the collected moving average of span w at time i. The expectation and variance of the DMA chart are

$$E(DMA_i) = \frac{\mu}{1-\alpha}$$

and

$$V(DMA_i) = \begin{cases} \displaystyle\sum_{j=1}^{i}\left(\frac{1}{j}\right)\frac{\mu}{i^2(1-\alpha)}; & i \le w, \\ \displaystyle\sum_{j=i-w+1}^{i}\left(\frac{1}{j}\right) + (i-w+1)\left(\frac{1}{w}\right)\frac{\mu}{w^2(1-\alpha)}; & w < i < 2w-1, \\ \dfrac{\mu}{w^2(1-\alpha)}; & i \ge 2w-1. \end{cases}$$

The upper and lower control limits of the DMA chart are

$$UCL_i/LCL_i = \begin{cases} \dfrac{\mu_0}{1-\alpha_0} \pm H\sqrt{\displaystyle\sum_{j=1}^{i}\left(\frac{1}{j}\right)\frac{\mu_0}{i^2(1-\alpha_0)}}; & i \le w \\ \dfrac{\mu_0}{1-\alpha_0} \pm H\sqrt{\begin{array}{c}\displaystyle\sum_{j=i-w+1}^{i}\left(\frac{1}{j}\right)+(i-w+1) \\ \times\left(\dfrac{1}{w}\right)\dfrac{\mu_0}{w^2(1-\alpha_0)}\end{array}}; & w < i < 2w-1 \\ \dfrac{\mu_0}{1-\alpha_0} \pm H\sqrt{\dfrac{\mu_0}{w^2(1-\alpha_0)}}; & i \ge 2w-1, \end{cases} \tag{5}$$

where H is coefficient of control limit of DMA chart or a constant to be chosen.

5. Derivative of Explicit Formulas for Evaluating the Average Run Length of a Double Moving Average Control Chart

In this section, the analytical ARL of the DMA chart in INAR(1) observations is derived. Based on the central limit theorem (CLT), the average run length of the DMA chart can be derived as follows:

Let $ARL = n.$ Then

$$\begin{aligned} \frac{1}{n} = \frac{1}{n}P & \text{ (out of control signal at time } i \le w) \\ & + \frac{1}{n}P \text{ (out of control signal at time } w < i < 2w-1) \\ & + \left[\frac{n-(2w-2)}{n}\right]P \text{ (out of control signal at time } i \ge 2w-1). \end{aligned} \tag{6}$$

According to equation (6), the DMA statistics in terms of out of control signals at time i state are replaced as follows:

$$\begin{aligned}
\frac{1}{n} = \frac{1}{n}&\left\{\sum_{i=1}^{w}\left[P\left(\frac{\sum_{j=1}^{i} MA_j}{i} > UCL_{i\leq w}\right) + P\left(\frac{\sum_{j=1}^{i} MA_j}{i} < LCL_{i\leq w}\right)\right]\right\} \\
&+ \frac{1}{n}\left\{\sum_{j=i-w+1}^{2w-2}\left[P\left(\frac{\sum_{j=i-w+1}^{i} MA_j}{w} > UCL_{w<i<2w-1}\right)\right.\right. \\
&\left.\left.+ P\left(\frac{\sum_{j=i-w+1}^{i} MA_j}{w} < LCL_{w<i<2w-1}\right)\right]\right\} \\
&+ \left[\frac{n-(2w-2)}{n}\right]\left\{P\left(\frac{\sum_{j=i-w+1}^{i} MA_j}{w} > UCL_{i\geq 2w-1}\right)\right. \\
&\left.+ P\left(\frac{\sum_{j=i-w+1}^{i} MA_j}{w} < LCL_{i\geq 2w-1}\right)\right\}. \qquad (7)
\end{aligned}$$

Substitute the control limit of the DMA statistic from equation (5) into equation (7). Then, equation (7) can be rewritten as follows:

$$\begin{aligned}
\frac{1}{n} = \frac{1}{n}&\left\{\sum_{i=1}^{w}\left[P\left(\frac{\sum_{j=1}^{i} MA_j}{i} > \frac{\mu_0}{1-\alpha_0} + H\sqrt{\sum_{j=1}^{i}\left(\frac{1}{j}\right)\frac{\mu_0}{i^2(1-\alpha_0)}}\right)\right.\right. \\
&\left.\left.+ P\left(\frac{\sum_{j=1}^{i} MA_j}{i} < \frac{\mu_0}{1-\alpha_0} - H\sqrt{\sum_{j=1}^{i}\left(\frac{1}{j}\right)\frac{\mu_0}{i^2(1-\alpha_0)}}\right)\right]\right\} \\
&+ \frac{1}{n}\left\{\sum_{j=i-w+1}^{2w-2}\left[P\left(\frac{\sum_{j=i-w+1}^{i} MA_j}{w} > \frac{\mu_0}{1-\alpha_0}\right.\right.\right.
\end{aligned}$$

$$+ H\sqrt{\sum_{j=i-w+1}^{i}\left(\frac{1}{j}\right) + (i-w+1)\left(\frac{1}{w}\right)\frac{\mu_0}{w^2(1-\alpha_0)}}\Bigg)$$

$$+ P\left(\frac{\sum_{j=i-w+1}^{i} MA_j}{w} < \frac{\mu_0}{1-\alpha_0}\right.$$

$$\left.- H\sqrt{\sum_{j=i-w+1}^{i}\left(\frac{1}{j}\right) + (i-w+1)\left(\frac{1}{w}\right)\frac{\mu_0}{w^2(1-\alpha_0)}}\right)\Bigg]\Bigg\}$$

$$+ \left[\frac{n-(2w-2)}{2}\right]\left\{P\left(\frac{\sum_{j=i-w+1}^{i} MA_j}{w} > \frac{\mu_0}{1-\alpha_0} + H\sqrt{\frac{\mu_0}{w^2(1-\alpha_0)}}\right)\right.$$

$$\left.+ P\left(\frac{\sum_{j=i-w+1}^{i} MA_j}{w} < \frac{\mu_0}{1-\alpha_0} - H\sqrt{\frac{\mu_0}{w^2(1-\alpha_0)}}\right)\right\}. \tag{8}$$

The central limit theorem is used to derive the explicit formulas. Therefore, equation (8) can be rewritten as

$$\frac{1}{n} = \frac{1}{n}\left\{\sum_{i=1}^{w}\left[P\left(Z_1 > \frac{UCL_{i\le w} + 0.5 - \frac{\mu_0}{1-\alpha_0}}{\sqrt{\frac{\mu_0}{i^2(1-\alpha_0)}\sum_{j=1}^{i}\frac{1}{j}}}\right)\right.\right.$$

$$\left.\left.+ P\left(Z_1 < \frac{LCL_{i\le w} - 0.5 - \frac{\mu_0}{1-\alpha_0}}{\sqrt{\frac{\mu_0}{i^2(1-\alpha_0)}\sum_{j=1}^{i}\frac{1}{j}}}\right)\right]\right\}$$

$$+\frac{1}{n}\left\{\sum_{j=i-w+1}^{2w-2}\left[P\left(Z_2 > \frac{UCL_{w<i<2w-1}+0.5-\frac{\mu_1}{1-\alpha_0}}{\sqrt{\frac{\mu_1}{w^2(1-\alpha_0)}\sum_{j=i-w+1}^{w-1}\frac{1}{j}+(i-w+1)\left(\frac{1}{w}\right)}}\right)\right.\right.$$

$$\left.\left.+P\left(Z_2 < \frac{LCL_{w<i<2w-1}-0.5-\frac{\mu_0}{1-\alpha_0}}{\sqrt{\frac{\mu_0}{w^2(1-\alpha_0)}\sum_{j=i-w+1}^{w-1}\frac{1}{j}+(i-w+1)\left(\frac{1}{w}\right)}}\right)\right]\right\}$$

$$+\left[\frac{n-(2w-2)}{n}\right]\left\{P\left(Z_3 > \frac{UCL_{i\geq 2w-1}+0.5-\frac{\mu_0}{1-\alpha_0}}{\sqrt{\frac{\mu_0}{w^2(1-\alpha_0)}}}\right)\right.$$

$$\left.+P\left(Z_3 < \frac{LCL_{i\geq 2w-1}-0.5-\frac{\mu_0}{1-\alpha_0}}{\sqrt{\frac{\mu_0}{w(1-\alpha_0)}}}\right)\right\}, \tag{9}$$

where

$$Z_1 = \frac{\sum_{j=1}^{i} MA_j - \frac{\mu}{1-\alpha}}{\sqrt{\frac{\mu}{i^2(1-\alpha)}\sum_{j=1}^{i}\frac{1}{j}}},$$

$$Z_2 = \frac{\sum_{j=i-w+1}^{i} MA_j - \frac{\mu}{1-\alpha}}{\sqrt{\frac{\mu}{w^2(1-\alpha)}\sum_{j=i-w+1}^{w-1}\frac{1}{j}+(i-w+1)\left(\frac{1}{w}\right)}} \text{ and}$$

$$Z_3 = \frac{\sum_{j=1}^{i} MA_j - \frac{\mu}{1-\alpha}}{\sqrt{\frac{\mu}{w^2(1-\alpha)}}}.$$

According to equation (9), let

$$A = \sum_{i=1}^{w}\left[P\left(Z_1 > \frac{UCL_{i\le w} + 0.5 - \frac{\mu_0}{1-\alpha_0}}{\sqrt{\frac{\mu_0}{i^2(1-\alpha_0)}\sum_{j=1}^{i}\frac{1}{j}}}\right)\right.$$

$$\left. + P\left(Z_1 < \frac{LCL_{i\le w} - 0.5 - \frac{\mu_0}{1-\alpha_0}}{\sqrt{\frac{\mu_0}{i^2(1-\alpha_0)}\sum_{j=1}^{i}\frac{1}{j}}}\right)\right],$$

$$B = \sum_{j=i-w+1}^{2w-2}\left[P\left(Z_2 > \frac{UCL_{w<i<2w-1} + 0.5 - \frac{\mu_0}{1-\alpha_0}}{\sqrt{\frac{\mu_0}{w^2(1-\alpha_0)}\sum_{j=i-w+1}^{w-1}\frac{1}{j} + (i-w+1)\left(\frac{1}{w}\right)}}\right)\right.$$

$$\left. + P\left(Z_2 < \frac{LCL_{w<i<2w-1} - 0.5 - \frac{\mu_0}{1-\alpha_0}}{\sqrt{\frac{\mu_0}{w^2(1-\alpha_0)}\sum_{j=i-w+1}^{w-1}\frac{1}{j} + (i-w+1)\left(\frac{1}{w}\right)}}\right)\right]$$

and

$$C = P\left(Z_3 > \frac{UCL_{i\ge 2w-1} + 0.5 - \frac{\mu_0}{1-\alpha_0}}{\sqrt{\frac{\mu_0}{w^2(1-\alpha_0)}}}\right)$$

$$+ P\left(Z_3 < \frac{LCL_{i \geq 2w-1} - 0.5 - \frac{\mu_0}{1-\alpha_0}}{\sqrt{\frac{\mu_0}{w^2(1-\alpha_0)}}} \right).$$

Then, the explicit formula of the ARL for the DMA chart is rewritten by substituting *A*, *B* and *C* into equation (9) as follows:

$$\frac{1}{n} = \frac{1}{n} A + \frac{1}{n} B + \frac{n - (2w-2)}{n} C,$$

$$nC = 1 - A - B + (2w - 2)C.$$

As $ARL = n,$ then

$$ARL = ((1 - A) - B)C^{-1} + (2w - 2). \tag{10}$$

Consequently, the explicit formula of ARL is

$$ARL = \left[1 - \sum_{i=1}^{w-1} \left[P\left(Z_1 > \frac{UCL_{i \leq w} + 0.5 - \frac{\mu}{1-\alpha}}{\sqrt{\frac{\mu}{i^2(1-\alpha)} \sum_{j=1}^{i} \frac{1}{j}}} \right) \right. \right.$$

$$\left. \left. + P\left(Z_1 < \frac{LCL_{i \leq w} - 0.5 - \frac{\mu}{1-\alpha}}{\sqrt{\frac{\mu}{i^2(1-\alpha)} \sum_{j=1}^{i} \frac{1}{j}}} \right) \right] \right]$$

$$- \sum_{j=w+1}^{2w-2} \left[P\left(Z_2 > \frac{UCL_{w<i<2w-1} + 0.5 - \frac{\mu}{1-\alpha}}{\sqrt{\frac{\mu}{w^2(1-\alpha)} \sum_{j=i-w+1}^{w-1} \frac{1}{j} + (i - w + 1)\left(\frac{1}{w}\right)}} \right) \right.$$

$$+ P\left(Z_2 < \frac{LCL_{w<i<2w-1} - 0.5 - \frac{\mu}{1-\alpha}}{\sqrt{\frac{\mu}{w^2(1-\alpha)} \sum_{j=i-w+1}^{w-1} \frac{1}{j} + (i-w+1)\left(\frac{1}{w}\right)}} \right)\Bigg]\Bigg]$$

$$\times \left[P\left(Z_3 > \frac{UCL_{i\geq 2w-1} + 0.5 - \frac{\mu}{1-\alpha}}{\sqrt{\frac{\mu}{w^2(1-\alpha)}}} \right) \right.$$

$$\left. + P\left(Z_3 < \frac{LCL_{i\geq 2w-1} - 0.5 - \frac{\mu}{1-\alpha}}{\sqrt{\frac{\mu}{w^2(1-\alpha)}}} \right) \right]^{-1} + (2w-2). \tag{11}$$

Since the in control process is given parameters $\mu = \mu_0$, $\alpha = \alpha_0$ and $\mu/1-\alpha = \mu_0/1-\alpha_0$, the explicit formula for the ARL_0 of the DMA chart is as follows:

$$ARL_0 = \left[1 - \sum_{i=1}^{w-1} \left[P\left(Z_1 > \frac{UCL_{i\leq w} + 0.5 - \frac{\mu_0}{1-\alpha_0}}{\sqrt{\frac{\mu_0}{i^2(1-\alpha_0)} \sum_{j=1}^{i} \frac{1}{j}}} \right) \right.\right.$$

$$\left.\left. + P\left(Z_1 < \frac{LCL_{i\leq w} - 0.5 - \frac{\mu_0}{1-\alpha_0}}{\sqrt{\frac{\mu_0}{i^2(1-\alpha_0)} \sum_{j=1}^{i} \frac{1}{j}}} \right) \right] \right]$$

$$- \sum_{j=w+1}^{2w-2} \left[P\left(Z_2 > \frac{UCL_{w<i<2w-1} + 0.5 - \dfrac{\mu_0}{1-\alpha_0}}{\sqrt{\dfrac{\mu_0}{w^2(1-\alpha_0)} \displaystyle\sum_{j=i-w+1}^{w-1} \frac{1}{j} + (i-w+1)\left(\frac{1}{w}\right)}} \right) \right.$$

$$\left. + P\left(Z_2 < \frac{LCL_{w<i<2w-1} - 0.5 - \dfrac{\mu_0}{1-\alpha_0}}{\sqrt{\dfrac{\mu_0}{w^2(1-\alpha_0)} \displaystyle\sum_{j=i-w+1}^{w-1} \frac{1}{j} + (i-w+1)\left(\frac{1}{w}\right)}} \right) \right]$$

$$\times \left[P\left(Z_3 > \frac{UCL_{i\geq 2w-1} + 0.5 - \dfrac{\mu_0}{1-\alpha_0}}{\sqrt{\dfrac{\mu_0}{w^2(1-\alpha_0)}}} \right) \right.$$

$$\left. + P\left(Z_3 < \frac{LCL_{i\geq 2w-1} - 0.5 - \dfrac{\mu_0}{1-\alpha_0}}{\sqrt{\dfrac{\mu_0}{w^2(1-\alpha_0)}}} \right) \right]^{-1} + (2w-2). \tag{12}$$

On the other hand, if the process is in an out of control state with parameter $\mu = \mu_1$, $\alpha = \alpha_1$ and $\mu/1-\alpha = \mu_1/1-\alpha_1$, where $\mu_1 = (1+\delta)\mu_0$, $\alpha_1 = (1+\delta)\alpha_0$ and $\mu_1/1-\alpha_1 = (1+\delta)\mu_0/1-\alpha_0$, where δ is a magnitude of shift, the explicit formula for the ADT of the DMA chart can be written as follows:

$$ADT = \left[1 - \sum_{i=1}^{w-1} \left[P\left(Z_1 > \frac{UCL_{i\leq w} + 0.5 - \dfrac{\mu_1}{1-\alpha_1}}{\sqrt{\displaystyle\sum_{j=1}^{i} \frac{1}{j} \frac{\mu_1}{i^2(1-\alpha_1)}}} \right) \right. \right.$$

$$+ P\left(Z_1 < \frac{LCL_{i \le w} - 0.5 - \frac{\mu_1}{1-\alpha_1}}{\sqrt{\sum_{j=1}^{i} \frac{1}{j} \frac{\mu_1}{i^2(1-\alpha_1)}}}\right)\Bigg]\Bigg]$$

$$- \sum_{j=w+1}^{2w-2}\left[P\left(Z_2 > \frac{UCL_{w<i<2w-1} + 0.5 - \frac{\mu_1}{1-\alpha_1}}{\sqrt{\sum_{j=i-w+1}^{w-1} \frac{1}{j} + (i-w+1)\left(\frac{1}{w}\right)\frac{\mu_1}{w^2(1-\alpha_1)}}}\right)\right.$$

$$\left. + P\left(Z_2 < \frac{LCL_{w<i<2w-1} - 0.5 - \frac{\mu_1}{1-\alpha_1}}{\sqrt{\sum_{j=i-w+1}^{w-1} \frac{1}{j} + (i-w+1)\left(\frac{1}{w}\right)\frac{\mu_1}{w^2(1-\alpha_1)}}}\right)\right]$$

$$\times \left[P\left(Z_3 > \frac{UCL_{i \ge 2w-1} + 0.5 - \frac{\mu_1}{1-\alpha_1}}{\sqrt{\frac{\mu_1}{w^2(1-\alpha_1)}}}\right)\right.$$

$$\left. + P\left(Z_3 < \frac{LCL_{i \ge 2w-1} - 0.5 - \frac{\mu_1}{1-\alpha_1}}{\sqrt{\frac{\mu_1}{w^2(1-\alpha_1)}}}\right)\right]^{-1} + (2w-2). \qquad (13)$$

6. Results

In this section, the numerical results of the ARL for the INAR(1) model obtained by the explicit formulas for ARL_0 and ADT of DMA chart are calculated from equation (12) and equation (13), and compared with the numerical results obtained from the MC. The parameter values for the DMA chart were chosen given the desired $ARL_0 = 370$ and $H = 3.$ The in control parameters are given $\mu_0 = 1$ and $\alpha_0 = 0.2,$ the out of control parameters are μ_1, α_1 and $\mu_1/1 - \alpha_1,$ and the given parameters $(\delta) = (0.1, 1.0)(0.1), 1.5,$ 2.0. Table 1 shows the ARL of the DMA chart considering a change in μ. The results show that the ARL values calculated from explicit formulas and approximated from MC are in excellent agreement. The performance of the DMA chart is sensitive to only a few of the out of control situations, i.e., when $\delta \leq 0.4,$ the DMA chart with $w = 5$ shows the best performance because of the minimum ADT value. Note that calculations with the explicit formulas are simple and very fast with computational times of less than 1 second while the MC are very time consuming. Tables 2 and 3 are presented in the same manner as Table 1 when considering changes in α and changes in $\mu/1 - \alpha.$ For the INAR(1) model, the ARL of the DMA chart when considering changes in μ, α and $\mu/1 - \alpha$ shows that when the shifts increase, the DMA performs better as the value of w decreases. Therefore, the proposed formulas for ARL_0 and ADT for the DMA chart are simple to use, accurate and efficient for optimal design when compared with the results obtained from the MC. Moreover, they are easy to implement and greatly reduce the computation times.

Table 1. Comparison of the ARL of the DMA chart using explicit formulas with the MC for INAR(1) model given $\mu_0 = 1$, $\alpha_0 = 0.2$ and considering a change in μ

δ	$w = 2$	$w = 5$	$w = 10$	$w = 15$
0.0	370.398	370.398	370.398	370.398
	(371.052 ± 1.682[a])	(370.885 ± 1.702)	(371.652 ± 1.637)	(370.424 ± 1.603)
0.1	201.525	119.819	52.267	37.871
	(201.627 ± 0.943)	(119.586 ± 0.937)	(52.476 ± 0.946)	(37.666 ± 0.905)
0.3	53.518	16.632	16.182	22.985
	(53.825 ± 0.766)	(16.723 ± 0.788)	(16.345 ± 0.704)	(22.431 ± 0.761)
0.5	19.475	8.296	12.952	15.033
	(30.854 ± 0.631)	(8.522 ± 0.513)	(12.345 ± 0.544)	(15.442 ± 0.537)
0.7	9.629	6.468	9.375	8.942
	(9.523 ± 0.349)	(6.387 ± 0.367)	(9.508 ± 0.334)	(8.945 ± 0.332)
0.9	5.933	5.519	6.593	6.163
	(5.299 ± 0.181)	(5.174 ± 0.177)	(6.875 ± 0.148)	(6.456 ± 0.172)
1.0	4.943	5.116	5.655	5.382
	(4.329 ± 0.095)	(5.674 ± 0.093)	(5.623 ± 0.097)	(5.348 ± 0.091)
1.5	2.816	3.443	3.355	3.347
	(2.755 ± 0.052)	(3.547 ± 0.055)	(3.377 ± 0.046)	(3.525 ± 0.039)
2.0	2.126	2.461	2.434	2.434
	(2.238 ± 0.037)	(2.946 ± 0.028)	(2.317 ± 0.030)	(2.684 ± 0.029)

[a] is standard deviation. ( ) is Monte Carlo simulation.

Table 2. Comparison of the ARL of the DMA chart using explicit formulas with the MC for INAR(1) model given $\mu_0 = 1$, $\alpha_0 = 0.2$ and considering a change in α

δ	$w = 2$	$w = 5$	$w = 10$	$w = 15$
0.0	370.398	370.398	370.398	370.398
	(370.526 ± 1.689[a])	(370.586 ± 1.756)	(370.528 ± 1.473)	(370.293 ± 1.597)
0.1	298.968	264.593	221.813	180.796
	(298.346 ± 0.994)	(264.382 ± 0.917)	(227.468 ± 0.974)	(180.376 ± 0.965)
0.3	185.959	110.495	57.390	42.074
	(185.747 ± 0.726)	(110.276 ± 0.775)	(57.446 ± 0.718)	(42.635 ± 0.746)
0.5	111.428	46.243	24.896	27.102
	(111.643 ± 0.528)	(46.529 ± 0.511)	(24.586 ± 0.583)	(27.568 ± 0.508)
0.7	66.141	22.663	17.577	24.022
	(65.379 ± 0.372)	(22.645 ± 0.338)	(17.607 ± 0.359)	(24.647 ± 0.384)
0.9	39.711	13.506	15.151	21.917
	(39.551 ± 0.167)	(13.824 ± 0.119)	(15.347 ± 0.185)	(21.748 ± 0.174)
1.0	31.043	11.155	14.401	20.649
	(31.229 ± 0.096)	(11.523 ± 0.093)	(14.376 ± 0.095)	(20.318 ± 0.092)
1.5	10.335	6.547	10.763	12.284
	(10.284 ± 0.052)	(6.284 ± 0.055)	(10.364 ± 0.052)	(12.638 ± 0.047)
2.0	4.494	4.893	6.494	6.268
	(4.442 ± 0.033)	(4.526 ± 0.036)	(6.538 ± 0.037)	(6.523 ± 0.034)

[a] is standard deviation. ( ) is Monte Carlo simulation.

Table 3. Comparison of the ARL of the DMA chart using explicit formulas with the MC for INAR(1) model given $\mu_0 = 1$, $\alpha_0 = 0.2$ and considering a change in $\mu/1-\alpha$

δ	$w = 2$	$w = 5$	$w = 10$	$w = 15$
0.0	370.398	370.398	370.398	370.398
	(370.526 ± 1.684[a])	(370.585 ± 1.704)	(370.869 ± 1.683)	(370.684 ± 1.624)
0.1	295.972	136.45	46.25	34.103
	(295.374 ± 0.927)	(136.451 ± 0.942)	(46.827 ± 0.937)	(34.527 ± 0.968)
0.3	100.394	16.300	15.998	21.593
	(100.763 ± 0.748)	(16.547 ± 0.773)	(15.472 ± 0.764)	(21.475 ± 0.756)
0.5	34.532	8.087	11.492	10.929
	(37.284 ± 0.507)	(8.044 ± 0.527)	(11.783 ± 0.527)	(10.427 ± 0.529)
0.7	14.474	6.402	7.031	6.605
	(14.663 ± 0.371)	(6.482 ± 0.362)	(7.286 ± 0.376)	(8.284 ± 0.382)
0.9	7.501	4.284	4.773	4.944
	(7.648 ± 0.176)	(4.372 ± 0.176)	(4.821 ± 0.118)	(4.985 ± 0.174)
1.0	5.820	3.602	3.796	4.392
	(5.626 ± 0.095)	(3.686 ± 0.093)	(4.388 ± 0.092)	(4.359 ± 0.094)
1.5	2.799	2.808	2.824	2.965
	(2.825 ± 0.052)	(2.678 ± 0.057)	(2.746 ± 0.057)	(2.967 ± 0.055)
2.0	2.037	2.060	2.060	2.060
	(2.071 ± 0.031)	(2.035 ± 0.040)	(2.467 ± 0.038)	(2.143 ± 0.033)

[a] is standard deviation. ( ) is Monte Carlo simulation.

The numerical results from Tables 1-3 can be analyzed as optimal parameters for process changes in each magnitude of change in which a pair of optimal parameters are the coefficient of the DMA chart and width of span (H, w). The results of the optimal parameters for the DMA chart are calculated from equations 10-11. The pair of optimal parameter values of the DMA chart are determined to correspond to the value of $ARL_0 = 370$ with minimum ADT; in control parameters are given $\mu_0 = 3$, $\alpha_0 = 0.2$ and $\mu_0/1-\alpha_0 = 3.75$, out of control parameters values are given μ_1, α_1 and $\mu_1/1-\alpha_1$ and shift parameters $(\delta) = (0.1, 1.0)(0.1), 1.5, 2.0.$ The results in terms of an optimal parameter of the DMA chart are H and span (w) with minimum ADT, which are shown in Table 4.

Table 4. Optimal parameter of the DMA chart for INAR(1) model when $ARL_0 = 370$ and $H = 3$ for given $\mu_0 = 3$, $\alpha_0 = 0.2$

δ	Change in μ		Change in α		Change in $\mu/1-\alpha$	
	w	ADT	w	ADT	w	ADT
0.1	14	37.87	17	173.5	19	153.5
0.2	11	20.21	13	71.39	17	60.55
0.3	7	15.36	12	38.32	14	27.45
0.4	5	10.62	11	25.38	13	15.99
0.5	5	8.296	9	22.11	11	13.10
0.6	4	6.879	8	18.93	9	8.312
0.7	4	5.883	7	17.32	8	7.932
0.8	3	5.219	6	16.01	7	6.547
0.9	3	4.530	5	13.50	5	6.883
1.0	3	4.057	5	11.15	5	5.704
1.5	2	2.816	4	6.141	4	3.282
2.0	2	2.216	3	3.863	3	2.263

7. Conclusion

The explicit formulas of the ARL of the DMA chart for the INAR(1) model were derived. The results obtained from the explicit formulas compared with the MC show that the accuracy of the explicit formulas is in excellent agreement with the MC and they take much less time to compute. In addition, the explicit formulas are able to find a pair of optimal parameters (H, w) with a minimum of ADT. The results show that when the magnitude of the shift increases, the DMA chart performs better as the value of w decreases for all case studies. Furthermore, these explicit formulas are simple and easy to implement, greatly reducing computational times to less than 1 second.

8. Discussion

The DMA chart has memory-less properties and the ability to detect small shifts. Without loss of generality, this chart can be relaxed due to its feasibility with the span of the control limit (w). The DMA chart performs better as the values of w increase for small shifts. However, the number of observations must be sufficiently large.

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